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Stabilization of Switched Linear Systems

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***Abstract*—In this note, we study the stabilization problem of systems that switch among a finite set of controllable linear systems with arbitrary switching frequency. For both cases of known and unknown switching functions, feedback laws are designed to achieve exponential stability. For the later case, a method combining on-line adaptive estimation and feedback stabilization is developed in the controller design.**

***Index Terms*—Estimation, excitation, stability, stabilization, switched systems.**

I. Introduction

In recent years, the switched systems have attracted considerable efforts; see, e.g., [2], [6], [10], [11], and [15], among many others. This is because switched systems have strong engineering backgrounds; see, for instance, [16] and [17]. When the switching laws are modeled as finite state Markov chains, the stabilization problem of switched stochastic systems has been investigated by many authors, and necessary and sufficient conditions have been given to solve the problem for both the nonadaptive case where the switchings are available (c.f. [7] and [9] and the adaptive case where the switchings are unavailable (c.f. [19]).

We will consider the stabilization problem for switched linear systems as follows:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m \quad (1)$$

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where the switching law $\sigma(t) : [0, \infty) \rightarrow \Lambda$ is a piecewise constant function that is continuous from the right, and where $\Lambda = \{1, 2, \ldots, N\}$ for some integer N.

When the switching law has no given mode (or is arbitrary), one way to investigate the stability and stabilization problems is to find a common Lyapunov function for all the switching models (c.f. [1], [3], [5], [10], and [14]). The conditions in such an approach tend to be strong because the existence of a common Lyapunov function guarantees the stability of a system under *all* possible switchings.

Another commonly used approach is to assume that a system remains unswitched for a period long enough to allow the overshoots of the closed-loop system in the transient phases to fade (c.f. [8] and [12]).

In this note, we will consider the stabilization problem of systems that switch among a finite set of controllable linear systems at *any* given frequency. To guarantee the stability of such a system at a given switching rate, it is certainly not enough to just stabilize each individual system for the obvious reason that the overshoots may destroy the stability. A feedback should be designed so that the magnitudes of the states of each individual system will decay by half on any interval of a given length. We will achieve this by first developing an estimation on the overshoots of the transition matrices (see Lemma 3.2), which can be considered as an enhancement of the Squashing Lemma in [13].

We will first present a preliminary result for the case when the switching functions are explicitly given. Our design in this case applies whenever the switching frequency is finite and known, in particular when the "average-dwell-time"[8] (instead of just the dwell time) is positive. The way the switching frequency is defined (see Definition 2.1) allows our result to apply to the case when the switching functions have some fast switchings on some intervals, provided that the switching frequency is "bounded on average" in the long run. We will then continue with the case when the switching frequency is finite but unknown. Finally, we will develop a method that combines online adaptive estimation and stabilization to treat the case when the switching functions are not given. We remark that even in the simplest case when the switching law is given the controllability condition cannot be relaxed to stabilizability. It is not hard to find an example of a system that switches between two stable systems that fails to be stable with certain switchings.

II. Main Results

Consider a system as in (1) with a switching function $\sigma(t)$. The switching moments $0 < t_1 < t_2 < \cdots$ of $\sigma(t)$ are defined recursively by $t_{k+1} = \inf\{t > t_k : \sigma(t) \neq \sigma(t_k)\}$, $t_0 = 0$. The switching duration δ_k is defined by $\delta_k = t_k - t_{k-1}$ $(k = 1, 2, \ldots)$.

Definition 2.1: Consider a switching function $\sigma(t) : [0, \infty) \rightarrow \Lambda$.

- The *switching frequency* f of $\sigma(t)$ is defined by

$$f = \overline{\lim_{t\rightarrow\infty}} \frac{\{\text{Number of switches of } \sigma(\cdot) \text{ in } [0, t]\}}{t}. \quad (2)$$

- The *dwell time* of $\sigma(t)$ is defined by $\tau = \inf_k \delta_k$.

□

Throughout this note, we will need the following standard assumption: H1) The models (A_i, B_i), $i = 1, \ldots, N$, are controllable.

Our first result is for the case when switching functions are explicitly given.

Theorem 2.1: Assume H1) holds for a switched system as in (1). Let $\alpha > 0$ be given. Then, there exist a set of gain matrices $\{K_i : i = 1, \ldots, N\}$ such that for any given switching law σ with a frequency $f \leq \alpha$, the switched linear system (1) under the switched feedback law $u(t) = K_{\sigma(t)}x(t)$ is exponentially stable.

In Theorem 2.1, the gain matrices are designed based on an upper bound α of the switching frequencies. The stability can still be achieved when such an upper bound on f is not given. This is the content of the next theorem.

Theorem 2.2: Let Assumption H1) hold for the switched linear system (1). Then, for any switching law σ with an unknown finite frequency f, a linear feedback control can be constructed such that the closed-loop system is exponentially stable.

Finally, we consider the case when the switching process $\sigma(t)$ itself is not available, that is, when the values of $\sigma(t)$ are not given. In this case, we will need to assume that the dwell time is positive rather than merely assuming the finiteness of the switching frequency.

Theorem 2.3: Let H1) be satisfied for a switched linear system as in (1). Then, for any unknown switching law with a positive dwell time, a switching linear feedback can be constructed so that the closed-loop system is exponentially stable.

The proofs of Theorems 2.1 and 2.2 will be given in Section III, and the proof of Theorem 2.3 will be given in Section IV.

III. Stabilization With Known Switchings

For a given switching function, let $\bar{\delta}_k = (1/k)\sum_{i=1}^{k}\delta_i$, and let $\delta = \underline{\lim}_{k\to\infty} \bar{\delta}_k$. The following lemma is an immediate consequence of the definitions.

Lemma 3.1: For any switching function $\sigma(\cdot)$, it holds that $\delta = 1/f$. □

To be more precise, Lemma 3.1 means that $f < \infty$ if and only if $\delta > 0$ and that $\delta = 1/f$ for all possible values (including ∞) of f.

To prove Theorems 2.1 and 2.2, we need the following estimation on transition matrices, which can be considered as a refinement of the Squashing Lemma in [13]; see also [7] for a related result.

Lemma 3.2: Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be two matrices such that the pair (A, B) is controllable. Then, there exists $M > 0$ such that for any $\lambda > 0$, there exists a matrix $K \in \mathbb{R}^{m\times n}$ for which the following holds:

$$\left\|e^{(A+BK)t}\right\| \leq M\lambda^L e^{-\lambda t} \qquad \forall t \geq 0 \tag{3}$$

where $L = (n-1)(n+2)/2$, and where hereafter $\|\cdot\|$ denotes the operator norm induced by the Euclidean norm on $\mathbb{R}^n$. □

The significance of Lemma 3.2 is that the estimation $M\lambda^L$ on the overshoot of the transition matrix $e^{(A+BK)t}$ is made explicitly. Roughly, the overshoot is dominated by λ^L which can be absorbed by the decay term $e^{-\lambda t}$ over any given interval if λ is large enough. We refer the reader to [4] for a detailed proof of Lemma 3.2 with a constructive calculation of M.

Proof of Theorem 2.1: By Lemma 3.2 and Assumption H1), we know that for any $\lambda > 0$, there exist a set of gain matrices $\{K_i : i = 1, 2, \ldots, N\}$ such that

$$\left\|e^{(A_i+B_iK_i)t}\right\| \leq M\lambda^L e^{-\lambda t} \qquad \forall t \geq 0,\ i = 1, 2, \ldots, N \tag{4}$$

where $L = (n-1)(n+2)/2$, and $M > 0$ is a constant depending on $\{(A_i, B_i), 1 \leq i \leq N\}$ and n only.

Let $\alpha > 0$, and let $\sigma(t)$ be a given switching function with a switching frequency $f \leq \alpha$. Consider the linear state feedback $u(t) = K_{\sigma(t)}x(t)$. Let $\Phi(t, s)$ denote the transition matrix of the closed-loop system $\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)})x(t)$.

Pick any $t > 0$, and let $k = \max\{i : t_i \leq t\}$ (i.e., k is the number of the switches on [0, t]). With $t_0 = 0$, one has $\Phi(t, 0) = (\prod_{i=0}^{k-1}\Phi(t_{i+1}, t_i)) \cdot \Phi(t, t_k)$, from which it follows that

$$\begin{aligned}\|\Phi(t,0)\| &\leq \left(\prod_{i=0}^{k-1} M\lambda^L e^{-\lambda(t_{i+1}-t_i)}\right)\cdot M\lambda^L e^{-\lambda(t-t_k)} \\ &= (M\lambda^L)^{k+1}e^{-\lambda t}.\end{aligned} \tag{5}$$

By the definition of f, it is not hard to see that $k/t \leq f+1$ for t large enough. Hence, $k \leq (\alpha+1)t$ for t large enough. Substituting this into (5), we have

$$\|\Phi(t,0)\| \leq (M\lambda^L)^{(\alpha+1)t+1}e^{-\lambda t} \leq Ce^{-\mu t} \tag{6}$$

if t is large enough, where $C = M\lambda^L$, and $\mu = \lambda - (\ln M + L\ln\lambda)(\alpha+1)$ which can certainly be made positive by choosing λ suitably large. Hence, we have proved Theorem 2.1 with the decay estimation (6). □

Proof of Theorem 2.2: Let $\sigma(\cdot)$ be a given switching function with $f < \infty$. We need to construct an online estimate of an upper bound on the frequency of switching first. By Lemma 3.1, we only need to consider an estimation of δ.

Let us introduce the following recursively defined sequence (where $\bar{\delta}_{k+1}$ is defined as in the beginning of this section):

$$\hat{\delta}_{k+1} = \min\{\hat{\delta}_k, \bar{\delta}_{k+1}\} \quad \hat{\delta}_0 = 1, \qquad k = 0, 1, 2\ldots. \tag{7}$$

Let $\hat{\delta}$ denote the limit of the nonincreasing sequence $\{\hat{\delta}_k\}$. It can be seen that $\hat{\delta} \leq \delta$. By Lemma 3.1, we have $f \leq 1/\hat{\delta}$. Also observe that since $\delta > 0$, it holds that $\hat{\delta} > 0$.

To construct a desired feedback law, let M and L be the "universal" constants given as in (4) (without loss of generality, we may assume that $M \geq e$). Choose $\lambda_k > 0$ to satisfy the following equation for each integer $k > 0$ [which is set to get the decay estimation in (12)]:

$$\frac{\lambda_k}{2} - (\ln M + L\ln\lambda_k)\left(\frac{1}{\hat{\delta}_k}+1\right) = 1. \tag{8}$$

For each λ_k defined as before, applying the inequality (4), we know that there exists a set of gain matrices $\{K_i(k), i = 1, 2, \ldots, N\}$ such that

$$\left\|e^{(A_i+B_iK_i(k))t}\right\| \leq M\lambda_k^L e^{-\lambda_k t} \qquad \forall t \geq 0,\ i = 1, 2, \ldots, N. \tag{9}$$

The feedback control law is then defined as follows:

$$u(t) = K_{\sigma(t)}(k)x(t), \qquad t \in [t_k, t_{k+1}). \tag{10}$$

Similar to the proof of Theorem 2.1, let $\Phi(t, s)$ be the transition matrix for the closed-loop equation of (1) under the state feedback (10). Then, for $t_k \leq t < t_{k+1}$

$$\|\Phi(t,0)\| \leq \left(\prod_{i=0}^{k-1} M\lambda_i^L e^{-\lambda_i(t_{i+1}-t_i)}\right)\cdot M\lambda_k^L e^{-\lambda_k(t-t_k)}. \tag{11}$$

It is not hard to see that since the sequence $\{\hat{\delta}_k\}$ is decreasing and converges to $\hat{\delta} > 0$, the sequence $\{\lambda_k\}$ is increasing and tends to a finite positive limit, say λ. Hence, there is an integer k_0 large enough such that $\lambda/2 < \lambda_k \leq \lambda$ for all $k \geq k_0$. Consequently, it follows from (11) that for $t_k \leq t < t_{k+1}$ with $k \geq k_0 + 1$

$$\begin{aligned}\|\Phi(t,0)\| &\leq M^{k+1}\left(\prod_{i=0}^{k}\lambda_i\right)^L e^{-\sum_{i=0}^{k-1}\lambda_i(t_{i+1}-t_i)}\cdot e^{-\lambda_k(t-t_k)} \\ &\leq C(M\lambda^L)^k e^{-\frac{1}{2}\lambda t} \leq C(M\lambda^L)^{(f+1)t}e^{-\frac{1}{2}\lambda t} \\ &= Ce^{-\left(\frac{1}{2}\lambda - (\ln M + L\ln\lambda)(f+1)\right)t} \leq Ce^{-t}\end{aligned} \tag{12}$$

where $C = M\lambda^L\exp(-\sum_{i=0}^{k_0-1}(\lambda_i - (\lambda/2))(t_{i+1}-t_i))$, and for the last inequality we have used (8) with the fact that $f \leq 1/\hat{\delta}$. □

IV. Stabilization With Unknown Switchings

This section is concerned with the case where the switching process $\sigma(t)$ is not directly available. Throughout this section, we assume the following trivial condition on the identifiability of models: $(A_i, B_i) \neq (A_j, B_j)$ for all $i \neq j$.

Intuitively, when the dwell time τ is positive, one would naturally try to first identify the switching signals at the beginning of each time interval $[t_k, t_{k+1})$ using a short time period (say, $\ll \tau$), and then control the identified system in the rest of the time interval.

Consider the switched linear system (1) on any interval $[t_k, t_{k+1})$ $(k \geq 0)$:

$$\dot{x}(t) = A_{\sigma(t_k)}x(t) + B_{\sigma(t_k)}u(t), \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m \quad (13)$$

which is obviously a time-invariant system on $[t_k, t_{k+1})$. Throughout the sequel, we will assume that $\|x(t_k)\| \neq 0$, since otherwise the system can be trivially stabilized by simply taking $u(t) = 0$ for $t \geq t_k$.

To identify the unavailable switching signal, we consider a short time period $t \in [t_k, t_k + h)$, with $h < \tau$, and introduce the following "filtered" signals $y(t)$, $\eta(t)$ and $\phi(t)$:

$$y(t) = x(t) - \eta(t) \quad (14)$$

$$\dot{\eta}(t) + \eta(t) = x(t), \qquad \eta(t_k) = x(t_k) \quad (15)$$

$$\dot{\phi}(t) + \phi(t) = \left(x(t)^T, u(t)^T\right)^T, \qquad \phi(t_k) = 0. \quad (16)$$

It is obvious that $y(t)$, $\eta(t)$ and $\phi(t)$ (which will be used to identify the value of $\sigma(t)$ on $[t_k, t_{k+1})$) are all available signals given the observations $\{(x(s), u(s)) : t_k \leq s \leq t\}$. Set

$$Y_k = \int_{t_k}^{t_k+h} \phi(t)y(t)^T dt \quad \Phi_k = \int_{t_k}^{t_k+h} \phi(t)\phi(t)^T dt. \quad (17)$$

Then, we can define an estimate for $\sigma(t_k)$ as

$$\hat{\sigma}(t_k) = \arg\min_{1\leq i\leq N} \left\| Y_k - \Phi_k \begin{pmatrix} A_i^T \\ B_i^T \end{pmatrix} \right\|. \quad (18)$$

We will see shortly that if the "information" matrix Φ_k is of full rank, then $\hat{\sigma}(t_k)$ can indeed correctly estimate the unknown $\sigma(t_k)$.

Remark 4.1: Note that $\hat{\sigma}(t_k)$ can also correctly estimate $\sigma(t_k)$ even if there are bounded noises in the systems, provided that Φ_k has certain level of excitation. The main reason that make this possible is that (A_i, B_i) are distinct matrix pairs of finite number. It is natural to see that in the bounded noise case, a suitable signal-to-noise ratio is required for feasibility of this identification.

We now introduce a class of excitation signals, under which the matrix Φ_k will have full rank. Let $C^n[0, h]$ be the space of $\mathbb{R}^m$-valued functions defined on [0, h], which have continuous derivatives up to order n. For any $u \in C^n[0, h]$, let us denote $U(t)$ as

$$U(t) = \left(u(t)^T, \dot{u}(t)^T, \ldots, u^{(n)}(t)^T\right)^T. \quad (19)$$

Consider the following class of functions:

$$H_0 = \left\{u \in C^n[0, h] | U(0) = 0, \ \lambda_{\min}\left(\int_0^h U(t)U(t)^T dt\right) > 0\right\} \quad (20)$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a square matrix. It is not difficult to show that H_0 is not empty (see the Appendix for more details). Now, let us take $u^0(t)$ as any fixed function in H_0. Then, we denote C as

$$C = \max_{1\leq i\leq N}\left(\|e^{A_i h}\| + \beta\left\|\int_0^h e^{A_i(h-s)}B_i u^0(s)ds\right\|\right) \quad (21)$$

where $\beta > 0$ is a constant which is large enough such that $\beta > \sqrt{(2h/b_1)}\eta_0$ for some constants b_1 and η_0 to be defined later in (30) and (36). Let λ be large enough so that the following holds:

$$\lambda(\tau - h) - [\ln(MC) + L\ln\lambda] > 1 \quad (22)$$

where M and L are the "universal" constant appeared in (4). By Lemma 2.1, there exist gain matrices K_i $(i = 1, 2, \ldots, N)$ such that (4) holds. We can now define the control law on each interval $[t_k, t_{k+1})$ by

$$u(t) = \begin{cases} \beta\|x(t_k)\|u^0(t - t_k), & t \in [t_k, t_k + h) \\ K_{\hat{\sigma}(t_k)}x(t), & t \in [t_k + h, t_{k+1}) \end{cases} \quad (23)$$

where $\hat{\sigma}(t_k)$ is the estimate of $\sigma(t_k)$ given by (18). To prove Theorem 2.3, it is enough to show that (1) is exponentially stable under the feedback (23). We sketch the proof via several lemmas, which are proved in the Appendix.

Lemma 4.1: If the matrix Φ_k defined in (17) is of full rank, then $\hat{\sigma}(t_k) = \sigma(t_k)$.

We will show that the matrix Φ_k is of full rank by the following steps.

Lemma 4.2: Let $z(\cdot)$ be defined by

$$\dot{z}(t) = Az(t) + Bu(t) \quad z(t_k) = 0 \quad (24)$$

where $u(t)$ is defined as in (23), and $(A, B) \in \mathcal{S} \triangleq \{(A_i, B_i) : 1 \leq i \leq N\}$. Then

$$\lambda_{\min}\left\{\int_{t_k}^{t_k+h}\begin{pmatrix} z(t) \\ u(t)\end{pmatrix}(z(t) \quad u(t))\,dt\right\} \geq b_1\beta^2\|x(t_k)\|^2. \quad (25)$$

Lemma 4.3: Consider the switched linear system (13) on the interval $[t_k, t_k + h)$ with the control law defined by (23). Then

$$\lambda_{\min}\left\{\int_{t_k}^{t_k+h}\begin{pmatrix} x(t) \\ u(t)\end{pmatrix}(x(t) \quad u(t))\,dt\right\} > 0. \quad (26)$$

Lemma 4.4: Under the conditions of Lemma 4.3, the matrix Φ_k defined by (17) is of full rank.

Finally, we are ready to prove Theorem 2.3.

Proof of Theorem 2.3: By (13), we know that

$$x(t_k + h) = e^{A_{\sigma(t_k)}h}x(t_k) + \int_{t_k}^{t_k+h} e^{A_{\sigma(t_k)}(t_k+h-s)}B_{\sigma(t_k)}u(s)ds.$$

Consequently, by the control law (23) and the definition of the constant C in (21), it follows that

$$\|x(t_k + h)\| \leq C\|x(t_k)\| \quad (27)$$

By Lemmas 4.1 and 4.4, we have $\hat{\sigma}(t_k) = \sigma(t_k)$. It then follows from (13), (23), (4), and (27) that

$$\begin{aligned}\|x(t_{k+1})\| &= \left\|e^{\left(A_{\sigma(t_k)}+K_{\sigma(t_k)}B_{\sigma(t_k)}\right)(t_{k+1}-t_k-h)}x(t_k+h)\right\| \\ &\leq M\lambda^L e^{-\lambda(t_{k+1}-t_k-h)}C\|x(t_k)\| \\ &\leq (MC\lambda^L)^{k+1}e^{-\lambda\left(t_{k+1}-h(k+1)\right)}\|x(0)\| \\ &\leq e^{[\ln(MC)+L\ln\lambda-\lambda(\tau-h)](k+1)}\|x(0)\| \\ &\leq e^{-(k+1)}\|x(0)\|\end{aligned}$$

where for the last two inequalities we have used (22) and the fact that $t_{k+1} \geq \tau(k + 1)$.

One can then complete the proof of Theorem 2.3 by following the same arguments in the proofs of Theorems 2.1 and 2.2 to show that $x(t)$ tends to zero exponentially. □

V. Concluding Remarks

We have investigated in this note the stabilization problem for systems that switch among a finite set of linear systems. The main conditions used are the controllability of each subsystem and the finiteness of the switching frequency of the switching signals. Both the cases where the switchings are available and unavailable are considered, and control laws are designed to make the closed-loop switched systems asymptotically exponentially stable. Based on our precise estimate on the norm growth of closed-loop linear system, the design for stabilizing control is presented.

We remark that some extensions of the results of this note are straightforward. For example, by suitably strengthening the conditions on the switching signals, one can design control laws to achieve the uniform exponential stability of the switched linear systems; and by modifying the design of controllers, one can also deal with bounded disturbances. However, there are also many challenging problems for future research. It would be nice to understand theoretically the capability of the switched linear models (or controllers) in dealing with uncertain nonlinear dynamical systems, and thus to help to understand the capability of the feedback mechanism in dealing with nonparametric uncertainties which are more general than those treated in, e.g., [18].

Appendix

Proof of Lemma 4.1: It is easy to see from (13)–(15) that $\dot{y}+y=\dot{x}=Ax+Bu$, where for sake of simplicity, $A=A_{\sigma(t_k)}$, $B=B_{\sigma(t_k)}$. Noticing (16), we have

$$y(t)=\int_{t_k}^{t} e^{-(t-s)}\left[(A\quad B)\begin{pmatrix}x(s)\\u(s)\end{pmatrix}\right]ds=(A\quad B)\phi(t).$$

It then follows that

$$\int_{t_k}^{t_k+h}\phi(t)y(t)^T dt=\int_{t_k}^{t_k+h}\phi(t)\phi(t)^T\begin{pmatrix}A^T\\B^T\end{pmatrix}dt.$$

By (17), we have $Y_k=\Phi_k\binom{A^T}{B^T}$. Substituting this into (18), we see that Lemma 4.1 holds true. □

Proof of Lemma 4.2: Denote the characteristic polynomial of A by $s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0$. Then, $A^n+a_{n-1}A^{n-1}+\cdots+a_1A+a_0I=0$. Now, by iterating the (24), we have

$$\begin{aligned}z^{(i)}(t)&=Az^{(i-1)}(t)+Bu^{(i-1)}(t)\\&=A^iz(t)+A^{i-1}Bu(t)+A^{i-2}Bu^{(1)}(t)\\&\quad+\cdots+Bu^{(i-1)}(t).\end{aligned}\tag{28}$$

Multiplying both sides by a_i $(i=0,1,\ldots,n)$ with $a_n\triangleq 1$ and summing up, we get

$$\begin{aligned}\sum_{i=0}^{n}a_iz^{(i)}(t)=&\sum_{i=0}^{n}a_iA^iz(t)\\&+Bu^{(n-1)}(t)+(AB+a_{n-1}B)u^{(n-2)}(t)\\&+(A^2B+a_{n-1}AB+a_{n-2}B)u^{(n-3)}(t)+\cdots\\&+(A^{n-1}B+a_{n-1}A^{n-2}B+\cdots+a_1B)u(t)\\\triangleq&\sum_{i=0}^{n}\bar{B}_iu^{(i)}(t)\end{aligned}$$

where $\bar{B}_n=0$, and $\bar{B}_i=(A^{n-i-1}B+a_{n-1}A^{n-i-2}B+\cdots+a_{i+1}B)$ $(1\le i\le n-1)$. Next, let $w(t)=(z(t)^T\ \ u(t)^T)^T$, $D_i=(\bar{B}_i^T\ \ a_iI)^T$. Then, $\sum_{i=0}^{n}a_iw^{(i)}(t)=\sum_{i=0}^{n}D_iu^{(i)}(t)$, that is

$$\sum_{i=0}^{n}a_iw^{(i)}(t)=QU(t)$$

where $U(t)$ is defined as before and $Q=(D_0\ \ D_1\ \ \ldots\ \ D_n)$.

For any $0\neq\alpha\in\mathbb{R}^{n+m}$, set $\alpha(t)=\alpha^Tw(t)$, $V(t)=\alpha^TQU(t)$. It can be seen that $\sum_{i=0}^{n}a_i\alpha^{(i)}(t)=V(t)$. Define $W(t)=(\alpha(t)\ \ \alpha^{(1)}(t)\ \ \cdots\ \ \alpha^{(n-1)}(t))^T$. Then, we can formally construct a linear system as

$$\begin{cases}\dot{W}(t)=FW(t)+bV(t),\\\alpha(t)=cW(t)\end{cases}\tag{29}$$

where

$$F=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\&&&\ddots&\\0&0&0&\cdots&1\\-a_0&-a_1&-a_2&\cdots&-a_{n-1}\end{pmatrix}\quad b=\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$$
$$c=(1,\quad 0,\quad\cdots,\quad 0).$$

Now, by the definition of $W(t)$, we know that $W(t_k)=0$. Hence, $\alpha(t)=\int_{t_k}^{t}ce^{(t-s)F}bV(s)ds$. Therefore, by the definition of $\alpha(t)$, $V(t)$ and $u(t)$ we have

$$\begin{aligned}&\lambda_{\min}\left\{\int_{t_k}^{t_k+h}w(t)w^T(t)dt\right\}\\&=\inf_{\|\alpha\|=1}\int_{t_k}^{t_k+h}\left(\alpha^Tw(t)\right)^2dt\\&=\inf_{\|\alpha\|=1}\int_{t_k}^{t_k+h}(\alpha(t))^2dt\\&=\inf_{\|\alpha\|=1}\int_{t_k}^{t_k+h}\left(\int_{t_k}^{t}ce^{(t-s)F}bV(s)ds\right)^2dt\\&=\beta^2\|x(t_k)\|^2\inf_{\|\alpha\|=1}\int_{t_k}^{t_k+h}\left(\int_{t_k}^{t}ce^{(t-s)F}bV^0(s-t_k)ds\right)^2dt\\&=\beta^2\|x(t_k)\|^2\inf_{\|\alpha\|=1}\int_{0}^{h}\left(\int_{0}^{t}ce^{(t-s)F}bV^0(s)ds\right)^2dt\end{aligned}$$

where $V^0(t)=\alpha^TQU^0(t)$ with $U^0(t)$ being defined in a similar way as $U(t)$ in (19) but with $u(t)$ replaced by $u^0(t)$. Therefore, to prove the lemma, we only need to show that

$$b_1\triangleq\inf_{(A,B)\in\mathcal{S}}\inf_{\|\alpha\|=1}\int_{0}^{h}\left(\int_{0}^{t}ce^{(t-s)F}bV^0(s)ds\right)^2dt>0.\tag{30}$$

Note that $\mathcal{S}$ is a finite set and $\{\alpha\in R^{n+m}:\|\alpha\|=1\}$ is a compact set, so to prove (30) we only need to show that for any $\alpha\in R^{n+m}$ with $\|\alpha\|=1$

$$\int_{0}^{h}\left(\int_{0}^{t}ce^{(t-s)F}bV^0(s)ds\right)^2dt>0.\tag{31}$$

Suppose this is not the case. Then, for all $t \in [0, h]$

$$\int_0^t ce^{(t-s)F} bV^0(s)ds \equiv 0. \quad (32)$$

Let $i = \min_{0 \le j \le n-1}\{j \ge 0 : cF^j b \ne 0\}$. By the observability of (F, c) and the fact that $b \ne 0$, we know that such an i must exist.

Now, differentiating both sides of (32) yields

$$cbV^0(t) + \int_0^t cFe^{(t-s)F} bV^0(s)ds \equiv 0. \quad (33)$$

Note that $cb = 0$, we then have

$$\int_0^t cFe^{(t-s)F} bV^0(s)ds \equiv 0.$$

Furthermore, differentiating both sides of the above equation up to i times, and using the definition of i, we get

$$cF^i bV^0(t) + \int_0^t cF^{i+1} e^{(t-s)F} bV^0(s)ds \equiv 0. \quad (34)$$

Next, let $a \triangleq |cF^i b|^{-1} \max_{0 \le s \le h} |cF^{i+1} e^{sF} b|$. Then, it follows from (34) that $|V^0(t)| \le a \int_0^t |V^0(s)|ds$ for all $t \in [0, h]$. Hence, by the Bellman–Gronwall Lemma, we must have

$$V^0(t) \equiv 0, \qquad t \in [0, h]. \quad (35)$$

However, by the controllability of (A, B) and the definition of the matrix Q, it can be verified that $\lambda_{\min}(QQ^T) > 0$. Consequently, by the definition of $V^0(t)$ and the fact that $u^0 \in H_0$, we have for any $\alpha \in R^{n+m}$ with $\|\alpha\| = 1$

$$\begin{aligned}\int_0^h \left|V^0(s)\right|^2 ds &= \alpha^T Q \int_0^h U^0(s)\left(U^0(s)\right)^T dsQ^T \alpha \\ &\ge \lambda_{\min}(QQ^T)\lambda_{\min}\left(\int_0^h U^0(s)\left(U^0(s)\right)^T ds\right) \\ &> 0\end{aligned}$$

which clearly contradicts (35). Hence, (31) must be true. □

Proof of Lemma 4.3: Let $z(t)$ and $\xi(t)$ be defined as follows

$$\begin{aligned}z(t) &= x(t) - \xi(t), \\ \dot{\xi}(t) &= A_{\sigma(t_k)}\xi(t), \qquad \xi(t_k) = x(t_k).\end{aligned}$$

It is obvious that $z(\cdot)$ satisfies all the conditions required in Lemma 4.2. We now denote

$$\eta_0 = \max_{1 \le i \le N} \max_{0 \le t \le h} \|e^{A_i t}\| \quad (36)$$

It then follows from the definition of $\xi(t)$ that

$$\sup_{t \in [t_k, t_k+h]} \|\xi(t)\| \le \eta_0 \|x(t_k)\| \qquad \forall k. \quad (37)$$

Hence, by the definition of $z(t)$ we have for any $\alpha \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}^m$ with $\|\alpha\|^2 + \|\gamma\|^2 = 1$

$$\begin{aligned}&\int_{t_k}^{t_k+h} \left(\alpha^T z(t) + \gamma^T u(t)\right)^2 dt \\ &= \int_{t_k}^{t_k+h} \left[\left(\alpha^T x(t) + \gamma^T u(t)\right) - \alpha^T \xi(t)\right]^2 dt \\ &\le 2\int_{t_k}^{t_k+h} \left(\alpha^T x(t) + \gamma^T u(t)\right)^2 dt + 2\int_{t_k}^{t_k+h} \left\|\alpha^T \xi_t\right\|^2 dt.\end{aligned}$$

Since α and γ are arbitrary, by Lemma 4.2 and (37) we have

$$\begin{aligned}&\lambda_{\min}\left(\int_{t_k}^{t_k+h} \begin{pmatrix}x(t)\\u(t)\end{pmatrix}(x(t) \quad u(t))\,dt\right) \\ &\ge \frac{1}{2}\lambda_{\min}\left(\int_{t_k}^{t_k+h} \begin{pmatrix}z(t)\\u(t)\end{pmatrix}(z(t) \quad u(t))\,dt\right) - h\eta_0^2 \|x(t_k)\|^2 \\ &\ge \left[\frac{1}{2}b_1\beta^2 - h\eta_0^2\right]\|x(t_k)\|^2 > 0\end{aligned}$$

where, for the last inequality, we have used the choice of the β made in (21). □

Proof of Lemma 4.4: Consider the signal $\phi(t)$ as defined in (16). For any $0 \ne \alpha \in \mathbb{R}^{n+m}$, we have $\alpha^T\phi(t) = \int_{t_k}^t e^{-(t-s)} f_\alpha(s)ds$, where $f_\alpha(t) = \alpha^T \binom{x(t)}{u(t)}$. Note that

$$\begin{aligned}\int_{t_k}^{t_k+h} \left(\alpha^T\phi(t)\right)^2 dt &= \int_{t_k}^{t_k+h}\left(\int_{t_k}^t e^{-(t-s)} f_\alpha(s)ds\right)^2 dt \\ &= \int_0^h \left(\int_{t_k}^{t+t_k} e^{-(t_k+t-s)} f_\alpha(s)ds\right)^2 dt \\ &\ge e^{-h}\int_0^h \left(\int_0^t f_\alpha(t_k+s)ds\right)^2 dt. \quad (38)\end{aligned}$$

By Lemma 4.3, the integral $\int_0^t f_\alpha(t_k+s)ds$ cannot be identically zero for any $\alpha \in \mathbb{R}^{n+m}$ with $\alpha \ne 0$. So there exists at least a $0 < \mu_\alpha < h$, such that $\int_0^{\mu_\alpha} f_\alpha(t_k+\lambda)d\lambda > 0$ (or < 0). It follows that for any nonzero $\alpha \in \mathbb{R}^{m+n}$, it holds that $\alpha^T \int_{t_k}^{t_k+h} \phi(t)\phi^T(t)dt\,\alpha > 0$, which is the desired result.

Proof of the Nonempty Property of H_0 in (20): Let $a^1, a^2, \ldots, a^J$ be $J = m(n+1)$ linearly independent vectors in $\mathbb{R}^J$. Choose $0 < \tau_1 < \tau_2 < \cdots < \tau_J < h$ and choose a C^n function u so that $U(\tau_i) = a^i$ $(i = 1, 2, \ldots J)$ and $U(0) = 0$. For any $v \in \mathbb{R}^J$, there exists some $i = 0, 1, \ldots J$ such that $v^T U(\tau_i) \ne 0$. By the continuity of U

$$v^T\left(\int_0^h U(t)U(t)^T dt\right)v > 0. \quad (39)$$

One can see that $u \in H_0$ since (39) is true for any $v \in \mathbb{R}^J$.

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Exponential Estimates for Neutral Time-Delay Systems: An LMI Approach

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***Abstract*—Exponential estimates and sufficient conditions for the exponential stability of linear neutral time delay systems are given. The estimates are obtained for the case of known parameters as well as the uncertain case, including uncertainties in the difference term. The proof is based on Lyapunov–Krasovskii functionals, and the conditions are expressed in terms of linear matrix inequalities (LMIs).**

***Index Terms*—Exponential estimate, linear matrix inequalities (LMIs), Lyapunov–Krasovskii functional, neutral time-delay system, uncertain systems.**

I. Introduction

A large number of linear matrix ineqaulity (LMI)-type stability conditions for linear time delay systems have been reported in the literature; see the survey papers [4] and [9]. Formally speaking, these conditions provide the asymptotic stability of time delay systems only. Having in mind the fact that in many cases the asymptotic stability is a synonym of the exponential stability one may ask if there exists a possibility to use the LMI approach for deriving exponential estimates for solutions of time delay systems, too.

For the case of retarded-type systems several approaches have been used in order to obtain exponential estimates for the solutions: Some exponential estimates based on the LMI approach were reported in [10]. Exponential estimates, based on a generalization of Bellman–Gronwall lemma and the matrix measure concept, have been presented in [8] and [7]. The Lyapunov–Razumikhin approach to derive exponential bounds for solutions has been developed in [3] and [12]. The same issue has been addressed recently in [6], where, based on the complete type Lyapunov–Krasovskii functionals proposed in [5], exponential bounds for solutions of exponentially stable retarded type time-delay systems were obtained.

In this paper, an LMI approach is presented to construct exponential estimates for solutions for the case of the neutral type time-delay systems. A certain modification of standard LMI-type stability conditions is needed, it consists of a slightly new form of estimation of the time derivative of the corresponding Lyapunov–Krasovskii functionals. Some auxiliary results on exponential estimates for difference systems in continuous time are also needed. These results are valid for systems which are stable independent of delay. On the other hand, both the exponential decay rate and the corresponding γ factor in the exponential estimates depend on the particular value of the delay. All these results are presented in Section II. In Section III, the previous results are extended to the case of uncertain time delay systems. It is worth to be mentioned that in contrast with some other known robust stability conditions for the neutral-type delay systems, here it is allowed uncertainty also in the corresponding difference term. Section IV contains an illustrative example.

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