A Note on Overshoot Estimation in Pole Placements

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Abstract

In this note we show that for a given controllable pair (A, B) and any $\lambda > 0$, a gain matrix K can be chosen so that the transition matrix $e^{(A+BK)t}$ of the system $\dot{x} = (A + BK)x$ decays at the exponential rate $e^{-\lambda t}$ and the overshoot of the transition matrix can be bounded by $M\lambda^L$ for some constants M and L that are independent of λ. As a consequence, for any $h > 0$, a gain matrix K can be chosen so that the magnitude of the transition matrix $e^{(A+BK)t}$ can be reduced by $\frac{1}{2}$ (or by any given portion) over $[0, h]$. An interesting application of the result is in the stabilization of switched linear systems with any given switching rate (see [1]).

Key words: Linear system, transition matrix, Squashing Lemma.

1 Introduction

Consider a linear system

$$\dot{x} = Ax + Bu, \tag{1}$$

where $x(\cdot)$ takes values in $\mathbb{R}^n$, $u(\cdot)$ takes values in $\mathbb{R}^m$, and where A and B are matrices of appropriate dimensions. Suppose (A, B) is a controllable pair. It is a well known fact that for any $\lambda > 0$, a gain matrix K can be chosen so that the transition matrix of the system $\dot{x} = (A+BK)x$ decays exponentially at the rate of $e^{-\lambda t}$, that is, for some $R > 0$,

$$\left\| e^{(A+BK)t} \right\| \le Re^{-\lambda t},$$

where and hereafter $\|\cdot\|$ denotes the operator norm induced by the Euclidean norm on $\mathbb{R}^n$. To get a faster decay rate, it is natural to consider a "higher gain" matrix K_1. However, such a gain matrix in general results in a bigger overshoot for the transition matrix $e^{(A+BK_1)t}$. In this note, we show that in the pole placement practice, a gain matrix K can be chosen so that the overshoot of the transition matrix $e^{(A+BK)t}$ can be bounded by $M\lambda^L$ for some constants M and L independent of λ. As a consequence, one sees that for any $h > 0$, a gain matrix K can be chosen so that the magnitude of the transition matrix $e^{(A+BK)t}$ can be reduced by $\frac{1}{2}$ (or by any given portion) over $[0, h]$. Note that this is a stronger requirement than merely requiring $e^{(A+BK)t}$ to decay at an exponential rate. An interesting application of the result is in the stabilization of switched linear systems with a given switching rate (see [1]).

The estimate of the overshoots of transition matrices in the practice of pole assignments has been studied widely (see e.g. [5], [9] and [7]). Our main result in this note can be considered an enhancement of the Squashing Lemma (see [7], [6] and [4]) which says the following: for any $\tau_0 > 0$, $\delta > 0$, any $\lambda > 0$, it is possible to find K such that

$$\left\| e^{(A+BK)t} \right\| \le \delta e^{-\lambda(t-\tau_0)}. \tag{2}$$

In the current note, we show that K can be chosen so that the estimate in (2) can be strengthened to

$$\left\| e^{(A+BK)t} \right\| \le M\lambda^L e^{-\lambda t}$$

for some constants M and L which are independent of λ. Our proof is constructive that shows explicitly how M and L are chosen.

⋆ This work was supported partly by the Chinese National Natural Science Foundation. The work of Wang was also supported partly by the US National Science Foundation (No.DMS-0072620).

2 Main Result

In this section we present our main result.

Proposition 2.1 Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be two matrices such that the pair (A, B) is controllable. Then for any $\lambda > 0$, there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that

$$\left\| e^{(A+BK)t} \right\| \le M\lambda^L e^{-\lambda t}, \quad \forall\, t \ge 0, \tag{3}$$

where $L = (n-1)(n+2)/2$ and $M > 0$ is a constant, which is independent of λ and can be estimated precisely in terms of A, B and n.

Comparing with the Squashing Lemma obtained in [7], Proposition 2.1 has two improvements: (i). In (2), the estimate on the transient overshoot is exponentially proportional to the decay rate λ, which resulted in an estimation of the transition matrix in terms of $e^{-\lambda(t-\tau_0)}$ instead of $e^{-\lambda t}$. In (3), the estimate on the transient overshoot is proportional to λ^L instead of $e^{\lambda\tau_0}$ as in (2). This distinction between the two types of estimations may be significant for some possible extensions of our results to systems with external inputs. (ii). The value of the constant M in estimate (3) can be precisely calculated by using our constructive proof (see equation (10) in the sequel). This is certainly a very desirable feature for practical purposes. See Example 3.1 for some illustrations.

Proposition 2.1 was primarily presented and applied to a stabilization problem of switched linear systems in [2]. It was found later that a recent paper [3] also provides a similar result with similar proofs. The difference is that [3] only considered the single input case and the upper bound $M\lambda^L$ in (3) was found to be a polynomial $p(\lambda)$ in [3] without an explicit expression. Hence, our result has obvious merits in control design.

Proof of Proposition 2.1. First we consider a linear system (A, b) of a single input. Without loss of generality, we assume that (A, b) is in the Brunovsky canonical form:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Let $\lambda_1, \ldots, \lambda_n$ be n distinct, negative real numbers. There exists some $k \in \mathbb{R}^{1\times n}$ such that the characteristic equation of the closed-loop system $A + bk$ is $p(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$. Note that the closed-loop system is given by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \ldots, \quad \dot{x}_{n-1} = x_n,$$
$$\dot{x}_n = \beta_1 x_1 + \beta_2 x_2 + \cdots \beta_n x_n$$

for some $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}$. Hence, x_1 satisfies the equation

$$x_1^{(n)} = \beta_1 x_1 + \beta_2 \dot{x}_1 + \cdots + \beta_n x_1^{(n-1)}, \tag{4}$$

whose characteristic equation is the same as $p(\lambda)$. Hence, the general solution of (4) is

$$x_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t},$$

where $c_1, c_2, \ldots, c_n$ are constants. From the equations $x_2 = \dot{x}_1, x_3 = \dot{x}_2, \ldots, x_n = \dot{x}_{n-1}$, we have $x(t) = \Lambda_0 e^{Dt} c$, where

$$\Lambda_0 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

and where $c = \begin{pmatrix} c_1 & c_2 & \cdots c_n \end{pmatrix}^T$. Now, observe that $x(0) = \Lambda_0 c$, that is, $c = \Lambda_0^{-1} x(0)$ (note that Λ_0 is an invertible Vandermonde matrix). Comparing this with the transition matrix of the system, one sees that

$$e^{(A+bk)t} = \Lambda_0\, e^{Dt}\, \Lambda_0^{-1}. \tag{5}$$

Let $\lambda_{\max} = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$. Without loss of generality, assume that $\lambda_{\max} \ge 1$. To get an estimate on $\|\Lambda_0\|$ and $\|\Lambda_0^{-1}\|$, we need the following simple fact: for an $n \times n$ matrix C, let $c_{\max} = \max_{1\le i,j\le n} |c_{ij}|$. It is not hard to see that

$$\|C\| \le n c_{\max}.$$

Hence, we have

$$\|\Lambda_0\| \le n\lambda_{\max}^{n-1}. \tag{6}$$

To get an estimate on Λ_0^{-1}, first note that

$$\Lambda_0^{-1} = \frac{1}{\det\Lambda_0}\, \mathrm{adj}\, \Lambda_0, \tag{7}$$

where $\mathrm{adj}\, \Lambda_0$ denotes the adjoint matrix of Λ_0, and that

$$\det \Lambda_0 = \prod_{j>i} (\lambda_j - \lambda_i).$$

Hence, if we choose $\lambda_1, \ldots \lambda_n$ in such a way that $\lambda_{i+1} \le \lambda_i - 1$ with $\lambda_1 < 0$, we get $|\det \Lambda_0| \ge 1$.

Taking the structure of $\text{adj}\Lambda_0$ into account, it is easy to see that for $C = \text{adj}\Lambda_0$,

$$\begin{aligned} c_{\max} &\le (n-1)!\lambda_{\max}{}^{1+2+\cdots+(n-1)} \\ &= (n-1)!\lambda_{\max}{}^{\frac{n(n-1)}{2}}. \end{aligned}$$

Hence, by (7), we have

$$\left\|\Lambda_0^{-1}\right\| \le \|\text{adj}\Lambda_0\| \le n(n-1)!\lambda_{\max}{}^{\frac{n(n-1)}{2}}. \tag{8}$$

Consequently, (6) and (8) yield that

$$\begin{aligned} \left\|\Lambda_0\, e^{Dt}\, \Lambda_0^{-1}\right\| &\le n\lambda_{\max}^{n-1}\, \left\|e^{Dt}\right\|\, n(n-1)!\lambda_{\max}^{n(n-1)/2} \\ &\le nn!\lambda_{\max}^{(n-1)(n+2)/2} e^{-\lambda_{\min} t}, \end{aligned}$$

where $\lambda_{\min} = \min\{|\lambda_1|, \ldots, |\lambda_n|\}$.

Suppose for some $\rho > 1$, $\lambda_{\max} \le \rho\lambda_{\min}$. Then, it follows that

$$\left\|\Lambda_0 e^{Dt} \Lambda_0^{-1}\right\| \le M\lambda_{\min}^{(n-1)(n+2)/2} e^{-\lambda_{\min} t}, \tag{9}$$

where

$$M = nn!\, \rho^{(n-1)(n+2)/2}. \tag{10}$$

In summary, we need the following conditions on the λ_i's:

- $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct, real, and negative;
- $\lambda_{i+1} \le \lambda_i - 1$ for $1 \le i \le n-1$, and hence, $\lambda_{\max} = |\lambda_n|$, $\lambda_{\min} = |\lambda_1|$;
- $|\lambda_n| \le \rho\, |\lambda_1|$, for some constant $\rho > 1$.

Obviously, for any given $\lambda > 0$, it is easy to choose $\lambda_1, \cdots, \lambda_n$ to satisfy all the above conditions together with the condition that $\lambda_1 \le -\lambda$. For example, one can choose $\lambda_1 < \min\{-1, -\lambda\}$, and let $\lambda_{i+1} = \lambda_i - 1$ for $1 \le i \le n-1$. Since $|\lambda_n| = |\lambda_1 - (n-1)| \le n\, |\lambda_1|$, we see that ρ can be set as $\rho = n$.

With such choices of $\lambda_1, \lambda_2, \ldots, \lambda_n$, we see from (5) and (9) that the desired result hold.

Now we consider the case when (A, b) is not in the Brunovsky canonical form. In this case, find an invertible $T \in \mathbb{R}^{n\times n}$ such that $(T^{-1}AT, T^{-1}b)$ is in the Brunovsky canonical form.

For any given $\lambda > 0$, the above proof has shown that for $A_1 = T^{-1}AT$, $b_1 = T^{-1}b$, one can find $k_0 \in \mathbb{R}^{1\times n}$ such that

$$e^{(A_1+b_1k_0)t} \le M\lambda^L e^{-\lambda t},$$

where M is given by (10) for some chosen ρ, and $L = (n-1)(n+1)/2$. Clearly, with $k = k_0T^{-1}$, one has

$$e^{(A+bk)t} = T(e^{(A_1+b_1k_0)t})T^{-1} \le M_1\lambda^L e^{-\lambda t}, \tag{11}$$

where $M_1 = M\, \|T\|\, \left\|T^{-1}\right\|$.

Finally, we consider the multi-input system

$$\dot{x} = Ax + Bu, \tag{12}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$. Suppose that the system is controllable. By Heymann's Lemma (c.f., e.g., page 187 of [8]), one sees that for any $v \in \mathbb{R}^m$ such that $b := Bv \ne 0$, there exists some $K_0 \in \mathbb{R}^{m\times n}$ such that $(A+BK_0,\, b)$ is itself controllable. Hence, the conclusion of single-input case that has just been proved above is applicable to the controllable pair $(A + BK_0,\, b)$, and one then sees that there exists some $k \in \mathbb{R}^{1\times n}$ such that $\left\|e^{(A+BK_0+bk)t}\right\| \le M\lambda^L e^{-\lambda t}$ for all $t \ge 0$. Hence, with $K = K_0 + vk$, it holds that

$$\left\|e^{(A+BK)t}\right\| \le M\lambda^L e^{-\lambda t} \qquad \forall\, t \ge 0. \tag{13}$$

This completes the proof. □

Remark 2.2 In the above proof, we have used the fact that for a single input system (A, b) which is controllable, when it is not in the Brunovsky canonical form, one can find an invertible matrix T such that $(T^{-1}AT, T^{-1}b)$ is in the canonical form. To be more precise, the matrix T can be chosen as (see e.g., [8]):

$$T = \begin{pmatrix} b & Ab & \cdots & A^{n-1}b \end{pmatrix} \begin{pmatrix} a_{n-1} & \cdots & a_1 & 1 \\ \vdots & \vdots & 1 & 0 \\ a_1 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix},$$

where $a_1, \ldots, a_{n-1}$ are as in the characteristic polynomial of A given by

$$\det(sI - A) = s^n + a_1s^{n-1} + \cdots + a_{n-2}s + a_{n-1}.$$

From this one can find an estimate of $\|T\|$ and $\left\|T^{-1}\right\|$, which in turn will lead to an estimate of M_1 in (11).

3 An Example

The design technique is demonstrated in the following example.

Example 3.1 Consider the following controllable linear system:

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

With the help of MATLAB, we first calculate the transfer matrix

$$T_1 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

With the transfer matrix T_1, one has

$$T_1^{-1}A_1T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \end{pmatrix}, \quad T_1^{-1}B_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Calculation shows that $\|T_1\| = 1.80193754431757$ and $\|T_1^{-1}\| = 2.24697960199992$. Taking $\rho = n(= 3)$, we have

$$L = \frac{(n-1)(n+2)}{2} = 5, \tag{14}$$

$$M = \|T_1\|\|T_1^{-1}\|nn!n^{(n-1)(n-2)/2} \approx 218.642. \tag{15}$$

Suppose for some design purpose, a decay constant $\lambda = 49.894$ is given. Choosing $\lambda_1 = -\lambda$, $\lambda_2 = \lambda_1 - 1$, $\lambda_3 = \lambda_2 - 1$, the feedback K_1 can be easily calculated (under the normal form) as

$$\tilde{K}_1 \approx \begin{pmatrix} -151.681 & -7769.474 & -131773.562 \end{pmatrix}.$$

Back to the original coordinate frame, we have

$$K_1 = \tilde{K}_1T_1^{-1} \approx \begin{pmatrix} -124155.769 & 7769.474 & -7617.793 \end{pmatrix}.$$

With such a choice of K_1, we get the desired decay estimate

$$\left\|e^{(A+BK)t}\right\| \le M\lambda^L e^{-\lambda t} \quad \forall t \ge 0,$$

for the given decay constant $\lambda = 49.894$ with L and M given as in (14)–(15). □

4 Conclusion

In this note we show that if (A, B) is controllable, then for any $\lambda > 0$, a gain matrix K can be chosen such that the transition matrix $e^{(A+BK)t}$ decays at the exponential rate $e^{-\lambda t}$ and the overshoot of $e^{(A+BK)t}$ can be bounded by $M\lambda^L$ for some constants M and L that are independent of the decay constant λ. The result provides a convenient tool for control design, particularly for switched systems, see [1].

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Erratum

There is a mild flaw in the statement of Proposition 2.1 in the above paper (cf. [1]). We restate it as follows.

Proposition Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be two matrices such that the pair (A, B) is controllable. Then for any $\lambda \ge 1$, there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that

$$\left\|e^{(A+BK)t}\right\| \le M\lambda^L e^{-\lambda t}, \quad \forall t \ge 0, \tag{16}$$

where $L = (n-1)(n+2)/2$ and $M > 0$ is a constant, which is independent of λ and can be estimated precisely in terms of A, B and n.

The proof of Proposition 2.1 in [1] is only valid for the case when $\lambda \ge 1$ (instead of the original version of $\lambda > 0$), because the eigenvalues $\lambda_1, \dots, \lambda_n$ were chosen to satisfy $\lambda_1 \le -1$, and $\lambda_k \le \lambda_1$ for $k \ge 1$. For more details, we refer the reader to the discussions that followed formula (10) in [1].

A main motivation of the work in [1] was for us to develop the results in [2]. As in most applications of overshoot estimation for pole placements, the parameter λ in [2] was chosen as a number of large value. Hence, the correction does not affect our results in [2].

Acknowledgment. The authors would like to thank Prof. Elena De Santis for pointing out the error.

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