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Brief Paper

A Lyapunov Formulation of the Nonlinear Small-gain Theorem for Interconnected ISS Systems*

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Key Words—Interconnected systems; nonlinear gain; Lyapunov function; stability.

Abstract—The goal of this paper is to provide a Lyapunov statement and proof of the recent nonlinear small-gain theorem for interconnected input/state-stable (ISS) systems. An ISS–Lyapunov function for the overall system is obtained from the corresponding Lyapunov functions for both the subsystems. Copyright © 1996 Elsevier Science Ltd.

1. *Introduction*

The notion of nonlinear gains has recently been acknowledged as being of interest by a number of authors. Its use in generalizing the classical small- (finite-) gain theorem for nonlinear feedback systems was pointed out by Hill (1991) and Mareels and Hill (1992) within the input–output context. A similar idea of nonlinear gains was also introduced in the independent work of Sontag (1989, 1990, 1995) in a state-space setting. Recently, Jiang *et al.* (1994) have combined the idea of nonlinear gains from the above two different areas and established an L_∞ version of the nonlinear small-gain theorem in which the role of the initial conditions is made explicit and asymptotic stability (in the Lyapunov sense) for the internal states is included. Related results and applications of the nonlinear small-gain theorem in nonlinear robust stability and nonlinear stabilization have been pursued by Jiang (1993), Praly and Jiang (1993), Jiang *et al.* (1994), Praly and Wang (1994), Jiang and Mareels (1995), Teel and Praly (1996) and Lin *et al.* (1996). These studies are based on the concept of gain functions. It is well known that Lyapunov functions play an important role in the analysis and design of nonlinear dynamical systems, and it is therefore natural to ask whether these nonlinear small-gain results can be derived using Lyapunov-like arguments.

In this paper, we report on some preliminary results in this direction. Our main contribution is to establish a Lyapunov-type nonlinear small-gain theorem whose proof relies upon the construction of appropriate Lyapunov functions.

The layout of the paper is as follows. We start with some mathematical preliminaries in which we introduce the basic definitions and recall some results. The main result is stated and illustrated in Section 3. Section 4 is devoted to the proof of the main result. We conclude in Section 5. The appendix contains some technical lemmas used in the main proof.

2. *Mathematical preliminaries*

2.1. *Notation.* We employ $|\cdot|$ to denote the usual Euclidean norm for vectors and $\|\cdot\|$ to denote the L_∞ norm for time functions. For a real-valued differentiable function V, ∇V stands for its gradient. x^T is the transpose of the vector $x \in \mathbb{R}^n$.

2.2. *ISpS–Lyapunov functions.* Before stating our main theorem in Section 3, we introduce in this section some stability notions and some basic results.

Consider the following controlled dynamical system:

$$\dot{x} = f(x, u), \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz map. Controls are measurable essentially bounded functions from $\mathbb{R}_+$ to $\mathbb{R}^m$.

Recall that a function $\gamma: \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is decreasing and tends to zero at infinity.

Definition 2.1. (Jiang (1993), Jiang *et al.* (1994)). The system (1) is said to be *input-to-state practically stable* (ISpS) if there exist a function β of class $\mathcal{KL}$, a function γ of class $\mathcal{K}$ and a nonnegative constant d such that, for each initial condition $x(0)$ and each measurable essentially bounded control $u(\cdot)$ defined on $[0, \infty)$, the solution $x(\cdot)$ of the system (1) exists on $[0, \infty)$ and satisfies

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(\|u\|) + d \quad \forall t \ge 0 \tag{2}$$

When (2) is satisfied with $d = 0$, the system (1) is said to be *input-to-state stable* (ISS), a notion originally introduced by Sontag (1989, 1990).

Definition 2.2. A smooth (i.e. C^∞) function V is said to be an ISpS-*Lyapunov function* for the system (1) if

- V is proper, positive-definite, that is, there exist functions ψ_1, ψ_2 of class $\mathcal{K}_\infty$ such that

$$\psi_1(|x|) \le V(x)\psi_2(|x|) \quad \forall x \in \mathbb{R}^n; \tag{3}$$

- there exist a positive-definite function α, a class $\mathcal{K}$ function χ and a nonnegative constant c such that the following implication holds:

$$\{|x| \ge \chi(|u|) + c\} \Rightarrow \nabla V(x) f(x, u) \le -\alpha(|x|). \tag{4}$$

When (4) holds with $c = 0$, V is called an ISS–*Lyapunov function* for the system (1).

Remark 2.1. Observe that this definition is slightly different from the original definition proposed by Sontag and Wang (1995a) in that α is only required to be positive-definite rather than class $\mathcal{K}$ as in Sontag and Wang (1995a). The

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equivalence of both definitions can be shown, see also Remark 4.2 in Lin *et al.* (1996).

Remark 2.2. One can define the ISpS–Lyapunov function in a slightly different way. Instead of requiring that (4) hold for V, one asks that the following hold for V:

$$\nabla V(x)f(x,u) \le -a(V(x)) + \theta(|u|) + d \quad (5)$$

for some functions $a \in \mathcal{K}_\infty$, $\theta \in \mathcal{K}$, and some constant $d \ge 0$. Correspondingly, one asks for $d = 0$ in (5) for V to be an ISS–Lyapunov function.

It is immediate that the system (1) admits an ISpS– (respectively ISS–) Lyapunov function satisfying (3) and (4) if and only if it admits an ISpS– (respectively ISS–) Lyapunov function satisfying (3) and (5) (cf. Sontag and Wang, 1995a).

Recently, the equivalence between the ISpS property and the existence of an ISpS–Lyapunov function was shown by Sontag and Wang (1995b), i.e. the following was proved.

Proposition 2.1. The system (1) is ISpS (respectively ISS) if and only if it has an ISpS– (respectively ISS–) Lyapunov function.

3. *Main result*

The main purpose of this section is to derive a Lyapunov-type nonlinear small-gain theorem, rather than the gain-functions-based small-gain theorem as in Jiang (1993) and Jiang *et al.* (1994), for interconnected systems:

$$\dot{x}_1 = f_1(x_1, x_2, u_1), \quad (6)$$

$$\dot{x}_2 = f_2(x_1, x_2, u_2), \quad (7)$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $f_i: \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ is locally Lipschitz.

Assume that, for $i = 1, 2$, there exists an ISpS–Lyapunov function V_i for the x_i subsystem such that the following hold:

(i) there exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ such that

$$\psi_{i1}(|x_i|) \le V_i(x_i) \le \psi_{i2}(|x_i|) \quad \forall x_i \in \mathbb{R}^{n_i}; \quad (8)$$

(ii) there exist functions $\alpha_i \in \mathcal{K}_\infty$, $\chi_i, \gamma_i \in \mathcal{K}$ and some constant $c_i \ge 0$ such that

$$V_1(x_1) \ge \max\{\chi_1(V_2(x_2)), \gamma_1(|u_1|) + c_1\}$$

implies

$$\nabla V_1(x_1)f_1(x_1, x_2, u_1) \le -\alpha_1(V_1), \quad (9)$$

and $V_2(x_2) \ge \max\{\chi_2(V_1(x_1)), \gamma_2(|u_2|) + c_2\}$ implies

$$\nabla V_2(x_2)f_2(x_1, x_2, u_2) \le -\alpha_2(V_2). \quad (10)$$

In the following, we shall give a nonlinear small-gain condition under which an ISpS–Lyapunov function for the interconnected system (6), (7) may be expressed in terms of ISpS–Lyapunov functions for the two subsystems.

Theorem 3.1. Assume that, for $i = 1, 2$, the x_i subsystem has an ISpS–Lyapunov function V_i satisfying (8)–(10). If there exists some $c_0 \ge 0$ such that

$$\chi_1 \circ \chi_2(r) < r \quad \forall r > c_0 \quad (11)$$

then the interconnected system (6), (7) is ISpS. Furthermore, if $c_0 = c_1 = c_2 = 0$ then the system is ISS. In particular, the zero solution of (6), (7) with no input (i.e. $u = 0$) is globally asymptotically stable.

Remark 3.1. The condition (11) is equivalent to

$$\chi_2 \circ \chi_1(r) < r \quad \forall r > \bar{c}_0, \quad (12)$$

where $\bar{c}_0 \ge 0$, and $\bar{c}_0 = 0$ if and only if $c_0 = 0$.

Proof. Assume that (11) holds. Define

$$\bar{c}_0 = \sup\{r : \chi_2 \circ \chi_1(r) \ge r\}. \quad (13)$$

First note that, with (11), we have

$$\chi_2 \circ \chi_1(r) < r \quad \forall r \in (\chi_2(c_0), \chi_2(\infty)). \quad (14)$$

It follows from this that $\bar{c}_0 \le \chi_2(c_0)$. Therefore (12) follows from (13) readily.

By symmetry, one knows that if (12) holds then (11) holds with $c_0 \le \chi_1(\bar{c}_0)$. □

Remark 3.2. If V_1 and V_2 are ISpS–Lyapunov functions for subsystems satisfying (8), and

$$\nabla V_1(x_1)f_1(x_1, x_2, u_1) \le -a_1(V_1(x_1)) + \theta_1^x(V_2(x_2)) + \theta_1^u(|u_1|) + d_1, \quad (15)$$

$$\nabla V_2(x_1)f_2(x_1, x_2, u_2) \le -a_2(V_2(x_2)) + \theta_2^x(V_1(x_1)) + \theta_2^u(|u_2|) + d_2 \quad (16)$$

for some $a_1 \in \mathcal{K}_\infty$, θ_i^x, $\theta_i^u \in \mathcal{K}$ and $d_i \ge 0$ $(i = 1, 2)$ then χ_1 and χ_2 can be chosen as

$$\chi_1(r) = a_1^{-1} \circ (\mathrm{Id} + \varepsilon) \circ \theta_1^x(r)$$
$$\chi_2(r) = a_2^{-1} \circ (\mathrm{Id} + \varepsilon) \circ \theta_2^x(r)$$

for any $\varepsilon > 0$, where Id stands for the identity function: $\mathrm{Id}(r) = r$ for all r. Thus the condition (11) becomes that there exist $\varepsilon > 0$ and $r_0 \ge 0$ such that

$$a_2^{-1} \circ (\mathrm{Id} + \varepsilon) \circ \theta_2^x \circ a_1^{-1} \circ (\mathrm{Id} + \varepsilon) \circ \theta_1^x(r) < r \quad \forall r \ge r_0. \quad (17)$$

Corollary 3.1. If, for $i = 1, 2$, V_i is an ISpS–Lyapunov function of the x_i subsystem satisfying (8), (15) and (16) with

$$\theta_1^x(s) = \kappa_1 a_2(s), \quad \theta_2^x(s) = \kappa_2 a_1(s)$$

for some $\kappa_1 > 0$ and $\kappa_2 > 0$ then the condition (17) is satisfied if $\kappa_1\kappa_2 < 1$. So the conclusion of Theorem 3.1 holds.

Note that Corollary 3.1 may be seen as an extension of Theorem 1 of Grujić and Šiljak (1973) in the case of two subsystems. Also note that, in this case, the composite functions $\lambda_1 V_1(x_1) + \lambda_2 V_2(x_2)$ form a family of smooth ISpS–Lyapunov functions for the overall system, provided that $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_1\kappa_1 < \lambda_2 < \lambda_1/\kappa_2$.

Remark 3.3. It is interesting to note that the condition (11) is very similar to the so-called nonlinear small-gain conditions utilized in Jiang (1993), Jiang *et al.* (1994) and Teel and Praly (1996). In fact, in our case, χ_1 or $a_1^{-1} \circ (\mathrm{Id} + \varepsilon) \circ \theta_1^x$ (respectively χ_2 or $a_2^{-1} \circ (\mathrm{Id} + \varepsilon) \circ \theta_2^x$) may be seen as an *input–output gain* (Jiang *et al.*, 1994; Jiang and Mareels, 1995) for the x_1 (respectively x_2) subsystem with V_2 (respectively V_1) as input and V_1 (respectively V_2) as output.

In order to illustrate the usefulness of Theorem 3.1 in testing the global asymptotic stability of nonlinear systems, we give an elementary example.

Example 3.1. Consider the three-dimensional nonlinear system

$$\begin{aligned} \dot{z}_1 &= -z_1 + z_1^3 z_2, \\ \dot{z}_2 &= -z_2 - z_1^6 z_2 + z_3^2, \\ \dot{z}_3 &= -z_3^3 + 0.5\,|z_1|^{3/2}. \end{aligned} \quad (18)$$

Let $x_1 = (z_1\ z_2)^{\mathrm{T}}$ and $x_2 = z_3$. As can be checked directly, $V_1(x_1) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$ is an ISS–Lyapunov function for the x_1 subsystem of (18) with x_2 as input and gains in (4) of the form

$$\chi_1(r) = \frac{8}{3 - 2\varepsilon_1} r^2$$

for sufficiently small $\varepsilon_1 > 0$. Also, $V_2(x_2) = \frac{1}{2}x_2^2$ is an ISS–Lyapunov function for the x_2 subsystem of (18) with x_1 as input and gains in (4) of the form

$$\chi_2(r) = \frac{2^{1/3}}{(4 - \varepsilon_2)^{2/3}} r^{1/2}$$

for sufficiently small $\varepsilon_2 > 0$. A simple calculation shows that there exist sufficiently small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\chi_1 \circ \chi_2(r) < r$ for all $r > 0$; namely the condition (11) holds with $c_0 = 0$. Therefore it follows from Theorem 3.1 that the system (18) is globally asymptotically stable at $(z_1, z_2, z_3) = (0, 0, 0)$.

4. *Proof of Theorem* 3.1

To simplify the proof of Theorem 3.1, the following observation is useful.

Lemma 4.1. For any χ_1, χ_2, c_1 and c_2 satisfying (9) and (10), if (11) holds with $c_0 > 0$ then we can always choose $\tilde{\chi}_1$, $\tilde{\chi}_2$, $\tilde{c}_1$ and $\tilde{c}_2$ such that (9) and (10) are satisfied and (11) holds with $\tilde{c}_0 = 0$. In addition, $\tilde{c}_1 = \tilde{c}_2 = 0$ if $c_0 = c_1 = c_2 = 0$.

Proof. Let $c^* = 2c_0$ and pick any $\mathscr{K}_\infty$ function $\tilde{\chi}_2$ with the property that $\tilde{\chi}_2(r) = \chi_2(r)$ for all $r \geq c^*$, and $\chi_1 \circ \tilde{\chi}_2(r) < r$ for all $r > 0$ (this is always possible, because $\chi_1 \circ \tilde{\chi}_2(r) = \chi_1 \circ \chi_2(r) < r$ for all $r \geq c^*$). With the new gain function $\tilde{\chi}_2$, it holds that $\chi_2(r) \leq \max\{\tilde{\chi}_2(r), \chi_2(c^*)\}$. By (10), it follows that $\nabla V_2(x_2) f_2(x_1, x_2, u) \leq -\alpha_2(V_2)$ whenever $V_2(x_2) \geq \max\{\tilde{\chi}_2(V_1(x_1)), \gamma_2(|u_2|) + \tilde{c}_2\}$, where $\tilde{c}_2 = c_2 + \chi_2(c^*) = c_2 + \chi_2(2c_0)$.

Defining $\tilde{\chi}_1 = \chi_1$ and $\tilde{c}_1 = c_1$ completes the proof. □

Proof of Theorem 3.1. In the light of Lemma 4.1, we may assume, without loss of generality, that $c_0 = 0$ in (11). Denote

$$b = \lim_{r \to \infty} \chi_1(r) \tag{19}$$

(note that $b = \infty$ if $\chi_1 \in \mathscr{K}_\infty$). Then χ_1^{-1} is defined on $[0, b)$, $\chi_1^{-1}(r) \to \infty$ as $r \to b^-$, and

$$\chi_2(r) < \chi_1^{-1}(r) \quad \forall r \in (0, b).$$

Now we let $\hat{\chi}_1$ be a function of $\mathscr{K}_\infty$ such that

- $\hat{\chi}_1(r) \leq \chi_1^{-1}(r)$ for each $r \in [0, b)$;
- $\chi_2(r) < \hat{\chi}_1(r)$ for all $r > 0$.

(Note that one can let $\hat{\chi}_1(r) = \chi_1^{-1}(r)$ if $\chi_1 \in \mathscr{K}_\infty$.) Applying Lemma A.1 in the Appendix to χ_2 and $\hat{\chi}_1$, one sees that there exists a $\mathscr{K}_\infty$ function σ continuously differentiable on $(0, \infty)$ with $\sigma'(r) > 0$ for all $r > 0$ such that

$$\chi_2(r) < \sigma(r) < \hat{\chi}_1(r) \quad \forall r > 0.$$

Now we define

$$V(x_1, x_2) = \max\{\sigma(V_1(x_1)), V_2(x_2)\}. \tag{20}$$

Clearly V is proper and positive-definite. Also note that $\sigma(V_1(x_1))$ is locally Lipschitz on $\mathbb{R}^{n_1}\backslash\{0\}$, and V_2 is locally Lipschitz everywhere. It is then a standard fact that V is locally Lipschitz on $\mathbb{R}^n\backslash\{0\}$, where $n = n_1 + n_2$. Therefore V is differentiable almost everywhere (a.e.).

Let $f(x, u) = (f_1(x_1, x_2, u_1)^{\mathrm{T}}\ f_2(x_1, x_2, u_2)^{\mathrm{T}})^{\mathrm{T}}$ and $u = (u_1^{\mathrm{T}}\ u_2^{\mathrm{T}})^{\mathrm{T}}$. In the following, we show that there exist a positive-definite function α, a $\mathscr{K}$ function γ and a constant $c \geq 0$ such that the following implication holds:

$$\{V(x) \geq \gamma(|u|) + c\} \Rightarrow \nabla_V(x) f(x, u) \leq -\alpha(V(x)) \quad \text{a.e.} \tag{21}$$

To this purpose, we define the following sets, as shown in Fig. 1:

$$A = \{(x_1, x_2) : V_2(x_2) < \sigma(V_1(x_1))\},$$
$$B = \{(x_1, x_2) : V_2(x_2) > \sigma(V_1(x_1))\},$$
$$\Gamma = \{(x_1, x_2) : V_2(x_2) = \sigma(V_1(x_1))\}.$$

Now fix any point $p = (p_1, p_2) \neq (0, 0)$, and a control value $v = (v_1, v_2)$. There are three cases.

Case 1. $p \in A$. In this case, $V(x_1, x_2) = \sigma(V_1(x_1))$ in a neighborhood of p, and consequently

$$\nabla V(p) f(p, v) = \sigma'(V_1(p_1)) \nabla V_1(p_1) f_1(p_1, p_2, v_1). \tag{22}$$

For $p \in A$, it holds that $V_2(p_2) < \sigma(V_1(p_1))$, and therefore $V_1(p_1) > \chi_1(V_2(p_2))$. This then implies

$$\nabla V_1(p_1) f_1(p_1, p_2, v_1) \leq -\alpha_1(V_1(p_1))$$

whenever $V_1(p_1) \geq \gamma_1(|v_1|) + c_1$. It follows from this that, for $p \in A$,

$$\nabla V(p) f(p, v) \leq -\hat{\alpha}_1(V(p)) \tag{23}$$

whenever $V(p) \geq \sigma(\gamma_1(|v_1|) + c_1$, where $\hat{\alpha}_1$ is a positive-definite function given by

$$\hat{\alpha}_1(s) = \sigma'(\sigma^{-1}(s)) \alpha_1(\sigma^{-1}(s)) \quad \forall s > 0. \tag{24}$$

We now let

$$\hat{\gamma}_1(r) = \sigma(\gamma_1(r + c_1)) - \sigma(\gamma_1(c_1)), \tag{25}$$

and let $\hat{c}_1 = \sigma(\gamma_1(c_1))$. Then (23) becomes

$$\nabla V(p) f(p, v) \leq -\hat{\alpha}_1(V(p)), \tag{26}$$

whenever $V(p) \geq \hat{\gamma}_1(|v_1|) + \hat{c}_1$.

Case 2. $p \in B$. Using exactly the same arguments as in Case 1, one shows that in this case,

$$\nabla V(p) f(p, v) \leq -\alpha_2(V(p)) \tag{27}$$

whenever $V(p) \geq \gamma_2(|v_2|) + c_2$.

Case 3. $p \in \Gamma$. First note that it holds for the locally Lipschitz function V that

$$\nabla V(p) f(p, v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} V(\varphi(t)) \quad \text{a.e.,}$$

where $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ is the solution of the initial-value problem

$$\dot{\varphi}(t) = f(\varphi(t), v), \quad \varphi(0) = p.$$

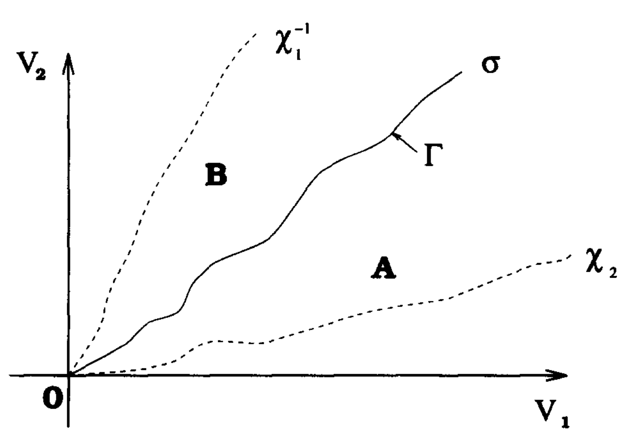

Fig. 1. Level sets A, B and Γ.

Assume $p = (p_1, p_2) \neq (0, 0)$ is such that

$$V_1(p_1) \geq \gamma_1(|v_1|) + c_1, \quad (28)$$

$$V_2(p_2) \geq \gamma_2(|v_2|) + c_2. \quad (29)$$

It then holds that

$$\nabla\sigma(V_1(p_1))f_1(p_1, p_2, v_1) \leq -\hat{\alpha}_1(V(p)),$$
$$\nabla V_2(p_2)f_2(p_1, p_2, v_2) \leq -\alpha_2(V(p)).$$

Note that in this case $p_1 \neq 0$ and $p_2 \neq 0$. Then, using the continuous differentiability of σ, V_1 and V_2 and the continuity of f, one sees that there exist neighborhoods $\mathcal{U}_1$ of p_1 and $\mathcal{U}_2$ of p_2 such that

$$\nabla\sigma(V_1(x_1))f_1(x_1, x_2, v_1) \leq -\tfrac{1}{2}\hat{\alpha}_1(V(p)),$$
$$\nabla V_2(x_2)f_2(x_1, x_2, v_2) \leq -\tfrac{1}{2}\alpha_2(V(p))$$

for all $(x_1, x_2) \in \mathcal{U}_1 \times \mathcal{U}_2$. Note also that there exists $\delta > 0$ such that $\varphi(t) \in \mathcal{U}_1 \times \mathcal{U}_2$ for all $0 \leq t < \delta$. Now pick $\Delta t \in (0, \delta)$. If $\varphi(\Delta t) \in A \cup \Gamma$ then

$$V(\varphi(\Delta t)) - V(p) = \sigma(V_1(\varphi_1(\Delta t))) - \sigma(V_1(p_1)) \leq -\tfrac{1}{2}\hat{\alpha}_1(V(p))\,\Delta t. \quad (30)$$

Similarly, if $\varphi(\Delta t) \in B \cup \Gamma$ then

$$V(\varphi(\Delta t)) - V(p) \leq -\tfrac{1}{2}\alpha_2(V(p))\,\Delta t. \quad (31)$$

Hence if V is differentiable at p then

$$\nabla V(p)f(p, v) \leq -\alpha(V(p)), \quad (32)$$

where $\alpha(r) = \min\{\tfrac{1}{2}\hat{\alpha}_1(r), \tfrac{1}{2}\alpha_2(r)\}$. Note that the assumptions (28) and (29) hold if $V(p) \geq \gamma(|v|) + c$, where $\gamma(r) = \hat{\gamma}_1(r) + \gamma_2(r)$ and $c = \hat{c}_1 + c_2$.

Combining (26), (27) and (32), one concludes that if V is differentiable at p then

$$\nabla V(p)f(p, v) \leq -\alpha(V(p)) \quad (33)$$

whenever $V(p) \geq \gamma(|v|) + c$.

Since V is differentiable almost everywhere, (33) holds almost everywhere. Note that V will be an ISpS–Lyapunov function for (6) and (7) if V is smooth. Though V is merely locally Lipschitz, the arguments used in the proof of the Claim on page 441 of Sontag (1989) are still valid to show that the existence of such a V implies the ISpS property. To make this work more self-contained, in what follows we prove the existence of a smooth ISpS–Lyapunov function.

First we remark that (33) implies that if V is differentiable at p then

$$\nabla V(p)f(p, v) \leq -\alpha(V(p)) \quad (34)$$

whenever $V(p) \geq \max\{\eta(|v|), 2c\}$, where $\eta(r) = \max\{2\gamma(r), r\}$. Clearly η is of class $\mathcal{K}_\infty$. Introducing $\rho(r) = \eta^{-1}(r)$, it follows from (34) that, at any point p where V is differentiable,

$$\nabla V(p)f(p, d\rho(V(p))) \leq -\alpha(V(p)) \quad (35)$$

for all p such that $V(p) \geq 2c$, and for all $d \in \mathbb{R}^m$ such that $|d| \leq 1$. Without loss of generality, we may assume that ρ is smooth (otherwise, we could always replace ρ by a smooth $\mathcal{K}_\infty$ function ρ_1 satisfying $\rho_1(r) \leq \rho(r)$ for all $r \geq 0$). By Theorem 4 of Lin *et al.* (1996), we know that there exists a function $\tilde{W}$ smooth on the set $\Theta := \{p : V(p) > 2c\}$ such that $\tfrac{1}{2}V(p) \leq \tilde{W}(p) \leq 2V(p)$ for all $p \in \mathbb{R}^n$, and

$$\nabla\tilde{W}(p)f(p, d\rho(V(p))) \leq -\tfrac{1}{2}\alpha(V(p)) \quad (36)$$

for all $p \in \Theta$, and all $|d| \leq 1$. To get an ISpS–Lyapunov function that is smooth everywhere, one finds a smooth, proper and positive-definite function W such that $W(p) = \tilde{W}(p)$ for all p such that $V(p) > \bar{c}$ for any $\bar{c} > 2c$. For such a choice of W, it holds that

$$\nabla W(p)f(p, d\rho(V(p))) \leq -\tfrac{1}{2}\alpha(V(p)) \quad (37)$$

for all p such that $V(p) \geq \bar{c}$, and all $|d| \leq 1$, which implies that $\nabla W(p)f(p, v) \leq -\tfrac{1}{2}\alpha(V(p))$ for all (p, v) such that $V(p) \geq \rho^{-1}(|v|)$ and $V(p) \geq \bar{c}$. Since both W and V are positive-definite and proper, there exist a positive definite function $\bar{\alpha}$, a $\mathcal{K}$ function $\bar{\gamma}$ and a constant $\bar{c} \geq 0$ such that the following implication holds:

$$\{W(p) \geq \bar{\gamma}(|v|) + \bar{c}\} \Rightarrow \nabla W(p)f(p, v) < -\bar{\alpha}(W(p)). \quad (38)$$

Finally note that if $c_1 = c_2 = c_0 = 0$ then (36) holds for all $p \neq 0$ and all $|d| \leq 1$. Then one can directly apply Proposition 4.2 of Lin *et al.* (1996) to $\tilde{W}$ and f to get a smooth, proper and positive-definite function W such that (37) holds for all $p \neq 0$ and $|d| \leq 1$. With this, one gets (38) with $\bar{c} = 0$. Hence V and W provide ISS–Lyapunov functions for the system. □

5. *Conclusions*

In this paper, we have given an alternative proof of the so-called nonlinear small-gain theorem for interconnected ISS systems by means of ISS–Lyapunov function arguments. An ISS–Lyapunov function for the total system is generated from the corresponding ISS–Lyapunov functions for the subsystems. The key technique was to modify appropriately the *gain* function (i.e. χ in (4)) for each subsystem. It complements the techniques of changing supply functions in a differential dissipation inequality (cf. (5)) proposed in the recent contribution by Sontag and Teel (1995) for single ISS systems.

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Appendix—Technical lemmas

The following technical lemma is used in the proof of Theorem 3.1.

Lemma A.1. Let $\sigma_1 \in \mathcal{K}$ and $\sigma_2 \in \mathcal{K}_\infty$ satisfy $\sigma_1(r) < \sigma_2(r)$ for all $r > 0$. Then there exists a $\mathcal{K}_\infty$ function σ such that

- $\sigma_1(r) < \sigma(r) < \sigma_2(r)$ for all $r > 0$;
- $\sigma(r)$ is C^1 on $(0, \infty)$, and $\sigma'(r) > 0$ for all $r > 0$.

Before proving this lemma, we first give an intermediate result.

Lemma A.2. Let ρ_0: $[0, \infty) \to [0, \infty)$ be a continuous function such that $\rho_0(0) = 0$ and $\rho_0(r) > 0$ for all $r > 0$. Then there exists a continuous function ρ: $[0, \infty) \to [0, \infty)$ such that

- $\rho(r) < \rho_0(r)$ for all $r > 0$;
- ρ is C^1 on $(0, \infty)$, and $\rho'(r) \le \frac{1}{2}$ for all $r > 0$.

Proof. We may assume that $\rho_0(r) \le \frac{1}{2}$ for all $r \ge 0$. Otherwise, we use $\min\{\frac{1}{2}, \rho_0(r)\}$ to replace $\rho_0(r)$. Let

$$\rho_1(r) = \begin{cases} \min_{s\in[r,2]} \rho_0(s) & \text{if } 0 \le r \le 1, \\ \min_{s\in[1,r+1]} \rho_0(s) & \text{if } r > 1. \end{cases}$$

Note then that ρ_1 is not increasing on $(1, \infty)$, and not decreasing on $(0, 1)$. Also note that $\rho_1(r) \le \rho_0(r)$ for all $r \ge 0$, and $\rho_1(r-1) \le \rho_0(r)$ for all $r \ge 1$. To get the desired function ρ, we let

$$\rho(r) = \begin{cases} \displaystyle\int_0^r \rho_1(s)\,\mathrm{d}s & \text{if } 0 \le r \le 1, \\ \displaystyle\int_{r-1}^r \rho_1(s)\,\mathrm{d}s & \text{if } r > 1. \end{cases}$$

It is easy to see that ρ is continuously differentiable, and $\rho'(r) \le \rho_1(r) \le \frac{1}{2}$ for all $r > 0$. Observe that $\rho(r) \le r\rho_1(r) \le \rho_0(r)$ for $r \in [0, 1]$; $\rho(r) \le \rho_1(r-1) \le \rho_0(r)$ for $r \ge 2$. Furthermore, for $r \in (1, 2)$, it holds that

$$\begin{aligned}\rho(r) &= \int_{r-1}^1 \rho_1(s)\,\mathrm{d}s + \int_1^r \rho_1(s)\,\mathrm{d}s \\ &\le \rho_1(1)(2-r) + \rho_1(1)(r-1) \\ &= \rho_1(1) \le \rho_0(r).\end{aligned}$$

Hence ρ has all the desired properties. □

Now, we return to the proof of Lemma A.1.

Proof of Lemma A.1. Let

$$\rho_0(r) = \tfrac{1}{2}[r - \sigma_2^{-1} \circ \sigma_1(r)].$$

Then $\sigma_2^{-1} \circ \sigma_1(r) < r - \rho_0(r)$, and consequently

$$\sigma_1(r) < \sigma_2(r - \rho_0(r)) \quad \forall r > 0.$$

By Lemma A.2, one knows that there exists a function ρ such that $0 < \rho(r) < \rho_0(r)$ and $\rho'(r) \le \frac{1}{2}$ for all $r > 0$. Again, without loss of generality, we may assume that $\rho(r) \le 1$ for all $r \ge 0$. Now we let $\sigma(0) = 0$ and

$$\sigma(r) = \frac{1}{\rho(r)} \int_{r-\rho(r)}^r \sigma_2(s)\,\mathrm{d}s \quad \forall r > 0.$$

Note then that $\sigma_1(r) < \sigma_2(r - \rho(r)) < \sigma(r) < \sigma_2(r)$ for all $r > 0$. Since ρ is C^1 on $(0, \infty)$, it follows that σ is C^1 on $(0, \infty)$, and

$$\begin{aligned}\sigma'(r) &= -\frac{\rho'(r)}{\rho^2(r)} \int_{r-\rho(r)}^r \sigma_2(s)\,\mathrm{d}s \\ &\quad + \frac{1}{\rho(r)}[\sigma_2(r) - \sigma_2(r-\rho(r))(1-\rho'(r))] \\ &= \frac{1}{\rho(r)}\Big[\sigma_2(r) - \sigma_2(r-\rho(r)) - \frac{\rho'(r)}{\rho(r)} \\ &\quad \times \int_{r-\rho(r)}^r \sigma_2(s)\,\mathrm{d}s + \rho'(r)\sigma_2(r-\rho(r))\Big].\end{aligned}$$

When $\rho'(r) \le 0$, one has

$$-\frac{\rho'(r)}{\rho(r)} \int_{r-\rho(r)}^r \sigma_2(s)\,\mathrm{d}s + \rho'(r)\sigma_2(r-\rho(r)) \ge 0. \tag{39}$$

If $\rho'(r) > 0$ then

$$\begin{aligned}&-\frac{\rho'(r)}{\rho(r)} \int_{r-\rho(r)}^r \sigma_2(s)\,\mathrm{d}s + \rho'(r)\sigma_2(r-\rho(r)) \\ &\quad \ge -\rho'(r)\sigma_2(r) + \rho'(r)\sigma_2(r-\rho(r)) \\ &\quad = -\rho'(r)[\sigma_2(r) - \sigma_2(r-\rho(r))],\end{aligned}$$

and therefore

$$\begin{aligned}\sigma'(r) &\ge \frac{1}{\rho(r)}[1-\rho'(r)][\sigma_2(r) - \sigma_2(\rho(r))] \\ &\ge \frac{1}{2\rho(r)}[\sigma_2(r) - \sigma_2(r-\rho(r))]. \end{aligned} \tag{40}$$

Combining (39) and (40), one gets $\sigma'(r) > 0$ for all $r > 0$. Hence σ is a strictly increasing function. To conclude that $\sigma \in \mathcal{K}_\infty$, note that $\sigma(r) \ge \sigma_2(r-1)$, and therefore $\sigma(r) \to \infty$ as $r \to \infty$. □