Small Gain Theorems on Input-to-Output Stability

Zhong-Ping Jiang*

Dept. of Electrical & Computer Engineering
Polytechnic University
Brooklyn, NY 11201, U.S.A.
zjiang@control.poly.edu

Yuan Wang†

Dept. of Mathematical Sciences
Florida Atlantic University
Boca Raton, FL 33431,U.S.A.
ywang@math.fau.edu

Abstract— This paper presents nonlinear small gain theorems for both continuous time systems and discrete time systems in a general setting of input-to-output stability. Most small gain theorems in the past literature were stated for interconnected systems of which the subsystems are either input-to-state stable, or are input-to-output stable and satisfy an additional condition of input/output-to-state stability. In this work we show that when the input-to-output stability is strengthened with an output Lagrange stability property, a small gain theorem can be obtained without the input/output-to-state stability condition.

I. Introduction

In the analysis of control systems, it is very often that stability properties are obtained as consequences of small gain theorems. Typically, a system is composed by two subsystems for which the output signals of one subsystem are the input signals of the other subsystem. For such interconnected systems, one would like to know how a stability property of an interconnected system can be determined by some gain conditions of the subsystems. Most of the classical work on the small gain theory applies to the linear (gain) case with a norm based approach. The work on small gain theorems for the nonlinear case began with the work [12], where a nonlinear generalization of the classical small-gain theorem was proposed for both continuous- and discrete-time feedback systems.

In [14], the notion of input-to-state stability (ISS) was introduced, which naturally encapsulates the "finite gain" concept and the classic stability notions used in ordinary differential equations, enabling one to give explicit estimates on transient behavior. In [6], a small gain theorem was given in the ISS-framework, which led an extended follow-up literature. In conjunction with the results in [16], a much simplified proof of the small gain theorem was provided in [1]. In [5], a small gain theorem was provided in terms of ISS-Lyapunov functions. In [2], the authors presented an ISS-type small gain theorem in terms of operators that recovers the case of state space form, with applications to serveral variations of the ISS property such as input-to-output stability, incremental stability and detectability for interconnected systems. Recently, some ISS-type small gain results for discrete time systems was discussed in [7].

Since its formulation in [14], the ISS notion has quickly become a foundational concept with wide applications in nonlinear feedback analysis and design. It has become an integrated part of several current textbooks and monographs, including [4, 9, 10, 11, 13]. A variation of the ISS notion, called input-to-output stability (IOS), was also introduced in [14]. In [17] and [18], several notions on input-to-output stability were studied. All the IOS notions require that the outputs of a system be asymptotically bounded by a gain function on the inputs, yet the notions differ from each other on how uniform the transient behavors are in terms of the initial values of state variables or the output variables. In this work, we will focus on small gain theorems for the so called output Lagrange IOS property. In addition to requiring that the outputs of a system be asymptotically bounded by a gain function on the inputs, the output Lagrange IOS property also require the overshoot be proportional to the size of the initial values of the outputs (instead of the size of the intial values of the states).

Our main contribution will be to provide small gain theorems on the output Lagrange IOS property for both continuous- and discrete-time systems. In the special case when the output variables of a system represent the full set of state variables, the output Lagrange IOS property becomes the ISS property. Thus, our results recover the ISS-type small gain theorems for both continuous- and discrete-time systems.

In the past work of the ISS-type smal gain theory, most of the results were obtained either for the ISS case, or for the IOS case but subject to an extra input/output-to-state stability condition (which roughly means that the state trajectories are bounded as long as the input and output are bounded. Clearly, this property is related to "detectability"). In the current work, we deal with the output Lagrange IOS case, and we show that for this case, small gain theorems can be developed without the above-mentioned input/output-to-state stability condition. Since the main ideas of the proofs, which are motivated by the approaches in [1], are the same for both continuous- and discrete-time systems, we present our results for both of the cases.

The paper is organized as follows. In Section II, we briefly discuss the output Lagrange IOS property. In Section III, we present our main results – an output Lagrange IOS small gain theorem for both the continuous- and discrete-time

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cases. In Section IV, we recover some ISS-small results for discrete-time systems. In Section V, we provid proofs of the main results presented in Section III.

II. Preliminaries

We consider, for the continuous time case, systems as in the following:

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)), \tag{1}$$

where the map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz continuous, and the output map $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuous. We also assume that $f(0, 0) = 0$ and $h(0) = 0$.

The function $u : [0, \infty) \to \mathbb{R}^m$ represents an input function, which is assumed to be measurable and locally essentially bounded. For a given input function $u(\cdot)$ and an initial state ξ, we use $x(\cdot, \xi, u)$ to denote the maximal solution (defined on some maximal interval) of (1) with the input u such that $x(0, \xi, u) = \xi$.

For the discrete time case, we consider systems of the following form:

$$x(k + 1) = f(x(k), u(k)), \quad y(k) = h(x(k)), \tag{2}$$

where the maps $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuous, and where $f(0, 0) = 0$ and $h(0) = 0$.

In the discrete time case, an input $u : \mathbb{Z}_+ \to \mathbb{R}^m$ is a map from the set of nonnegative integers $\mathbb{Z}_+$ to $\mathbb{R}^m$. Slightly absuing the notations, we again use $x(\cdot, \xi, u)$ to denote the solution of (2) with the input function u and the initial state ξ.

In both the continuous case and the discrete case, we use $y(t, \xi, u)$ to denote $h(x(t, \xi, u))$.

We will use the symbol $|\cdot|$ for Euclidean norms in $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^p$, and $\|\cdot\|$ to denote the L_∞ norms of a function defined on either $[0, \infty)$ or $\mathbb{Z}_+$, taking values in $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^p$.

Below we briefly discuss some input-to-output stability properties. For detailed discussions on the properties, see [17] and [18].

Definition 1 A forward-complete continuous system as in (1) is:

- *input to output stable* (IOS) if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that

$$|y(t, \xi, u)| \le \beta(|\xi|, t) + \gamma(\|u\|) \tag{3}$$

  for all $t \ge 0$;

- *output-Lagrange input to output stable* (OL-IOS) if it is IOS <u>and</u> there exist some $\mathcal{K}$-functions σ_1, σ_2 such that

$$|y(t, \xi, u)| \le \max\{\sigma_1(|h(\xi)|), \sigma_2(\|u\|)\} \tag{4}$$

  for all $t \ge 0$.

A discrete time system as in (2) is:

- IOS if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that (3) holds for all $t \in \mathbb{Z}_+$;
- OL-IOS if it is IOS and there exist some $\mathcal{K}$-functions σ_1, σ_2 such that (4) holds for all $t \in \mathbb{Z}_+$.

Remark 2 It was shown in [17] that a system (1) is OL-IOS if and only if there exist some $\beta \in \mathcal{KL}$, some $\rho \in \mathcal{K}$ and some $\gamma \in \mathcal{K}$ such that

$$|y(t, \xi, u)| \le \beta\left(|h(\xi)|, \frac{t}{1 + \rho(|\xi|)}\right) + \gamma(\|u\|)$$

for all $t \in \mathbb{R}_{\ge 0}$ in the continuous-time case. This property nicely encapsulates both the IOS and the output-Lagrange aspects of the OL-IOS notion. It is not hard to see that the property also applies to discrete time systems (2) for all $t \in \mathbb{Z}_+$. □

The OL-IOS property can also be characterized in terms of output Lagrange stability property and the asymptotic gains.

Lemma 3 A forward-complete continuous time system as in (1) is OL-IOS if and only if the following holds:

- the output-Lagrange stability property holds, that is, for some $\sigma_1, \sigma_2 \in \mathcal{K}$, (4) holds for all $t \in \mathbb{R}_{\ge 0}$; and
- the *asymptotic gain* property holds: for some $\gamma \in \mathcal{K}$,

$$\overline{\lim_{t\to\infty}} |y(t, \xi, u)| \le \gamma(\|u\|). \tag{5}$$

A discrete time system as in (2) is OL-IOS if and only if the following holds:

- for some $\sigma_1, \sigma_2 \in \mathcal{K}$, (4) holds for all $t \in \mathbb{Z}_+$;
- for some $\rho \in \mathcal{K}$,

$$\overline{\lim_{k\to\infty}} |y(k, \xi, u)| \le \rho(\|u\|). \tag{6}$$

Remark 4 Combining Remark 2 with a similar argument as in the proofs of the corresponding results for ISS(see [15] for the continuous case and [7] for the discrete case), one can show that if a system (1) (or (2) respectively) admits the OL-IOS estimates as in (3) and (4), then the asymptotic gain function ρ in (5) (or (6) respectively) can be chosen to be γ. □

Remark 5 It is not hard to see that it results in an equivalent property if one strengthens (5) or (6) to:

$$\overline{\lim_{t\to\infty}} |y(t, \xi, u)| \le \overline{\lim_{t\to\infty}} \gamma(|u(t)|).$$

For the proof of Lemma 3, we refer to Theorem 1 of [3] for the continuous time case, where the proof was based on a theorem about approximations of relaxed trajectories by regular trajectories. The proof of the discrete time case is rather straightforward: as the limit function of a sequence of trajectories of (12) is also a trajectory of (12) (see the proofs of Theorem 1 of [7] and Proposition 3.2 of [8]), while in the continuous time case, the limit of a sequence of trajectories of (1) is in general a trajectory of the relaxed system of (1).

III. Main Results

In this section, we present an OL-IOS small gain theorem for both continuous time and discrete time systems.

A. A Small Gain Theorem – the Continuous Case

Consider a interconnected system as in the following:

$$\begin{array}{rcl} \dot{x}_1(t) &=& f_1(x_1(t), v_1(t), u_1(t)) \\ y_1(t) &=& h_1(x_1(t)), \\ \dot{x}_2(t) &=& f_2(x_2(t), v_2(t), u_2(t)) \\ y_2(t) &=& h_1(x_2(t)), \end{array} \tag{7}$$

subject to the interconnection constraints

$$v_1(t) = y_2(t), \quad v_2(t) = y_1(t), \tag{8}$$

where for $i = 1, 2$, $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, and $y_i(t) \in \mathbb{R}^{p_i}$, and where f_i is continuously differentiable, h_i is continuous.

Theorem 1 *Suppose both of the subsystems in (7) are complete and are* OL-IOS*, and thus, there exist some* $\beta_i \in \mathcal{KL}$ *and some* $\mathcal{K}$*-functions* γ_i^y, γ_i^u, σ_i^o, $(i = 1, 2)$*, such that the following holds for all* $t \in \mathbb{R}_{\geq 0}$*:*

$$\begin{array}{rcl} |y_1(t)| &\leq& \max\Big\{\beta_1(|x_1(0)|, t), \\ && \qquad \gamma_1^y(\|v_1\|),\ \gamma_1^u(\|u_1\|)\Big\}, \\ |y_2(t)| &\leq& \max\Big\{\beta_2(|x_2(0)|, t), \\ && \qquad \gamma_2^y(\|v_2\|),\ \gamma_2^u(\|u_2\|)\Big\}; \end{array} \tag{9}$$

and

$$\begin{array}{rcl} |y_1(t)| &\leq& \max\{\sigma_1^o(|h_1(\xi_1)|), \gamma_1^y(\|v_1\|), \gamma_1^u(\|u_1\|)\}, \\ |y_2(t)| &\leq& \max\{\sigma_1^o(|h_2(\xi_2)|), \gamma_2^y(\|v_2\|), \gamma_2^u(\|u_2\|)\}, \end{array} \tag{10}$$

where we have used $y_i(t)$ *to denote* $y_i(t, \xi_i, v_i, u_i)$*. If the small gain condition*

$$\gamma_1^y \circ \gamma_2^y(s) < s \qquad \forall\, s > 0, \tag{11}$$

then the interconnected system (7)-(8) is OL-IOS *with* (y_1, y_2) *as the output and* (u_1, u_2) *as the input.*

B. A Small Gain Theorem – the Discrete Case

In this section, we present a discrete analogue of the small gain theorem for systems as in the following:

$$\begin{array}{rcl} x_1(k+1) &=& f_1(x_1(k), v_1(k), u_1(k)) \\ y_1(k) &=& h_1(x_1(k)), \\ x_2(k+1) &=& f_2(x_2(k), v_2(k), u_2(k)) \\ y_2(k) &=& h_2(x_2(k)), \end{array} \tag{12}$$

subject to the interconnection constraints

$$v_1(k) = y_2(k)\ , \quad v_2(k) = y_1(k) \tag{13}$$

where for $i = 1, 2$ and for each $k \in \mathbb{Z}_+$, $x_i(k) \in \mathbb{R}^{n_i}$, $u_i(k) \in \mathbb{R}^{m_i}$, $v_i(k) \in \mathbb{R}^{p_i}$, and where f_i and h_i are continuous maps.

Theorem 2 *Assume that both of the subsystems of (12) are* OL-IOS*, and thus, there exist some* $\mathcal{KL}$*-functions* $\beta \in \mathcal{KL}$ *and some* $\mathcal{K}$*-functions* σ_i^o, γ_i^y *and* γ_i^u*,* $(i = 1, 2)$*, such that (9)–(10) hold for all* $t \in \mathbb{Z}_+$ *for any trajectory with* $y_i(t) = y_i(t, \xi_i, v_i, u_i)$ $(i = 1, 2)$*. If the small gain condition (11) holds, then the interconnected system (12)-(13) is* OL-IOS *with* (y_1, y_2) *as the output and* (u_1, u_2) *as the input.*

IV. The Case of Discrete ISS

Consider a discrete time system as in (2). When $h(\xi) = \xi$, the OL-IOS property becomes the input-to-state stability (ISS) property. The ISS property for discrete time systems was studied in detail in our previous work [7], where we have presented two ISS-small gain theorems (c.f. Theorems 2 and 3 of [7]) for interconnected systems. Although Theorems 2 and 3 of [7] are correct, their proofs carried in [7] were flawed with an incomplete statement of Lemma 4.1 of [7]. One of the motivations of the current work is to fix the flaw appeared in [7], and to show that, indeed, Theorems 2 and 3 of [7] can be seen as corollaries of Theorem 2 presented in the previous section. Observe that, when applying Theorem 2 to the case when $h_1(\xi_1) = \xi_1, h_2(\xi_2) = \xi_2$, the output Lagrange stability property (10) becomes redundant, as it is a direct consequence of (9).

Theorem 3 *Consider an interconnected system*

$$\begin{array}{rcl} x_1(k+1) &=& f_1(x_1(k), v_1(k), u_1(k)), \\ x_2(k+1) &=& f_2(x_2(k), v_2(k), u_2(k)), \end{array} \tag{14}$$

subject to the interconnection constraints

$$v_1(k) = x_2(k), \quad v_2(k) = x_1(k). \tag{15}$$

Suppose both the subsystems in (14) are ISS *in the sense that*

$$|x_1(k, \xi, v_1, u_1)| \leq \max\Big\{\beta_1(|\xi_1|, k),\ \gamma_1^x(\|v_1\|),\ \gamma_1^u(\|u_1\|)\Big\},$$

$$|x_2(k, \xi, v_2, u_2)| \leq \max\Big\{\beta_2(|\xi_2|, k),\ \gamma_2^x(\|v_2\|),\ \gamma_1^u(\|u_2\|)\Big\}.$$

If $\gamma_1^x \circ \gamma_2^x(s) < s$ *(equivalently,* $\gamma_2^x \circ \gamma_1^x(s) < s$*) for all* $s > 0$*, then the interconnected system (14)-(15) is* ISS *with* (u_1, u_2) *as input.* ∎

Invoking Theorem 2, one can also derive a Lyapunov small gain theorem for interconnected systems. Assume that both of the subsystems in (14) are ISS. By Theorem 1 of [7], both of the subsystems admit ISS-Lyapunov functions. That is, for each $i = 1, 2$, there exists a continuous map $V : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ such that

- for $i = 1, 2$, there exist $\overline{\alpha}_i, \underline{\alpha}_i \in \mathcal{K}_\infty$,

$$\underline{\alpha}_i(|\xi_i|) \leq V_i(\xi_i) \leq \overline{\alpha}_i(|\xi_i|) \quad \forall\, \xi; \tag{16}$$

- there exist $\sigma_i \in \mathcal{K}_\infty$ and ρ_i^x, ρ_i^u such that

$$\begin{aligned} V_1(f_1(\xi_1,\xi_2,\mu_1)) - V_1(\xi_1) &\le -\sigma_1(V_1(\xi_1)) \\ &\quad + \max\Big\{\rho_1^x(V_2(\xi_2)), \rho_1^u(|\mu_1|)\Big\}, \\ V_2(f_2(\xi_2,\xi_1,\mu_2)) - V_2(\xi_2) &\le -\sigma_2(V_2(\xi_2)) \\ &\quad + \max\Big\{\rho_2^x(V_1(\xi_1)), \rho_2^u(|\mu_1|)\Big\}. \end{aligned} \tag{17}$$

Without loss of generality, we assume that $\mathrm{Id} - \sigma_i \in \mathcal{K}$ for $i = 1, 2$ (c.f. Lemma B.1 of [7]).

Theorem 4 *Assume that x_1- and x_2-subsystems of (14) admit* ISS*-Lyapunov functions V_1 and V_2 respectively that satisfy (16)-(17), with $\mathrm{Id} - \sigma_i \in \mathcal{K}$ for $i = 1, 2$. If there exists a $\mathcal{K}_\infty$-function ρ such that*

$$\sigma_1^{-1} \circ \rho_1^x \circ \sigma_2^{-1} \circ \rho_2^x \ < \ \mathrm{Id}, \tag{18}$$

then the interconnected system (14)-(15) is ISS *with (u_1, u_2) as the input.*

Proof: For $i = 1, 2$, Let $\chi_i^x = \sigma_i^{-1} \circ \rho_i^x$, $\chi_i^u = \sigma_i^{-1} \circ \rho_i^u$. By Remark 3.7 of [7], one sees that

$$V_1(x_1(k)) \le \max\{V_1(x_1(0)),\ \chi_1^x(\|V_2(x_2)\|),\ \chi_1^u(\|u_1\|)\},$$

$$V_2(x_2(k)) \le \max\{V_2(x_2(0)),\ \chi_2^x(\|V_1(x_1)\|),\ \chi_2^u(\|u_2\|)\}.$$

To get the asymptotic gain conditions for $V_1(x_1(k))$ and $V_2(x_2(k))$, we apply Lemma 3.13 of [7] to get

$$\overline{\lim_{k\to\infty}} V_1(x_1(k)) \le \max\left\{\chi_1^x(\limsup_{k\to\infty} V_2(x_2(k))), \chi_1^u(\|u_1\|)\right\}$$

$$\overline{\lim_{k\to\infty}} V_2(x_1(k)) \le \max\left\{\chi_2^x(\limsup_{k\to\infty} V_1(x_1(k))), \chi_2^u(\|u_2\|)\right\}.$$

Consequently,

$$\begin{aligned} \overline{\lim_{k\to\infty}} V_1(x_1(k)) &\le \max\Big\{\chi_1^x \circ \chi_2^x(\limsup_{k\to\infty} V_1(x_2(k))), \\ &\qquad \chi_1^x \circ \chi_2^u(\|u_2\|),\ \chi_1^u(\|u_1\|)\Big\}, \\ \overline{\lim_{k\to\infty}} V_2(x_2(k)) &\le \max\Big\{\chi_1^x \circ \chi_2^x(\limsup_{k\to\infty} V_2(x_1(k))), \\ &\qquad \chi_2^x \circ \chi_1^u(\|u_1\|),\ \chi_1^u(\|u_2\|)\Big\}. \end{aligned}$$

With the small gain condition $\chi_1^x \circ \chi_2^x < \mathrm{Id}$, we conclude that

$$\overline{\lim_{k\to\infty}} V_1(x_1(k)) \le \max\left\{\chi_1^x \circ \chi_2^u(\|u_2\|),\ \chi_1^u(\|u_1\|)\right\},$$

$$\overline{\lim_{k\to\infty}} V_2(x_1(k)) \le \max\left\{\chi_2^x \circ \chi_1^u(\|u_1\|),\ \chi_2^u(\|u_2\|)\right\}.$$

By Theorem 2, the interconnected system (14)-(15) is OL-IOS with $(V_1(x_1), V_2(x_2))$ as output and (u_1, u_2) as input. By property (16), one sees that the system is ISS. ∎

V. Proofs of the Small Gain Theorems

The main idea of the proof in the continuous time case and the main idea in the proof in the discrete time case are the same. Both proofs are based on Lemma 3. Since much work has been done for continuous time samll gain theorems, and not so much has been done for the discrete counter part, we provide here a proof for the discrete time case.

Proof of Theorem 2. Suppose for some $\mathcal{KL}$-function β_i and some $\mathcal{K}$-function $\sigma_i^o, \gamma_i^y, \gamma_i^u$ $(i = 1, 2)$, (9)–(10) hold for all $t \in \mathbb{Z}_+$ along any trajectory with $y_i(t) = y_i(t, \xi_i, v_i, u_i)$. Assume the small gain condition (11) holds.

To show that the interconnected system (12)–(13) satisfies the output Lagrange stability property, we consider a trajectory $x(k) = (x_1(k), x_2(k))$ and the corresponding output function $y(k) = (y_1(k), y_2(k))$. By causality, for any $k \ge 1$,

$$|y_1(t)| \le \max\Big\{\sigma_1^o(|y_1(0)|, \gamma_1^y\left(\|y_2\|_{[k-1]}\right),\ \gamma_1^u(\|u_1\|)\Big\},$$

$$|y_2(t)| \le \max\Big\{\sigma_2^o(|y_2(0)|, \gamma_2^y\left(\|y_1\|_{[k-1]}\right),\ \gamma_2^u(\|u_2\|)\Big\},$$

where we have let $\|w\|_{[l]} = \max_{0\le j\le l} |w(j)|$ for any function w defined on $\mathbb{Z}_+$. Therefore,

$$\begin{aligned} |y_1(t)| &\le \max\Big\{\sigma_1^o(|y_1(0)|, \gamma_1^y \circ \sigma_2^o(|y_2(0)|), \\ &\qquad \gamma_1^y \circ \gamma_2^y\left(\|y_1\|_{[k-1]}\right),\ \gamma_1^y \circ \gamma_2^u(\|u_2\|),\ \gamma_1^u(\|u_1\|)\Big\}, \\ |y_2(t)| &\le \max\Big\{\sigma_2^o(|y_2(0)|, \gamma_2^y \circ \sigma_1^o(|y_1(0)|), \\ &\qquad \gamma_2^y \circ \gamma_1^y\left(\|y_2\|_{[k-1]}\right),\ \gamma_2^y \circ \gamma_1^u(\|u_1\|),\ \gamma_2^u(\|u_2\|)\Big\} \end{aligned}$$

for all $t \in \mathbb{Z}_+$. Taking the maximal values for $t = 0, 1, \dots k$, on both sides of the above inequality, we get:

$$\begin{aligned} \|y_1\|_{[k]} &\le \Big\{\sigma_1^o(|y_1(0)|), \gamma_1^y \circ \sigma_2^o(|y_2(0)|), \gamma_1^u(\|u_1\|) \\ &\qquad \gamma_1^y \circ \gamma_2^y\left(\|y_1\|_{[k-1]}\right),\ \gamma_1^y \circ \gamma_2^u(\|u_2\|), \Big\}, \\ \|y_2\|_{[k]} &\le \max\Big\{\sigma_2^o(|y_2(0)|, \gamma_2^y \circ \sigma_1^o(|y_1(0)|), \gamma_2^u(\|u_2\|) \\ &\qquad \gamma_2^y \circ \gamma_1^y\left(\|y_2\|_{[k-1]}\right),\ \gamma_2^y \circ \gamma_1^u(\|u_1\|),\ \Big\}. \end{aligned}$$

With the small condition (11), we get

$$\begin{aligned} |y_1(k)| &\le \max\Big\{\sigma_1^o(|y_1(0)|, \gamma_1^y \circ \sigma_2^o(|y_2(0)|), \\ &\qquad \gamma_1^y \circ \gamma_2^u(\|u_2\|),\ \gamma_1^u(\|u_1\|)\Big\} \\ |y_2(k)| &\le \max\Big\{\sigma_2^o(|y_2(0)|, \gamma_2^y \circ \sigma_1^o(|y_1(0)|), \\ &\qquad \gamma_2^y \circ \gamma_1^u(\|u_1\|),\ \gamma_2^u(\|u_2\|)\Big\}. \end{aligned}$$

Hence, one sees that the interconnected system (12)–(13) satisfies the following output Lagrange stability estimates:

$$|y_1(k)| \le \max\Big\{\tilde{\sigma}_1(|y(0)|),\ \tilde{\gamma}_1(\|u\|)\Big\},$$

$$|y_2(k)| \le \max\Big\{\tilde\sigma_2(|y(0)|),\ \tilde\gamma_2(\|u\|)\Big\},$$

for some $\mathcal{K}$-functions $\tilde\sigma_1, \tilde\sigma_2, \tilde\gamma_1$ and $\tilde\gamma_2$.

Next we show that the interconnected system admits an asymptotic gain. We have shown that for any given trajectory, there exists some $c \ge 0$ (depending on $|y(0)|$ and $\|u\|$) such that

$$|y_1(k)| \le c \quad \text{and} \quad |y_2(k)| \le c \qquad \forall\, k \in \mathbb{Z}_+.$$

Hence, both $\overline{\lim}_{k\to\infty} |h_1(x_1(k))|$ and $\overline{\lim}_{k\to\infty} |h_2(x_2(k))|$ are finite. Applying Lemma 3 together with Remarks 4 and 5 to get the following:

$$\overline{\lim_{k\to\infty}}\, |y_1(k)| \le \max\left\{\gamma_1^y\left(\overline{\lim_{k\to\infty}}\, |y_2(k)|\right),\ \gamma_1^u\left(\overline{\lim_{k\to\infty}}\, |u_1(k)|\right)\right\},$$

$$\overline{\lim_{k\to\infty}}\, |y_2(k)| \le \max\left\{\gamma_2^y\left(\overline{\lim_{k\to\infty}}\, |y_1(k)|\right),\ \gamma_2^u\left(\overline{\lim_{k\to\infty}}\, |u_2(k)|\right)\right\},$$

Consequently,

$$\begin{aligned}
\overline{\lim_{k\to\infty}}\, |y_1(k)| \le \max\Big\{&\gamma_1^y\circ\gamma_2^y\left(\overline{\lim_{k\to\infty}}\, |y_1(k)|\right), \\
&\gamma_1^y\circ\gamma_2^u\left(\overline{\lim_{k\to\infty}}\, |u_2(k)|\right),\ \gamma_1^u\left(\overline{\lim_{k\to\infty}}\, |u_1(k)|\right)\Big\},\\
\overline{\lim_{k\to\infty}}\, |y_2(k)| \le \max\Big\{&\gamma_2^y\circ\gamma_1^y\left(\overline{\lim_{k\to\infty}}\, |y_2(k)|\right), \\
&\gamma_2^y\circ\gamma_1^u\left(\overline{\lim_{k\to\infty}}\, |u_1(k)|\right),\ \gamma_2^u\left(\overline{\lim_{k\to\infty}}\, |u_2(k)|\right)\Big\}.
\end{aligned}$$

Again, with the small gain condition (11), we get the following asymptotic gain estimates:

$$\overline{\lim_{k\to\infty}}\, |y_1(k)| \le \rho_1(\|u\|),$$

$$\overline{\lim_{k\to\infty}}\, |y_2(k)| \le \rho_2(\|u\|)$$

for some $\rho_1, \rho_2 \in \mathcal{K}$.

We have proved that the interconnected system (12)–(13) satisfies both the output Lagrange stability property and the asymptotic gain property. By Lemma 3, the system is OL-IOS.

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