ALGEBRAIC DIFFERENTIAL EQUATIONS AND RATIONAL CONTROL SYSTEMS[1]

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ABSTRACT

An equivalence is shown between realizability of input/output operators by rational control systems and high order algebraic differential equations for input/output pairs. This generalizes, to nonlinear systems, the equivalence between autoregressive representations and finite dimensional linear realizability.

1. Introduction. In this paper we prove an equivalence between realizability of input/output operators by rational control systems and the existence of high order algebraic differential equations relating derivatives of inputs and outputs.

In many experimental situations involving systems, it is often the case that one can model system behavior through differential equations, which are referred to as *input/output ("i/o") equations* in this work, of the type

$$E\left(u(t), u'(t), u''(t), \ldots, u^{(r)}(t), y(t), y'(t), y''(t), \ldots, y^{(r)}(t)\right) \;=\; 0 \tag{1}$$

where $u(\cdot)$ and $y(\cdot)$ are the input and output signals respectively, and E is a polynomial. An i/o operator $F:\; u(\cdot) \mapsto y(\cdot)$ is said to *satisfy* the equation (1) if the equation holds for each sufficiently differentiable input u and the corresponding output $y = F[u]$ of F. (Precise definitions will be given later.)

The functional relation E is usually estimated, for instance through least squares techniques, if a parametric general form (e.g. polynomials of fixed degree) is chosen. For example, in linear systems theory one often deals with degree-one polynomials E:

$$y^{(k)}(t) = a_1 y(t) + \ldots + a_k y^{(k-1)}(t) + b_1 u(t) + \ldots + b_k u^{(k-1)}(t) \tag{2}$$

(or their frequency-domain equivalent, transfer functions; the difference equation analogue is sometimes called an "autoregressive moving average" representation). In the linear case, such representations form the basis of much of modern systems analysis and identification theory.

State-space formalisms are more popular than i/o equations in nonlinear control, however. There, one assumes that inputs and outputs are related by a system of *first* order differential equations

$$x'(t) = f(x(t)) + G(x(t))u(t) \;,\;\; y(t) = h(x(t)) \tag{3}$$

where the state $x(t)$ is now a vector, and no derivatives of controls are allowed. These descriptions are central to the modern nonlinear control theory, as they permit the application of techniques from differential equations, dynamical systems, and optimization theory. Thus a basic question is that of deciding when a given i/o operator admits a representation of this form. This is the area of *realization theory*, which is closely related, especially when stochastic effects are included, to *systems identification*. Roughly speaking, if such a state space description does exist for a given i/o operator, then we say that the i/o operator is *realizable*. More precisely, we shall be interested in realizations in which the entries of f and G, as well as the function h, can be expressed in terms of rational functions of the state, but due to the technical problems that arise in the definition because of possible poles of these rational functions, we will give the precise definition in terms of "singular polynomial systems" and we will also study realizability by (nonsingular) polynomial systems.

One knows that an equation such as (2) can be reduced, by adding state variables for enough derivatives of the output y, to a system (3) of first order equations, with $f(x)$ linear and $G(x)$ constant,

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i.e., a linear finite-dimensional system. In frequency-domain terms, rationality of the transfer function is equivalent to realizability. (For references on the linear theory, see eg [14], [23] and [32].) One of the methods for obtaining a linear realization from a given linear i/o equation relies on Lord Kelvin's principle for solving differential equations by means of mechanical analog computers (cf. [14]). The principle, which was suggested a hundred years ago, provided a way for simulating a system without using differentiators.

For nonlinear systems this reduction presents a far harder problem, one that is to a great extent unsolved. The problem is basically that of in some sense replacing a nontrivial equation (1) by a system of first-order equations (3) which does not involve derivatives of the inputs. A number of results were already available about the relation between (1) and (3); see for instance [4], [12], or [26]. It is easy to show, by elementary arguments involving finite transcendence degree, that any i/o operator realizable by a rational state space system satisfies some i/o equation of type (1), with E a polynomial. In [6] it was remarked —as a consequence of theorems from differential algebra,— that in order to characterize the i/o behavior of a state space system *uniquely*, one needs to add inequality constraints to (1). In [18] and [27] it was shown that, under some constant rank conditions, the outputs of an observable smooth state space system can be described by an equation of type (1) for which E is a smooth function, and local i/o equations were shown to exist, for generic initial states of (3), in [3].

1.1. Our Approach. The discrete-time work reported in [20] and [21] provided one approach to relating these two types of representations —with difference equations appearing instead,— based on the idea of dealing with existence of realizations separately from the question of "well-posedness" of the equation (in the sense to be described). This work has been developed further and it was for example used as a basis of identification algorithms by other authors; see for instance [15] and [5]. (The former reference shows also how to include stochastic effects.) These results have recently been extended to continuous-time for the very special case of *bilinear* systems: A theorem showed that realizability by such systems is equivalent to the existence of an E of a special form, namely affine on y (see [22]). However, the techniques in [22] were linear-algebraic, and hence not powerful enough to handle the extension of [21] to the general nonlinear case. The present work completes the development of the extension of the main realizability result in [21] to continuous-time.

The separation into "well-posedness" and realizability can be illustrated with the simple example

$$u(t)y'(t) = 1 \ .$$

This can never be satisfied by all the input/output pairs corresponding to a state space system, as remarked in [22]. Moreover, it cannot even be satisfied by any "input/output map" of the type that we shall consider, realizable or not. Indeed, our main result gives that if the equation would have been well-posed, in the sense that it is an equation satisfied by all input/output pairs corresponding to what we will call a *Fliess operator* —i.e. one described by a convergent generating series— and if E is a polynomial, then it is always realizable by a singular polynomial system, or a rational system with possible poles. (Singular systems appear naturally in control theory, for instance in robotics; see [17] for many examples.) In the special case when equation (1) is recursive —i.e. the coefficient of the highest derivative of y in (1) does not depend on the lower derivatives of y,— our construction will provide a polynomial realization (no poles).

Our formalism is based on the *generating series* suggested by Fliess in the late 70's, who was in turn motivated by Chen's work on power series solutions of differential equations. The i/o operators induced by convergent generating series form a very general class of causal operators, capable of representing a variety of nonlinear systems. We shall call them "Fliess operators". For instance, any i/o operator induced by an initialized analytic state space system affine in controls can be described in this manner. In the paper [29] we develop the basic analytic properties of Fliess operators, and results from there will be freely used here.

The proofs are based on a careful analysis of the concept of *observation space*, introduced in [16] (and [21] for discrete-time), developed further in [11], and later rediscovered by many authors. One of the central technical results relates two different definitions of this space, one in terms of smooth controls and another in terms of piecewise constant ones: these two definitions are seen to coincide. One of them immediately relates to i/o equations, while the other is related to realizability through the notion of *observation algebras* and *observation fields*. The latter are the analogues of the corresponding discrete-time concepts studied in [21]. For differential equations they were first employed in [1] and [2]; the results there related finiteness properties of the various algebraic objects to realizability, in strict analogy to the relations that hold in discrete time ([21]).

In addition to single operators, it is also natural to study *families of i/o maps*, defined by a *family of convergent generating series*. To study a single i/o map is natural as a formal description of a initialized *black box*, but in general, a system may induce more than one i/o map. For example, a system described by an ordinary differential equation on a manifold may induce infinitely many i/o maps, each of them corresponding to some initial state. One should study all the i/o maps induced by the system simultaneously rather than individually, unless a fixed initial state is of particular interest. This leads to the concept of families of i/o maps. One question arises naturally: when can a family of i/o maps be realized by *one* state space system? i.e., when can all the members of the family be realized by some singular polynomial system in such a way that each member of the family is associated to some initial state of the system? We will prove that a family of i/o maps is realizable in this sense if and only if all the members of the family satisfy a common i/o equation.

The paper is organized as follows. After introducing an algebraic structure on series, the shuffle product, we consider observation spaces. Then we study i/o equations satisfied by i/o operators, showing that the existence of an i/o equation implies that the observation field is a finitely generated field extension of $\mathbb{R}$. In the next section, realizability by polynomial systems and singular polynomial systems is considered; the result there is that realizability by singular polynomial systems is guaranteed by the condition that the observation field is a finitely generated extension of $\mathbb{R}$. The approach pursued there is to use the generators of the field as state variables and use the equalities which hold among the generators to construct the needed vector fields. In the main section, based on the previous results, we establish the equivalence between equations and realizability. We also show there that a special kind of equations, recursive i/o equations, lead to realization by polynomial systems. However, as opposed to the general case, the converse of this fact is not true in general, and a counterexample is provided to illustrate the fact that realizability by a polynomial system may not lead to a recursive i/o equation. Finally, we extend our main result to families of i/o operators.

This paper has a heavily algebraic flavor. All analytic properties needed are quoted from [28] and [29], and will not be proved here. The latter paper also shows how, using analytic function theory as well as differential-geometric nonlinear realization tools, an analogous theory can be developed for local realizability provided that an equation with E analytic (not necessarily polynomial) exist for the given operator.

2. Preliminaries. Let m be a fixed integer and consider the "alphabet" set

$$P = \{\eta_0,\, \eta_1,\, \dots,\, \eta_m\}$$

and P^*, the free monoid generated by P, where the neutral element of P^* is the empty word, denoted by 1, and the product is concatenation. Let

$$P^k = \{\eta_{i_1}\eta_{i_2}\dots\eta_{i_k} :\ 0 \le i_s \le m,\ 1 \le s \le k\}$$

for each $k \ge 0$. We define $\mathcal{P}$ to be the $\mathbb{R}$-algebra generated by P^*, i.e, the set of all polynomials in the variables η_i's. A *power series in the noncommutative variables* $\eta_0, \eta_1, \dots, \eta_m$ is a formal power series

$$c = \sum_{\iota \in I^*} \langle c,\ \eta_\iota\rangle \eta_\iota, \tag{4}$$

where

$$\eta_\iota = \eta_{i_1}\eta_{i_2}\cdots\eta_{i_l} \quad \text{if } \iota = i_1 i_2 \cdots i_l,$$

and $\langle c,\ \eta_\iota\rangle \in \mathbb{R}$ for each multiindex ι. Note that c is a polynomial if and only if there are only finitely many $\langle c,\ \eta_\iota\rangle$'s which are non-zero. A power series is nothing more than a mapping from I^* to $\mathbb{R}$; as we shall see later, however, the algebraic structures suggested by the series formalism are very important. We use $\mathcal{S}$ to denote the set of all power series (over a fixed but arbitrary alphabet P).

For $c,\ d \in \mathcal{S}$ and $\gamma \in \mathbb{R}$, $\gamma c + d$ is the series defined as follows:

$$\langle \gamma c + d,\ \eta_\iota \rangle = \gamma \langle c,\ \eta_\iota \rangle + \langle d,\ \eta_\iota \rangle.$$

With these operations, $\mathcal{S}$ forms a vector space over $\mathbb{R}$. In addition, we can introduce an algebra structure on $\mathcal{S}$ by defining the *shuffle product* on $\mathcal{S}$. First of all, we define the shuffle product on words,

$$\sqcup\!\sqcup\ :\ P^* \times P^* \longrightarrow \mathcal{P}$$

inductively on length in the following way:

$$1 \sqcup\!\sqcup\, \eta = \eta \sqcup\!\sqcup\, 1 = \eta \quad \text{for any} \quad \eta \in P\ ,$$

$$\eta_i \eta_\iota \sqcup\!\sqcup\, \eta_j \eta_\kappa = \eta_i (\eta_\iota \sqcup\!\sqcup\, \eta_j \eta_\kappa) + \eta_j (\eta_i \eta_\iota \sqcup\!\sqcup\, \eta_\kappa) \quad \text{for any} \quad \eta_\iota, \eta_\kappa \in P^*, \eta_i, \eta_j \in P. \tag{5}$$

It can be proved by induction that an equivalent way to define the shuffle product is to replace (5) by the following:

$$\eta_\iota \eta_i \sqcup\!\sqcup\, \eta_\kappa \eta_j = (\eta_\iota \sqcup\!\sqcup\, \eta_\kappa \eta_j) \eta_i + (\eta_\iota \eta_i \sqcup\!\sqcup\, \eta_\kappa) \eta_j \quad \text{for any} \quad \eta_\iota, \eta_\kappa \in P^*, \eta_i, \eta_j \in P. \tag{6}$$

Then we extend the shuffle product to power series in the following way: For

$$c = \sum \langle c,\ \eta_\iota \rangle \eta_\iota \quad \text{and} \quad d = \sum \langle d,\ \eta_\kappa \rangle \eta_\kappa,$$

we define

$$c \sqcup\!\sqcup\, d = \sum \langle c,\ \eta_\iota \rangle \langle d,\ \eta_\kappa \rangle \eta_\iota \sqcup\!\sqcup\, \eta_\kappa. \tag{7}$$

With the operations "+" and "$\sqcup\!\sqcup$" defined as above, $\mathcal{S}$ forms a commutative $\mathbb{R}$-algebra.

Remark 2.1. One can also define a comultiplication $M : \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ and a counit ε over $\mathcal{S}$. First, for $z \in P^*$, define

$$M(z) = \sum_{z_1 z_2 = z} (z_1,\ z_2),$$

$$\varepsilon(z) = \begin{cases} 0 & \text{if } z \neq 1, \\ 1 & \text{if } z = 1\ . \end{cases}$$

Then extend M and ε to $\mathcal{S}$. It can be shown that $\mathcal{S}$ forms a Hopf algebra with the antipode σ defined by

$$\sigma(\eta_{i_1} \eta_{i_2} \cdots \eta_{i_s})\ =\ (-1)^s \eta_s \cdots \eta_{i_2} \eta_{i_1}$$

for any s and $\eta_{i_1} \eta_{i_2} \cdots \eta_{i_s} \in P^*$ (cf [25]). Though $\mathcal{S}$ possesses both an algebra structure and a coalgebra structure, in this work, however, only the algebra structure of $\mathcal{S}$ will be studied. □

Lemma 2.1. *The algebra $\mathcal{S}$ is an integral domain.*

Proof. First we order the basis elements $(\eta_{i_1}, \ldots, \eta_{i_k})$ of P^* lexicographically with respect to

$$k, i_1, i_2, \ldots, i_k.$$

Then take two nonzero series c and d and let

$$z_1 = \eta_{i_1} \cdots \eta_{i_m} \quad \text{and} \quad z_2 = \eta_{j_1} \cdots \eta_{j_n}$$

be the smallest basis element of P^* appearing in c and d, respectively, with nonzero coefficients. Let

$$w := \eta_{l_1} \cdots \eta_{l_{m+n}}$$

be the smallest basis elements of P^* appearing in $z_1 \sqcup\!\sqcup\, z_2$. Then the coefficient of w in $c \sqcup\!\sqcup\, d$ is:

$$\langle c \sqcup\!\sqcup\, d,\ w\rangle = \sum_{\iota,\kappa} \langle c,\ \eta_\iota\rangle\langle d,\ \eta_\kappa\rangle\langle \eta_\iota \sqcup\!\sqcup\, \eta_\kappa,\ w\rangle.$$

Using the minimality property of w, z_1, z_2, we get

$$\langle c \sqcup\!\sqcup\, d,\ w\rangle = \langle c,\ z_1\rangle\langle d,\ z_2\rangle\langle z_1 \sqcup\!\sqcup\, z_2,\ w\rangle,$$

which is nonzero since $\langle c,\ z_1\rangle$, $\langle d,\ z_2\rangle$, $\langle z_1 \sqcup\!\sqcup\, z_2,\ w\rangle$ are all nonzero. □

The method used in the above proof is similar to the method used in [19], where the author proved that the ring of polynomials in $\eta_0, \eta_1, \ldots, \eta_m$ is an integral domain. In [19], the author used the greatest basis elements (the "degree") for polynomials while here we used the smallest basis elements (the "order") for power series. Alternatively, one could prove this elementary fact by establishing an isomorphism with a ring of power series in (infinitely many) *commuting* variables, along the lines of the discussion in pages 46-47 in [21].

In order to define operators associated to series, one needs a notion of convergence. We follow [8], [13] and [29] and say that c is *convergent* if there exist some nonnegative real numbers K and M so that the estimate

$$|\langle c,\ \eta_\iota\rangle| \le KM^k k! \tag{8}$$

holds for each multiindex $\iota \in I^k$, and each $k \ge 0$. As in [29], we denote by $\mathcal{U}_T$ the set of all essentially bounded measurable functions $u : [0, T] \to \mathbb{R}^m$, for each fixed $T > 0$. It is convenient to think of $\mathcal{U}_T$ as a space with the L_1 norm ($\|u\|_1 := \max\{\|u_i\|_1 : 1 \le i \le m\}$), but we also use at times the norm $u_\infty := \max\{\|u_i\|_\infty : 1 \le i \le m\}$.

By induction of l, one defines, for each input $u \in \mathcal{U}_T$, and each $\iota \in I^l$,

$$V_\phi := 1, \quad V_{i_1 \cdots i_{l+1}}[u](t) = \int_0^t u_{i_1}(s) V_{i_2 \cdots i_{l+1}}(s)\, ds. \tag{9}$$

Here u_i denotes the i-th coordinate of u, if $i = 1, 2, \ldots, m$, and we make the convention $u_0(t) \equiv 1$. Using these notations, to each convergent power series c in $\eta_0, \eta_1, \ldots, \eta_m$, one can associate the *input/output operator*

$$F_c[u](t) = \sum \langle c,\ \eta_\iota\rangle V_\iota[u](t). \tag{10}$$

This is well-defined for any T admissible for c, i.e, $T < (Mm + M)^{-1}$; see for [8], [13] and [29] for details (the series (10) converges uniformly and absolutely for all $t \in [0, T]$ and all those $u \in \mathcal{U}_T$ such that $\|u\|_\infty < 1$; we denote $\mathcal{V}_T = \{u \in \mathcal{U}_T : \|u\|_\infty < 1\}$, the set of all such controls).

The correspondence between series and operators is one-to-one in the following sense: Assume that c and d are two convergent series and F_c coincides with F_d on $\mathcal{V}_T$ for some $T > 0$, then the two power series c and d coincide; see [30], [29] for these facts as well as further properties of generating series and their associated operators.

Assume that c and d are two convergent power series and T is admissible for both c and d, then T is admissible for both $c + d$ and $c \sqcup\!\sqcup\, d$ (cf [28]). Now for any positive integer n, denote

$$c^n = \underbrace{c \sqcup\!\sqcup\, c \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c}_{n},$$

and $c^0 = 1$. In [7] it was shown that for any polynomial $p \in \mathbb{R}[X_1, X_2, \ldots, X_s]$ and any s convergent power series $c_1, \ldots, c_s$,

$$p\left(F_{c_1}, F_{c_2}, \ldots, F_{c_s}\right) = F_{p(c_1, c_2, \ldots, c_s)}, \tag{11}$$

that is, the assignment $c \mapsto F_c$ is a homomorphism from the set of all convergent series, seen as an algebra under the shuffle product, into the set of input/output operators. (More precisely, identifying operators with their restrictions to smaller time intervals.) By the previous discussion, this homomorphism in one-to-one.

Assume c is a convergent series and pick up a T admissible for c. We show in [29] that F_c is a continuous operator from $\mathcal{V}_T$ to $\mathcal{C}[0,\,T]$ with respect to the L_1 norm in $\mathcal{V}_T$ and the $\mathcal{C}^0$ norm in $\mathcal{C}[0,\,T]$. Furthermore, F_c maps functions of class $\mathcal{C}^{k-1}$ to functions of class $\mathcal{C}^k$, for all $k = 1, 2, \ldots$, and analytic functions to analytic functions. See also [10] for the proof of the following formula:

$$\frac{d}{dt} F_c[u](t) \;=\; F_{\eta_0^{-1}c}[u](t) \,+\, \sum_{j=1}^{m} u_j(t)\, F_{\eta_j^{-1}c}[u](t)\;, \tag{12}$$

where $\langle z^{-1}c,\ \eta_\iota\rangle := \langle c,\ z\eta_\iota\rangle$ is defined for each $z \in P^*$ and each $\eta_\iota \in P^*$. (It is known, cf [22] that $z^{-1}c$ is convergent if c is, and in fact the same T remains admissible.)

3. Observation Space. In realization theory and many other areas of nonlinear control, the concept of observation space plays a central role. Observation spaces were first defined in [16] and [11] for continuous-time systems, and in [21] for discrete-time. The solution of many problems for systems, such as the the "bilinear immersion" problem treated in [11], are characterized by properties of these spaces. One may define observation spaces in two very different ways, as discussed in this section. Roughly, one possibility is to take the functions corresponding to derivatives with respect to switching times in piecewise constant controls, and the other is to take high-order derivatives at the final time, if smooth controls are used. We will show however that both definitions lead to the same concept, and this equivalence will provide one of the main technical tools that we use in order to establish the main result.

For each power series c, we define the first type of observation space $\mathcal{F}_1$ as the linear subspace of the set of all power series spanned by all the elements of the form $z^{-1}c$, i.e.,

$$\mathcal{F}_1(c) = \operatorname{span}_{\mathbb{R}}\{z^{-1}c :\ z \in P^*\}. \tag{13}$$

Then $F_1(c)$ consists of convergent series if c is a convergent series (cf [22]).

For a convergent power series c, the elements of $\mathcal{F}_1(c)$ are closely related to the derivatives of $F_c[u]$ with respect to switching times in piecewise constant controls, in the sense to be made precise next.

For any $\mu \in \mathbb{R}^m$, we define $P^\mu : \mathbf{F} \to \mathbf{F}$, where $\mathbf{F}$ is the set of all germs of i/o operators induced by convergent generating series, in the following way:

$$(P^\mu \circ F_c)[u](t) \;=\; \frac{d}{dt}\Big|_{\tau=0^+} F_c[u\#_t\omega^\mu](t+\tau),$$

where $u\#_\tau v$ denotes the concatenated control:

$$(u\#_t v)(\sigma) = \begin{cases} u(\sigma) & \text{if } 0 \le \sigma \le t, \\ v(\sigma - t) & \text{if } t < \sigma \le T \end{cases}$$

for any u and v, and $\omega^\mu(\tau) \equiv \mu$, a constant control. Note that $(P^\mu \circ F_c)[u]$ is defined if u is in the domain of F_c. In fact, by formula (12), one has the following easy relation:

$$P^\mu \circ F_c \;=\; F_{\eta_0^{-1}c} + \sum_{j=1}^{m} \mu_j F_{\eta_j^{-1}c},$$

for any $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m$.

For a convergent power series c, let $\mathcal{G}_1(c)$ be the smallest subspace of operators which contains F_c and which is invariant under P^μ for any $\mu \in \mathbb{R}^m$. By Lemma 2.1 in [30], $\mathcal{G}_1(c)$ is isomorphic to $\mathcal{F}_1(c)$.

To introduce the second type of observation space, we need to introduce more notations. Consider, for each $q \ge 1$, the following set of $2 \times q$ matrices:

$$S_q = \left\{ \begin{pmatrix} j_1 & j_2 & \cdots & j_q \\ i_1 & i_2 & \cdots & i_q \end{pmatrix} : \right.$$
$$i_s, j_s \in \mathbb{Z},\ 1 \le i_s \le m,\ j \ge 0,\ (1,0) \le (i_1, j_1) \le \cdots \le (i_q, j_q) \Big\}\,, \tag{14}$$

where "$\leq$" is the lexicographic order on the set $\{(i,\, j) :\ i,\, j \in \mathbb{Z}\}$. For each element

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_q \\ i_1 & i_2 & \cdots & i_q \end{pmatrix}$$

in S_q and each $n \geq q + \sum j_r$, we define

$$\Gamma^{j_1 \cdots j_q}_{i_1 \cdots i_q}(n) = \eta_0^{(k)} ш \eta_{i_1} X^{(j_1)} ш \eta_{i_2} X^{(j_2)} ш \cdots ш \eta_{i_q} X^{(j_q)} \Big|_{X=1}, \tag{15}$$

where $k = n - q - \sum j_s$. The evaluation is interpreted as follows: first introduce a new variable X, then perform all shuffles, and finally delete X from the result. Note that (15) is different from $\eta_{i_1} ш \eta_{i_2} ш \cdots ш \eta_{i_q}$, for example,

$$\eta_0 ш \eta_1 X \big|_{X=1} = \eta_0\eta_1 + 2\eta_1\eta_0 \ ,$$

while

$$\eta_0 ш \eta_1 = \eta_0\eta_1 + \eta_1\eta_0.$$

For any word $w \in P^*$ and and each series $c \in \mathcal{S}$, we define $\psi_c(w) = w^{-1}c$, and more generally, for any polynomial $d = \sum \langle d,\ \eta_\kappa \rangle \eta_\kappa$, we let

$$\psi_c(d) = \sum \langle d,\ \eta_\kappa \rangle {\eta_\kappa}^{-1} c.$$

Now let $X_j = (X_{1j}, \ldots, X_{mj})$ be m indeterminates over $\mathbb{R}$, for $j \geq 0$. For any $n > 0$, let

$$c_n(X_0, \ldots, X_{n-1}) = \psi_c(\eta_0^{(n)}) + \sum_{q=1}^{n} \sum \frac{1}{s_1! \cdots s_p!}\, \psi_c\left(\Gamma^{j_1 \cdots j_q}_{i_1 \cdots i_q}(n)\right) X_{i_1 j_1} \cdots X_{i_q j_q}, \tag{16}$$

where the second sum is taken over the set of all those

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_q \\ i_1 & i_2 & \cdots & i_q \end{pmatrix} \in S_q$$

such that $\sum j_s + q \leq n$, and where $s_1, \ldots, s_p$ are integers so that

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_q \\ i_1 & i_2 & \cdots & i_q \end{pmatrix} = \begin{pmatrix} \beta_1 \cdots \beta_1 & \beta_2 \cdots \beta_2 & \cdots & \beta_p \cdots \beta_p \\ \underbrace{\alpha_1 \cdots \alpha_1}_{s_1} & \underbrace{\alpha_2 \cdots \alpha_2}_{s_2} & \cdots & \underbrace{\alpha_p \cdots \alpha_p}_{s_p} \end{pmatrix}$$

and $(\alpha_1, \beta_1) < (\alpha_2, \beta_2) < \cdots < (\alpha_p, \beta_p)$. For $n = 0$, we simply define

$$c_0 := c.$$

It was shown in [30] that for each integer n and every $u \in \mathcal{V}_T$ such that T is admissible for c, we have

$$\frac{d^n}{dt^n} F_c[u](t) = F_{c_n(u(t), \ldots, u^{n-1}(t))}[u](t). \tag{17}$$

Hence, for any $\mu_0, \ldots, \mu_{n-1} \in \mathbb{R}^m$,

$$\frac{d^n}{d\tau^n}\bigg|_{\tau=0^+} F_c[u \#_t w_\mu](t+\tau) \ = \ F_{c_n(\mu_0, \ldots, \mu_{n-1})}[u](t), \tag{18}$$

where $w_\mu(t) = \mu_0 + \mu_1 t + \cdots + \mu_{s-1} \dfrac{t^{s-1}}{(s-1)!}$.

The second type of observation space associated to c, $\mathcal{F}_2(c)$, is defined as follows:

$$\mathcal{F}_2(c) = \operatorname{span}_{\mathbb{R}} \{c_n(\mu_0, \ldots, \mu_{n-1}) : \ \mu_i \in \mathbb{R}^m, 0 \le i \le n-1, \ n \ge 0\}. \tag{19}$$

Let $\mathcal{G}_2(c)$ be the subspace of operators spanned by $F_{c_n(\mu_0 \mu_1 \cdots \mu_{n-1})}$ for all n and all choices of $\mu_0, \ldots, \mu_{n-1}$. Then $\mathcal{F}_2(c)$ is isomorphic to $\mathcal{G}_2(c)$ (cf [30]).

Clearly for any power series c, $\mathcal{F}_2(c) \subseteq \mathcal{F}_1(c)$ since for each integer n, $c_n(X_0, \ldots, X_{n-1})$ is a polynomial on the X_i's with coefficients belonging to $\mathcal{F}_1(c)$. A less trivial conclusion is that $\mathcal{F}_1(c) \subseteq \mathcal{F}_2(c)$. The following is an outline of the proof of this conclusion, for the detailed proof, we refer the readers to [30].

For any fixed positive integers k and $i_1, i_2, \ldots, i_q$ such that

$$1 \le i_1 \le i_2 \le \cdots \le i_q \le m,$$

let

$$S^k(i_1, i_2, \ldots, i_q) = \left\{ \sigma(\underbrace{0, \ldots, 0}_{k}, i_1, i_2, \ldots, i_q) : \ \sigma \in S_n \right\},$$

where $n = k + q$ and S_n is the permutation group on a set of n elements. Let

$$T_k(i_1, i_2, \ldots, i_q) = \left\{ w = \eta_{l_1} \eta_{l_2} \cdots \eta_{l_n} \ : \ (l_1, \ldots, l_n) \in S^k(i_1, i_2, \ldots, i_q) \right\},$$

and order the elements of $T_k(i_1, i_2, \ldots, i_q)$ as $W_1, W_2, \ldots, W_r$. Then for any $j_1, \ldots, j_q$ given,

$$\begin{aligned} \Upsilon^{j_1 \cdots j_q}_{i_1 \cdots i_q}(k) &:= \Gamma^{j_1 \cdots j_q}_{i_1 \cdots i_q}(j_1 + \cdots + j_q + k + q) \\ &= \eta_0^{(k)} ш \eta_{i_1} X^{(j_1)} ш \eta_{i_2} X^{(j_2)} ш \cdots ш \eta_{i_q} X^{(j_q)} \Big|_{X=1} \end{aligned}$$

is a linear combination of the elements in $T_k(i_1, i_2, \ldots, i_q)$. We now define

$$\Delta_k(i_1, \ldots, i_q) = \left\{ \Upsilon^{j_1 \cdots j_q}_{i_1 \cdots i_q}(k) : \ j_s \ge 0, \ 1 \le s \le q \right\}.$$

Our conclusion can be proved by showing that every element of $T_k(i_1, i_2, \ldots, i_q)$ is a linear combination of elements in $\Delta_k(i_1, i_2, \ldots, i_q)$ for any $i_1, \ldots, i_q$ and k given.

For each fixed k and q and fixed $i_1, i_2, \ldots, i_q$, we order the elements of $\Delta_k(i_1, i_2, \ldots, i_q)$ as $Q_1, Q_2, \ldots$. Then for each Q_i, there exist $a_{ij}, j = 1, \ldots, r$ such that

$$Q_i = \sum_{j=1}^{r} a_{ij} W_j .$$

Let A be the matrix of r columns and infinitely many rows whose (i,j)-th entry is a_{ij}, i.e, $A = (a_{ij})$.

We claim that A is of full column rank in the sense that there is no nonzero vector $v \in \mathbb{R}^r$ such that $Av = 0$. Suppose there is some $v \ne 0$ such that $Av = 0$. Let a be the polynomial defined by

$$a = v_1 W_1 + v_2 W_2 + \cdots, + v_r W_r$$

where v_i is the i-th component of v. Then for any $w \in P^*$,

$$\langle w^{-1} a, \ \phi \rangle \ne 0$$

if and only if $w = W_i$ for some i. Hence

$$\langle \psi_a \left(\Upsilon^{j_1 \cdots j_p}_{s_1 \cdots s_p}(l) \right), \ \phi \rangle = 0 \tag{20}$$

if $l \neq k$, or $p \neq q$ or $s_t \neq i_t$ for some t. In the other words, (20) holds if

$$\Upsilon_{s_1 \cdots s_p} j_1 \cdots j_p(k) \notin \Delta_k(i_1, i_2, \ldots, i_q)\ .$$

For $Q_i \in \Delta_k(i_1, i_2, \ldots, i_q)$, we have

$$\langle \psi_a(Q_i),\ \phi \rangle \ = \sum_{j=1}^{r} a_{ij} \langle W_j^{-1} a,\ \phi \rangle = \sum_{j=1}^{r} a_{ij} \langle a,\ W_j \rangle \ = \sum_{j=1}^{r} a_{ij} v_j\ .$$

But by assumption, $\sum a_{ij} v_j = 0$ for any i. Therefore, (20) holds for any choice of $s_1, \ldots, s_p, j_1, \ldots, j_p$ and any l. It then follows directly from the definition of $a_n(X_0, \ldots, X_{n-1})$ that

$$\langle a_n(\mu_0, \mu_1, \ldots, \mu_{n-1}),\ \phi \rangle \ = \ 0 \tag{21}$$

for any n and any value of $\mu_0, \ldots, \mu_{n-1}$, which, by (17), implies that

$$\frac{d^l}{dt^l} F_a[u](0) = F_{a_n(\mu_0, \ldots, \mu_{n-1})}[u](0) \ = \ \langle a_n(\mu_0, \mu_1, \ldots, \mu_{n-1}),\ \phi \rangle$$

for any analytic control u. Thus $F_a[u] \equiv 0$ for any analytic control. It then follows from the continuity of F_a and the density property of analytic controls in L_1 controls that $F_a \equiv 0$, which in turn implies that $a = 0$, a contradiction to the assumption that $v \neq 0$. Hence, A is of full column rank.

It is not hard to see that there exists some submatrix A_1 of A with finitely many rows such that A_1 is full column rank, which implies that each W_i is a linear combination of finitely many Q_j's.

The above discussion shows that following conclusion:

Theorem 3.1. *For any power series c, $\mathcal{F}_1(c) = \mathcal{F}_2(c)$.*

4. Input/Output Equations. In this section, we study high–order differential equations satisfied by inputs and outputs arising from i/o operators. To carry out this study, we find it useful to introduce the algebraic concepts of observation algebra and observation field corresponding to any given series c.

The *observation algebra* $\mathcal{A}_2(c)$ is defined as the $\mathbb{R}$-algebra generated by the elements of $\mathcal{F}_2(c)$. By Lemma 2.1, $\mathcal{A}_2(c)$ is an integral domain, so its quotient field is well defined; we define the *observation field* of c as this quotient field. We shall see later that elementary properties of these algebraic objects serve to characterize the existence of i/o equations.

4.1. Definitions. By an *algebraic input/output equation of order k* we shall mean an equation of the type

$$P\left(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)\right) = 0, \tag{22}$$

where

$$P \in \mathbb{R}[S_0, \ldots, S_k, L_0, \ldots, L_k]$$

is a polynomial nontrivial in L_k, and S_i denotes the set of m variables

$$(S_{1i}, \ldots, S_{mi}).$$

Definition 4.1. *We say that a polynomial P as above is*

(a) rational when $P(S_0, \ldots, S_k, L_0, \ldots, L_k) \ =$

$$P_0(S_0, \ldots, S_{k-1}, L_0, \ldots, L_{k-1})\, L_k + P_1(S_0, \ldots, S_k, L_0, \ldots, L_{k-1}); \tag{23}$$

(b) recursive when $P(S_0, \ldots, S_k, L_0, \ldots, L_k) \ =$

$$= P_0(S_0, \ldots, S_{k-1})\, L_k + P_1(S_0, \ldots, S_k, L_0, \ldots, L_{k-1}); \tag{24}$$

□

DEFINITION 4.2. *Assume that c is a convergent power series. We say that the i/o operator F_c satisfies an algebraic i/o equation (22) if (22) holds for every possible $\mathcal{C}^k$ i/o pair*

$$(u(t),\, y(t)) := (u(t), F_c[u](t))$$

of F_c for all $t \in [0,\, T]$ and for any T admissible for c. In such a case, (22) is called an i/o equation of F_c.

An i/o operator F_c satisfies a rational i/o equation if P can be chosen rational so that $P_0 = 0$ is not an i/o equation of F_c, in another word, there exists some i/o pair $(u,\, y)$ of F_c such that

$$P_0(u(t),\, u'(t),\, \ldots,\, u^{(k)}(t),\, y(t),\, y'(t),\, \ldots,\, y^{(k-1)}(t)) \,\neq\, 0, \tag{25}$$

for some t. An i/o operator F_c satisfies a recursive equation if there is some such equation for which P is recursive. □

The following Lemma was proved in [28]; a detailed proof in the more general analytic case is given in [29].

LEMMA 4.3. *F_c satisfies i/o equation (22) if and only if*

$$P\left(\mu_0, \ldots, \mu_k, F_c, F_{c_1(\mu_0)}, \ldots, F_{c_k(\mu_0,\ldots,\mu_{k-1})}\right) \,=\, 0, \tag{26}$$

for any $\mu_0, \mu_1, \ldots, \mu_k \in \mathbb{R}^m$. □

4.2. Properties of I/O Equations. We now introduce the field

$$K = \mathbb{R}(\{S_{ij},\ i = 1, \ldots, m, j \geq 1\})$$

obtained by adjoining the indeterminates S_{ij} to $\mathbb{R}$. Let $\mathcal{F}^K$, $\mathcal{A}^K$ be the K*-space* and K*-algebra* generated by $c_n(S_0, \ldots, S_{n-1})$ for all n. Let $\mathcal{Q}^K$ be the quotient field of $\mathcal{A}^K$. Note that the field $\mathcal{Q}^K$ is defined since $\mathcal{A}^K$ is an integral domain. The reason for this is essentially because $\mathcal{A}^K$ can be naturally identified to the tensor product $\mathcal{A}_2 \otimes K$.

LEMMA 4.4. *Let F_c be the i/o operator corresponding to the series c. The following properties then hold:*

(a) If F_c satisfies a recursive i/o equation, then $\mathcal{A}^K$ is a finitely generated K-algebra.

(b) If F_c satisfies an algebraic i/o equation, then $\mathcal{Q}^K$ is a finitely generated field extension of K.

Proof. Consider $\hat{\mathcal{A}}^K$, the K-algebra generated by $F_{c_n(S_0,\ldots,S_{n-1})}$ for all n. Since the assignment $\psi:\, c_n(\mu_0, \ldots \mu_{n-1}) \,\mapsto\, F_{c_n(\mu_0,\ldots,\mu_{n-1})}$ is a isomorphism from $\mathcal{A}_2(c)$ onto $\hat{\mathcal{A}}_2(c)$; the $\mathbb{R}$-algebra generated by $F_{c_n(\mu_0,\ldots,\mu_{n-1})}$. thus ψ induces an isomorphism from $\mathcal{A}^K$ onto $\hat{\mathcal{A}}^K$. Consequently, $\hat{\mathcal{Q}}^K$, the quotient field of $\hat{\mathcal{A}}^K$, is isomorphic to $\mathcal{Q}^K$. We shall prove conclusion (b) by showing that $\hat{\mathcal{Q}}^K$ is a finite generated field extension of K, when F_c satisfies some algebraic equation.

It is not hard to show, by taking the derivative with respect to time t on both sides of an algebraic i/o equation, that existence of an algebraic i/o equation for F_c implies that F_c also satisfies a rational i/o equation. Thus

$$\begin{aligned} &P_0(u(t),\, \ldots,\, u^{(k)}(t),\, y(t),\, \ldots,\, y^{(k-1)}(t))\, y^{(k)}(t) \\ &\qquad = -P_1(u(t),\, \ldots,\, u^{(k)}(t),\, y(t),\, \ldots,\, y^{(k-1)}(t)), \end{aligned} \tag{27}$$

for some polynomials P_0 and P_1, where $P_0 = 0$ is not an i/o equation of F_c. (See [28] for details, as well as [29] for an analogous result for result for analytic i/o equations). By Lemma 4.3, we know that

$$\begin{aligned} &P_0\left(S_0, \ldots, S_{k-1}, F_c, \ldots, F_{c_{k-1}(S_0,\ldots,S_{k-2})}\right) F_{c_k(S_0,\ldots,S_{k-1})} \\ &= -P_1\left(S_0, \ldots, S_k, F_c, \ldots, F_{c_{k-1}(S_0,\ldots,S_{k-2})}\right). \end{aligned}$$

Notice that since $P_0 = 0$ is not an i/o equation of F_c, there must exist some vector $(\mu_0, \ldots, \mu_{k-1})$ such that

$$P_0\left(\mu_0, \ldots, \mu_{k-1}, F_c, \ldots, F_{c_{k-1}(\mu_0,\ldots,\mu_{k-2})}\right) \neq 0$$

which in turn implies that

$$P_0\left(S_0,\ldots,S_{k-1},F_c,\ldots,F_{c_{k-1}(S_0,\ldots,S_{k-2})}\right)\neq 0$$

as a polynomial in $S_0,\ldots,S_{k-1}$. It follows from this discussion that

$$F_{c_k(S_0,\ldots,S_{k-1})}\in\hat{\mathcal{Q}}^K_{k-1},$$

where $\hat{\mathcal{Q}}^K_r$ denotes the field obtained by adjoining $F_c,F_{c_1(S_0)},\,\ldots,F_{c_r(S_0,\ldots,S_{r-1})}$ to K.

Taking the derivative with respect to t on both sides of (27), we get

$$\begin{aligned}&P_0(u(t),\,\ldots,\,u^{(k)}(t),\,y(t),\,\ldots,\,y^{(k-1)}(t))\,y^{(k+1)}(t)\\ &\qquad=P_2(u(t),\,\ldots,\,u^{(k+r)}(t),\,y(t),\,\ldots,\,y^{(k+r-1)}(t)),\end{aligned}\tag{28}$$

where P_2 is some polynomial. By using the same argument as before, one shows that

$$F_{ck+1(S_0,\,\ldots,S_k)}\in\hat{\mathcal{Q}}^K_k\subset\hat{\mathcal{Q}}^K_{k-1}.$$

By induction, one shows that

$$\hat{\mathcal{Q}}^K=\hat{\mathcal{Q}}^K_{k-1}.$$

Since $\hat{\mathcal{Q}}^K_{k-1}$ is a finitely generated field extension of K, — the generators are the coefficients of S_{ij} $i=1,\ldots,m;$ $j=0,1,\ldots,k-2$, in $F_c,F_{c_1},\ldots,F_{c_{k-1}}$,— we get our conclusion that $\mathcal{Q}^K$ is also a finitely generated field extension of K. This completes the proof of (b); property (a) is proved in a similar fashion. ∎

Lemma 4.5. *Let F_c be the i/o operator corresponding to the series c. The following properties then hold:*

(a) If $\mathcal{A}^K$ is a finitely generated K-algebra, then $\mathcal{A}_2(c)$ is a finitely generated $\mathbb{R}$-algebra.

(b) If $\mathcal{Q}^K$ is a finitely generated field extension of K, then $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$.

Proof. Again, we shall only provide the proof for part (b). Part (a) can proved similarly.

Assume that $\mathcal{Q}^K$ is a finitely generated field extension of K. Then there exists some $n>0$ so that for any $r\geq 0$, there exist two polynomials Q_0, Q_1 over K with

$$Q_0\left(c_0,c_1(S_1),\ldots,c_{n-1}(S_0,\ldots,S_{n-2})\right)\neq 0$$

such that

$$\begin{aligned}&Q_0\left(c_0,c_1(S_0),\ldots,c_{n-1}(S_0,\ldots,S_{n-1}\right)\;c_{n+r}(S_0,\ldots,S_{n+r-1})\\ &\qquad=\;Q_1\left(c_0,c_1(S_0),\ldots,c_{n-1}(S_0,S_1,\ldots,S_{n-2})\right)\end{aligned}$$

After clearing denominators and getting rid of extra μ_j's there results an equation

$$\begin{aligned}&P_0\left(S_0,\ldots,S_{n+r-1},c_0,c_1(S_0),\ldots,c_{n-1}(S_0,\ldots,S_{n-2})\right)\;c_{n+r}(S_0,\ldots,S_{n+r-1})\\ &=\;P_1\left(S_0,\ldots,S_{n+r-1},c_0,c_1(S_0),\ldots,c_{n-1}(S_0,\ldots,S_{n-2})\right)\end{aligned}$$

with

$$P_0\left(S_0,\ldots,S_{n+r-1},c_0,c_1(S_0),\ldots,c_{n-1}(S_0,\ldots,S_{n-2})\right)\neq 0,$$

which implies that there exists some $(\mu_0,\ldots,\mu_{n+r-1})$ so that

$$P_0\left(\mu_0,\ldots,\mu_{n+r-1},c_0,c_1(\mu_0),\ldots,c_{n-1}(\mu_0,\ldots,\mu_{n-2})\right)\neq 0,$$

or equivalently,

$$P_0\left(\mu_0,\ldots,\mu_{n+r-1},F_c,F_{c_1(\mu_0)},\ldots,F_{c_{n-1}(\mu_0,\ldots,\mu_{n-2})}\right)\neq 0.$$

This is an equation involving operators. Its meaning is that there exists some $u \in \mathcal{V}_T$, where T is admissible to c, and t such that

$$P_0\left(\mu_0, \ldots, \mu_{n+r-1}, F_c[u](t), \ldots, F_{c_{n-1}(\mu_0,\ldots,\mu_{n-2})}[u](t)\right) \neq 0.$$

It follows from the fact that

$$P_0\left(\mu_0, \ldots, \mu_{n+r-1}, F_c[u](t), \ldots, F_{c_{n-1}(\mu_0,\ldots,\mu_{n-2})}[u](t)\right)$$

is a polynomial in $\mu_0, \ldots, \mu_{n+r-1}$, the set

$$\Omega_1 := \left\{\mu^{n+r-1} : \ P_0\left(\mu^{(n+r-1)}, F_c[u](t), \ldots, F_{c_{n-1}(\mu^{n-2})}[u](t)\right) \neq 0.\right\}$$

is dense in $\mathbb{R}^{m\times(n+r)}$, where $\mu^l = (\mu_0, \ldots, \mu_l)$ for any l. Define

$$\Omega = \left\{\mu^{n+r-1} : \ P_0\left(\mu^{n+r-1}, c_0, \ldots, c_{n-1}(\mu^{n-2})\right) \neq 0\right\}.$$

Then $\Omega_1 \subseteq \Omega$. Thus Ω is dense in $\mathbb{R}^{n+r}$.

Clearly if $\mu^{n+r-1} \in \Omega$, then $F_{c_{n+r}(\mu^{n+r-1})} \in \mathcal{T}_{n-1}$, the field obtained by adjoining all the coefficients of X_{ij} in $c_p(X_1, \ldots, X_{p-1})$ for $p \leq n-1$ to $\mathbb{R}$. Applying Lemma 12.11 in [21], one sees that $F_{c_{n+r}(\mu^{n+r-1})} \in \mathcal{T}_{n-1}$ for any $\mu^{n+r-1} \in \mathbb{R}^{n+r}$. Since r can be chosen arbitrarily, it follows that $\mathcal{Q}_2(c) = \mathcal{T}_{n-1}$, from which it follows that $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$. ☐

Combining Lemma 4.4 and 4.5, we get our main result in this section:

THEOREM 4.6. *Let F_c be the i/o operator corresponding to the series c. The following properties then hold:*

(a) If F_c satisfies a recursive i/o equation, then $\mathcal{A}_2(c)$ is a finitely generated $\mathbb{R}$-algebra.

(b) If F_c satisfies an algebraic i/o equation, then $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$. ∎

REMARK 4.1. Generally speaking, a field extension over $\mathbb{R}$ with finite transcendence degree is not necessarily a finitely generated field extension of $\mathbb{R}$. But by using Theorem 4.6, one can show that if the transcendence degree of $\mathcal{Q}_2(c)$ is finite, then it follows that $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$. The reasoning is as follows:

Assume that

$$\operatorname{trdeg}_{\mathbb{R}} \mathcal{Q}_2(c) < \infty,$$

where $\operatorname{trdeg}_{\mathcal{K}} \mathcal{Q}$ denotes the transcendence degree of $\mathcal{Q}$ over $\mathcal{K}$ for any fields $\mathcal{Q}$ and $\mathcal{K}$.

Now let $\mathcal{L}_n$ be the set of all the coefficients of $c_n(S_0, \ldots, S_{n-1})$, seen as a polynomial in $S_0, \ldots, S_{n-1}$ over $\mathcal{S}$, the ring of all series. Let

$$\mathcal{L} = \bigcup_n \mathcal{L}_n .$$

Then $\mathcal{Q}_2(c) = \mathbb{R}(\mathcal{L})$. On the other hand, $\mathcal{Q}^K = K(\mathcal{L})$. Therefore,

$$\operatorname{trdeg}_{\mathbb{R}} \mathcal{Q}_2(c) < \infty$$

implies that

$$\operatorname{trdeg}_K \mathcal{Q}^K < \infty. \tag{29}$$

If (29) holds, then there exists some n such that

$$c, \ c_1(S_0), \ \ldots, \ c_n(S_0, \ldots, S_{n-1})$$

are algebraically dependent over K, i.e., there exists some polynomial P over K such that

$$P\left(c, \ c_1(S_0), \ldots, \ c_n(S_0, \ldots, S_{n-1})\right) = 0.$$

After clearing denominators and getting rid of the extra S_{ij}, one gets the following equation:

$$Q\left(S_0, \ldots, S_k, \ c, \ c_1(S_0), \ \ldots, \ c_n(S_0, \ldots, S_{n-1})\right) = 0 . \tag{30}$$

Notice that if a convergent series c satisfies (30), then (30) is an algebraic i/o equation of F_c, which, by Theorem 4.6, implies that $\mathcal{Q}_2(c)$ is a finitely generated field extension of $\mathbb{R}$. ☐

5. Realizability. We wish to study realization by "rational" systems, such as those studied in Bartosiewicz [1]. However, the question of possible poles in the right-hand side of the equation is very delicate, and it seems better to study instead a "singular" polynomial model, as we do next.

Just as i/o equations turn out to be related to the structure of $\mathcal{A}_2(c)$ and $\mathcal{Q}_2(c)$, realizability forces the study of the observation algebra and observation field corresponding to the other type of observation space, $\mathcal{F}_1(c)$. For a given power series c, we associate with it an *observation algebra* $\mathcal{A}_1(c)$ defined as the $\mathbb{R}$-algebra generated by the elements of $\mathcal{F}_1(c)$, and associate with it an *observation field* $\mathcal{Q}_1(c)$ defined as the quotient field of $\mathcal{A}_1(c)$. Again, we know that $\mathcal{Q}_1(c)$ is defined since $\mathcal{A}_1(c)$ is an integral domain. It turns out, because of previous results, that $\mathcal{A}_1 = \mathcal{A}_2$ and $\mathcal{Q}_1 = \mathcal{Q}_2$ for every c, but the facts in this Chapter do not depend on the equality, and they are more readily understood in terms of $\mathcal{A}_1$ and $\mathcal{Q}_1$.

DEFINITION 5.1. *Suppose that c is a convergent series and T is admissible for c. The i/o operator F_c is realizable by a singular polynomial state-space system*

$$\Sigma = ((g_0, \ldots, g_m),\, x_0,\, q,\, h)$$

if there exists an integer n, some $x_0 \in \mathbb{R}^n$, polynomial vector fields

$$g_0,\, g_1,\, \ldots,\, g_m$$

on $\mathbb{R}^n$, and two polynomial functions

$$q,\ h : \mathbb{R}^n \to \mathbb{R}$$

such that the following properties hold:

(a) For each $u \in \mathcal{V}_T$ and $y = F_c[u]$, there is some absolutely continuous function $x(\cdot)$ defined on $[0,\, T]$ and satisfying $x(0) = x_0$, such that

$$q(x(t))x'(t) \;=\; g_0(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t))$$

for almost all $t \in [0,\, T]$, and,

$$y(t) \;=\; h(x(t))\;,$$

for all $t \in [0,\, T]$.

(b) The solution $x(\cdot)$ in part (a) is of class $\mathcal{C}^\omega$ if u is of class $\mathcal{C}^\omega$, and $x(\cdot)$ is of class $\mathcal{C}^{k+1}$ if u is of class $\mathcal{C}^k$.

(c) There holds the following regularity condition: there exists some set Ω of analytic inputs which is dense in $\mathcal{C}^\infty[0,\, T]$ (with respect to the Whitney topology) such that for any $u \in \mathcal{V}_T \cap \Omega^m$, there exists some $\mathcal{C}^\omega$ solution $x(\cdot)$ as in (a) so that $q(x(\cdot)) \neq 0$.

If F_c can be realized by a singular polynomial system with $q(x) \equiv 1$, we say that F_c is realizable by a polynomial state-space system. □

It can be seen from the definition (5.1) that if $q(x) \neq 0$ for any $x \in \mathbb{R}^n$, then F_c is realizable (globally) by an analytic system in the usual sense. If $q(x_0) \neq 0$, then F_c is realizable locally by an analytic system.

The nondegeneracy condition turns out to be equivalent (as shown in the proof below) to the fact that for "almost every" i/o pair it holds that $q(x(t)) \neq 0$ for almost every t. It could happen, however, that q vanishes along some trajectories.

The following Theorem is our main result in this section. It constitutes a converse to Theorem 4.6, but in terms of different algebraic objects.

THEOREM 5.2. *Let F_c be the i/o operator corresponding to the series c. The following properties then hold:*

(a) If $\mathcal{A}_1(c)$ is a finitely generated $\mathbb{R}$-algebra, then F_c is realizable by a polynomial system.

(b) If $\mathcal{Q}_1(c)$ is a finitely generated field extension of $\mathbb{R}$, then F_c is realizable by a singular polynomial system.

Proof. As in the proof of Theorem 4.6, we shall only provide proof of part (b). Part (a) can be proved by the same argument without involving regularity property.

Suppose that $\mathcal{Q}_1(c)$ is a finitely generated field extension of $\mathbb{R}$, i.e, there exist some $c_1, c_2, \ldots, c_n$ such that

$$\mathcal{Q}_1(c) \;=\; \mathbb{R}(c_1, c_2, \ldots, c_n).$$

Without loss of generality, we may assume that $c_i \in \mathcal{A}_1(c)$ for $i = 1, 2, \ldots, n$ and $c_1 = c$. For each c_i and η_j, there exist some $q_{ij}, g_{ij} \in \mathbb{R}[X_1, X_2, \ldots, X_n]$ such that

$$q_{ij}(c_1, c_2, \ldots, c_n)\,(\eta_j^{-1} c) \;=\; g_{ij}(c_1, c_2, \ldots, c_n),$$

for $i = 1, 2, \ldots, n,\;\; j = 0, 1, \ldots, m$, and $q_{ij}(c_1, c_2, \ldots, c_n) \neq 0$. Without loss of generality, we may assume that $q_{ij} = q$ for all i, j. Otherwise, we may let

$$q(c_1,\, c_2,\, \ldots,\, c_n) \;=\; \prod_{i,j} q_{ij}\,(c_1,\, c_2,\, \ldots, c_n)$$

and change the g_{ij} accordingly. It follows from the fact that $\mathcal{S}$ is an integral domain that

$$q(c_1, c_2, \ldots, c_n) \;\neq\; 0. \tag{31}$$

For $j = 0, 1, \ldots, m$, let $g_j = (g_{1j}, g_{2j}, \ldots, g_{nj})'$, where " ' " denotes the transpose. Let $x_0 = (\langle c_1,\ \phi\rangle, \langle c_2,\ \phi\rangle, \ldots, \langle c_n,\ \phi\rangle)'$ and $h(x) = x_1$. For $u \in \mathcal{V}_T$, let

$$x(t) \;=\; (F_{c_1}[u](t), F_{c_2}[u](t), \ldots, F_{c_n}[u](t))'\,, \tag{32}$$

Then $x(0) = x_0$,

$$q(x(t))x'(t) \;=\; g_0(x(t)) \,+\, \sum_{j=1}^{m} u_j(t) g_j(x(t))$$

for almost all $t \in [0,\ T]$ and

$$y(t) \;=\; h(x(t)).$$

Thus the system

$$\begin{aligned} q(x)x' &\;=\; g_0(x) \,+\, \sum g_j(x) u_j \\ x(0) &\;=\; x_0 \\ y &\;=\; h(x) \end{aligned}$$

realized F_c if regularity property of the system holds. To verify the regularity condition for this realization, let $d = q(c_1, c_2, \ldots, c_n)$. Then $F_d \neq 0$. Note that polynomial controls are dense in $\mathcal{V}_T$ with respect to the L_1 norm and F_d is a continuous operator. Hence there is at least one polynomial control $p \in \mathbb{R}[t]$ such that $F_d[p] \neq 0$. It follows from the fact that, for any t, $F_d[p](t)$ depends analytically on the coefficients of t in $p(t)$ (cf [28]) that $F_d[u] \neq 0$ for all polynomial controls u in a dense set of $\mathcal{V}_T$, which is the desired regularity property. ∎

6. Main Results. In this section we establish the equivalence between realizability and the existence of input/output equations. Recall that any convergent series c induces an i/o operator F_c on $\mathcal{V}_T$ for which T is admissible for c. The following is our main result in this work.

THEOREM 6.1. *Assume that c is a convergent power series, let $T > 0$ be admissible for c, and let F_c be the i/o operator induced by c on $\mathcal{V}_T$. Then:*

(a) The following statements are equivalent:

(i) F_c satisfies an algebraic i/o equation;

(ii) F_c *satisfies a rational i/o equation;*

(iii) F_c *is realizable by a singular polynomial system.*

(b) F_c *is realizable by a polynomial system if* F_c *satisfies a recursive i/o equation.*

The realizability implications follow from Theorems 3.1, 4.6 and 5.2. The converses, i.e. the existence of equations assuming realizability, are quite straightforward exercises in elimination theory, and the details are given next.

LEMMA 6.2. *Assume that c is a convergent power series. Then F_c satisfies an algebraic i/o equation if F_c is realizable by a singular polynomial system.*

Proof. Assume c is a convergent power series. We need to prove that F_c satisfies some i/o equation

$$P\left(u(t),\, \ldots,\, u^{(k)}(t),\, y(t),\, \ldots, y^{(k)}(t)\right) \;=\; 0 \tag{33}$$

valid for all $\mathcal{C}^k$ i/o pairs $(u,\, y)$ with $u \in \mathcal{V}_T$, and any T admissible for c. Form now on we shall fix such a T and we assume that F_c is realized by the singular polynomial system

$$q(x)x' = g_0(x) + \sum_{j=0}^{m} u_j\, g_j(x), \qquad x \in \mathbb{R}^n \tag{34}$$

$$x(0) = x_0, \qquad\qquad x_0 \in \mathbb{R}^n \tag{35}$$

$$y = h(x), \qquad\qquad y \in \mathbb{R}\ . \tag{36}$$

Assume for now that $q(x_0) \neq 0$. Then there exists some neighborhood $\mathcal{N}$ of x_0 in $\mathbb{R}^n$ such that $q(x) \neq 0$ for all $x \in \mathcal{N}$. Note that on $\mathcal{N}$, equation (34) can be written as

$$x' \;=\; p_0(x) \;+\; \sum_{j=0}^{m} u_j\, p_j(x), \tag{37}$$

where $p_j \;=\; \frac{g_j}{q}$, for $j = 0, 1, \ldots, m$.

Let $\varphi(t, x, u)$ to denote the solution of (37) corresponding to the control u with the initial condition $x(0) = x$. Let $y_x(t) = h(\varphi(t, x, u))$. Then

$$y_x(0),\, y_x'(0),\, \ldots,\, y_x^{(n)}(0)$$

are rational functions of x over the field of K, the field obtained by adjoining μ_{ij} $(i = 0, \ldots, n-1,\ j = 0,\, \ldots,\, m)$ to $\mathbb{R}$. Since the transcendence degree of $K(x)$ over K is n, the $n+1$ rational functions $y_x(0),\, y_x'(0),\, \ldots,\, y_x^{(n)}(0)$ are algebraically dependent over K, i.e., there exists some nontrivial polynomial Q over K such that

$$Q\left(y_x(0),\; y_x'(0),\; \ldots,\; y_x^{(n)}(0)\right) \;=\; 0.$$

Clearing the denominators in the coefficients (rational functions in the variables $\mu_0,\, \ldots,$ μ_{n-1}), one gets

$$P\left(\mu_0,\, \ldots,\, \mu_{n-1},\, y_x(0),\, \ldots,\, y_x^{(n)}(0)\right) \;=\; 0$$

where $P \in \mathbb{R}\,[Y,\, \mu_0,\, \ldots,\, \mu_{n-1}]$ is some polynomial over $\mathbb{R}$. Note here that P is nontrivial in Y since Q is nontrivial.

Since P was chosen independent of the initial state x, it follows that, for any $u \in \mathcal{V}_T$, there exists some $\delta > 0$ such that

$$P\left(u(t),\, \ldots,\, u^{(n-1)}(t),\, y(t),\, \ldots, y^{(n)}(t)\right) \;=\; 0 \tag{38}$$

for $t < \delta$. By principle of analytic continuation, (38) holds for all $t \in [0,\, T]$ and analytic controls in $\mathcal{V}_T$. Since analytic controls are dense in $\mathcal{V}_T$ and F_c is continuous, (38) holds for all controls in $\mathcal{V}_T$.

Finally, we show how to overcome the restriction $q(x_0) \neq 0$. Assume now $q(x_0) = 0$. Then by definition, there exists a set Ω of analytic inputs in $\mathcal{C}^\infty$, open dense with respect to the Whitney topology, so that for each $u \in \Omega \cap \mathcal{V}_T$, there exists some analytic function $\varphi(t)$ satisfying (34) and (35) such that $q(\varphi(\cdot)) \neq 0$ and $F_c[u](t) = h(\varphi(t))$. It follows from analyticity that there exists some $\delta > 0$ such that $q(\varphi(t)) \neq 0$ for $t \in (0, \delta)$. From the previous argument one sees that $(u(t), y(t))$ satisfies equation (38) for any $t \in (0, \delta)$. Using analyticity again, one knows that $(u(t), F_c[u](t))$ satisfies (38) for all $t \in [0, T]$.

Since Ω is dense in $\mathcal{C}^\infty$ controls and $\mathcal{C}^\infty$ controls are dense in $\mathcal{C}^n$ controls with respect to the Whitney topology, it follows that (38) holds for all $\mathcal{C}^n$ controls in $\mathcal{V}_T$. □

Note that, in contrast to the cases of rational i/o equation the converse of part (b) does not hold in general, i.e, realizability by polynomials system does not necessarily imply the existence of a recursive i/o equation. This can be illustrated by the following example:

Example 6.1. Consider the following system

$$x_1' = x_1 x_2, \; x_1(0) = x_{10} = 1; \qquad (39)$$
$$x_2' = u, \quad x_2(0) = x_{20} = 0;$$
$$y = x_1.$$

Then there exists some $T > 0$ such that for all $u \in \mathcal{V}_T$, $y(t) = F_c[u](t)$, where c is given by

$$\langle c, \, \eta_{i_1} \eta_{i_2} \cdots \eta_{i_l} \rangle = L_{g_{i_l}} \cdots L_{g_{i_2}} L_{g_{i_1}} h(x_0)$$

where $g_0 = x_1 x_2 \dfrac{\partial}{\partial x_1}$, $g_1 = \dfrac{\partial}{\partial x_2}$ and $h(x) = x_1$. (cf [13]). In the other words, F_c is realizable by the polynomial system (39).

To show that the operator F_c does not satisfy any recursive i/o equation, we need first establish the following fact: To a general analytic state space system

$$x' = g_0(x) + \sum_{i=1}^{m} g_i(x), \qquad x \in \mathcal{M}$$
$$y = h(x), \qquad (40)$$

we associate an observation space F_1 defined as $\mathbb{R}$-space spanned by all the functions

$$L_{g_{i_1}} L_{g_{i_2}} \cdots L_{g_{i_k}} h(x) \,, \qquad k \geq 0, \;\; 0 \leq i_1, \, i_2, \, \ldots, \, i_k \leq m \, .$$

We define the *observation algebra* $\mathcal{A}$ of (40) as the $\mathbb{R}-$algebra generated by the elements of F_1.

For each $x_0 \in \mathcal{M}$, let c_h be the generating series defined by

$$\langle c_h, \, \eta_{i_1} \eta_{i_2} \cdots \eta_{i_l} \rangle = L_{g_{i_l}} \cdots L_{g_{i_2}} L_{g_{i_1}} h(x_0) \, . \qquad (41)$$

We say that the system (40) is *accessible* at x_0 if for any neighborhood $\mathcal{B}$ of x_0, there exists an open subset of $\mathcal{U}$ of $\mathcal{B}$ such that for any $p \in \mathcal{U}$, there exist some $\tau \geq 0$ and some $u \in L_\infty^m[0, \tau]$ such that $x(\tau, x_0, u) = p$. The following Lemma was provided in [28]:

Lemma 6.3. *Assume that the analytic system (40) is accessible at x_0 and that $\mathcal{M}$ is connected. Let c_h be the series defined by (41). Then the observation algebra $\mathcal{A}_1(c_h)$ associated with c_h is isomorphic to the observation algebra $\mathcal{A}$ associated with (40).* □

The system (34) is accessible at $x_0 = (1, 0)$ since the accessibility rank condition (see for instance [24]) holds:

$$\text{rank} \; (\, g_0(x_0) \quad [g_0, \, g_1](x_0) \,) = 2 \, .$$

If F_c would satisfy some recursive i/o equation, then the observation algebra $\mathcal{A}_2(c)$ would be finitely generated, which by Lemma 6.3, would imply that $\mathcal{A}$ is also finitely generated as an $\mathbb{R}$-algebra. But this is false as $\mathcal{A}$ is the algebra generated by

$$x_1, \, x_1 x_2, \, x_1 x_2^2, \, \ldots, \, x_1 x_2^k, \ldots \;\; k \geq 0 \;\; .$$

Thus F_c cannot satisfy any recursive i/o equation, even though it is realized by the polynomial system (39). □

7. Families of I/O Operators. In this section we study families of power series and i/o operators. Let Λ be an index set. We say that $\underline{\mathbf{c}}$ is a *family of power series* (parameterized by $\lambda \in \Lambda$) if

$$\underline{\mathbf{c}} = \{c^\lambda : \ \lambda \in \Lambda\},$$

where c^λ is a power series for each fixed λ. A family $\underline{\mathbf{c}}$ can also be viewed as a power series with coefficients belonging to a ring of functions from Λ to $\mathbb{R}$, i.e,

$$\underline{\mathbf{c}} = \sum \langle \underline{\mathbf{c}}, \ \eta_\iota \rangle \eta_\iota,$$

where $\langle \underline{\mathbf{c}}, \ \eta_\iota \rangle : \Lambda \to \mathbb{R}, \ \lambda \mapsto \langle c^\lambda, \ \eta_\iota \rangle$ is a function defined on Λ.

Thus one may treat families of power series as power series over some ring R. We use $\mathcal{S}_R$ to denote the set of all power series over R. Then $\mathcal{S}_R$ is a ring with "+" and "$\sqcup\!\sqcup$" defined as the following:

$$\gamma \underline{\mathbf{c}} + \underline{\mathbf{d}} \ = \ \{\gamma c^\lambda + d^\lambda : \ \lambda \in \Lambda\},$$

$$\underline{\mathbf{c}} \sqcup\!\sqcup \lambda \ = \ \{\underline{\mathbf{c}}^\lambda \sqcup\!\sqcup \underline{\mathbf{d}}^\lambda : \ \lambda \in \Lambda\},$$

for all $\underline{\mathbf{c}}, \ \underline{\mathbf{d}} \in \mathcal{S}_R$, $\gamma \in \mathbb{R}$.

Unlike the set $\mathcal{S}$ of power series over $\mathbb{R}$, $\mathcal{S}_R$ may not be an integral domain. This is due to the fact that ring R may not be an integral domain. However, by following the same steps in the proof of Lemma 2.1, one can get the following conclusion:

LEMMA 7.1. *The ring $\mathcal{S}_R$ is an integral domain if R is an integral domain.* □

It follows from the principle of analytic continuation that any ring of analytic functions from a connected analytic manifold to $\mathbb{R}$ is an integral domain. So we have the following fact:

COROLLARY 7.2. *If Λ is a connected analytic manifold and R is a ring of analytic functions from Λ to $\mathbb{R}$, then $\mathcal{S}_R$ is an integral domain.* □

DEFINITION 7.3. *We say a family $\underline{\mathbf{c}}$ is a convergent family if:*

(a) Each member of the family is convergent;

(b) Λ is a topological space, $\langle c^\lambda, \ \eta_\iota \rangle$ depends on λ continuously, for each $\eta_\iota \in P^*$, and the constants K_λ, M_λ as in (8) can be chosen continuously depending on λ. ∎

Since each convergent series induces an i/o operator, each convergent family c of power series induces a family of i/o operators

$$\{F_{c^\lambda} : \ \lambda \in \Lambda\}$$

which we denote by $\mathbf{F}_{\underline{\mathbf{c}}}$. The following result was provided in [28].

LEMMA 7.4. *Assume that $\underline{\mathbf{c}}$ is a convergent family. If T is admissible for c^{λ_0}, then T is admissible for c^λ for all λ in a small neighborhood of λ_0, and, $F_c^\lambda[u](t)$ depends (jointly) continuously on t and λ.* □

7.1. Observation Spaces for Families of I/O Operators. For a family $\underline{\mathbf{c}}$ of power series, we define $z^{-1}c$ to be the family $\{z^{-1}c : \ \lambda \in \Lambda\}$, for any $z \in P^*$. For any $n \geq 0$, $\underline{\mathbf{c}}_n(X_0, \ldots, X_{n-1})$ is defined to be the family

$$\left\{c_n^\lambda(X_0, \ldots, X_{n-1}) : \ \lambda \in \Lambda\right\},$$

where $X_i = (X_{i1}, \ldots, X_{im})$ are m indeterminates over $\mathbb{R}$, $i \geq 0$.

As in the case of single power series, we associate to $\underline{\mathbf{c}}$ two types of observation spaces in the following way:

$$\tilde{\mathcal{F}}_1(\underline{\mathbf{c}}) := \operatorname{span}_{\mathbb{R}} \left\{\alpha^{-1}\underline{\mathbf{c}} : \ \alpha \in P^*\right\}.$$

$$\tilde{\mathcal{F}}_2(\underline{\mathbf{c}}) := \operatorname{span}_{\mathbb{R}} \left\{\underline{\mathbf{c}}_n(\mu_0, \ldots, \mu_{n-1}) : \ \mu_i \in \mathbb{R}^m, 0 \leq i \leq n-1, \ n \geq 0\right\}.$$

Note here the elements of $\tilde{\mathcal{F}}_1(\underline{\mathbf{c}})$ and $\tilde{\mathcal{F}}_2(\underline{\mathbf{c}})$ are families of series. For instance, if $\underline{\mathbf{c}}$ is given by

$$c^\lambda = \lambda^2 + 2\lambda\,\eta_0 - \lambda^3\,\eta_1, \quad \lambda \in \mathbb{R}.$$

Then $\tilde{\mathcal{F}}_1(\underline{\mathbf{c}})$ is spanned by three elements: $\underline{\mathbf{c}}$, 2λ and λ^3, thus, $\tilde{\mathcal{F}}_1(\underline{\mathbf{c}})$ is a three dimensional $\mathbb{R}$-space.

Treating families of series as single series over a ring and following the same steps in the proof of Theorem 3.1, one can obtain an analogue of Theorem 3.1 for families:

THEOREM 7.5. *For any family $\underline{\mathbf{c}}$ of power series, $\tilde{\mathcal{F}}_1(\underline{\mathbf{c}}) = \tilde{\mathcal{F}}_2(\underline{\mathbf{c}})$.* ∎

7.2. I/O Equations for Families of I/O operators. We say that a family $\mathbf{F}_{\underline{\mathbf{c}}}$ satisfies an algebraic i/o equation of order k if there exists some polynomial $P \in \mathbb{R}[S_0, \ldots, S_k, L_0, \ldots, L_k]$, nontrivial in L_k such that

$$P\left(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)\right) = 0 \tag{42}$$

is an i/o equation for F_{c^λ} for each $\lambda \in \Lambda$.

If (42) is recursive, then we say that $\mathbf{F}_{\underline{\mathbf{c}}}$ satisfies a recursive equation. We say that (42) is a rational i/o equation for $\mathbf{F}_{\underline{\mathbf{c}}}$ if

$$\begin{aligned} &P\left(S_0, \ldots, S_k, L_0, \ldots, L_k\right) \\ &= P_0\left(S_0, \ldots, S_k, L_0, \ldots, L_{k-1}\right) L_k + P_1\left(S_0, \ldots, S_k, L_0, \ldots, L_{k-1}\right) \end{aligned}$$

for some polynomials P_0 and P_1, and P_0 is not an i/o equation for $\mathbf{F}_{\underline{\mathbf{c}}}$, i.e., there exists some $\lambda \in \Lambda$ and some i/o pair (u, y) of F_{c^λ} which does not satisfy equation (42).

For a family of generating series $\underline{\mathbf{c}}$, we associate with it an observation algebra $\tilde{\mathcal{A}}_2(\underline{\mathbf{c}})$ defined as the $\mathbb{R}$-algebra generated by the elements of $\tilde{\mathcal{F}}_2(\underline{\mathbf{c}})$. Recall that $\tilde{\mathcal{F}}_2(\underline{\mathbf{c}})$ is the $\mathbb{R}$-space generated by

$$\underline{\mathbf{c}}_n(\mu_0, \ldots, \mu_{n-1})$$

for all n and all μ.

To be able to define the observation field, we need the assumption that $\tilde{\mathcal{A}}_2(\underline{\mathbf{c}})$ is an integral domain.

DEFINITION 7.6. *We say that a convergent family*

$$\underline{\mathbf{c}} = \{c^\lambda : \lambda \in \Lambda\}$$

is an analytic family if Λ is a connected analytic manifold and $\langle c^\lambda, \eta_\iota\rangle$ is an analytic function defined on Λ for all $\iota \in P^*$. □

By Corollary 7.3, $\tilde{\mathcal{A}}_2(\underline{\mathbf{c}})$ is an integral domain, therefore, its quotient field is well defined.

For an analytic family $\underline{\mathbf{c}}$, we define the observation field $\tilde{\mathcal{Q}}_2(\underline{\mathbf{c}})$ of $\underline{\mathbf{c}}$ as the quotient field of $\tilde{\mathcal{A}}_2(\underline{\mathbf{c}})$.

By using the same ideas used in §4, we get the following conclusion:

THEOREM 7.7. *Assume that $\underline{\mathbf{c}}$ is an analytic family of power series. Then*

(a) $\tilde{\mathcal{A}}_2(\underline{\mathbf{c}})$ is a finitely generated $\mathbb{R}$-algebra if $\tilde{\mathcal{F}}_{\underline{\mathbf{c}}}$ satisfies a recursive i/o equation.

(b) $\tilde{\mathcal{Q}}_2(\underline{\mathbf{c}})$ is finitely generated field extension of $\mathbb{R}$ if $\tilde{\mathcal{F}}_{\underline{\mathbf{c}}}$ satisfies an algebraic i/o equation. ∎

7.3. Realizability for Families of I/O Operators. DEFINITION 7.8. *We say that a family $\mathbf{F}_{\underline{\mathbf{c}}}$ of i/o operators is realizable by a singular polynomial state space system*

$$\Sigma = ((g_0, g_1, \ldots, g_m), X, q, h)$$

where

$$g_0, g_1, \ldots, g_m$$

are polynomial vector fields of $\mathbb{R}^n$, X is a subset of $\mathbb{R}^n$, q and h are polynomial functions defined on $\mathbb{R}^n$, if the following properties hold:

(a) For each $\lambda \in \Lambda$ and each $u \in \mathcal{V}_{T_\lambda}$, where T_λ is admissible for c^λ, there exists some absolutely continuous function $x^\lambda(\cdot)$ defined on $[0, T]$ satisfying $x^\lambda(0) = x_0^\lambda$ for some $x_0^\lambda \in X$, such that

$$q(x^\lambda(t))\,(x^\lambda(t))' \;=\; g_0(x^\lambda(t)) \;+\; \sum_{j=1}^{m} g_j(x^\lambda(t))u_j(t)$$

for almost all $t \in [0, T]$, and,

$$F_{c^\lambda}[u](t) \;=\; h(x^\lambda(t))$$

for all $t \in [0, T]$ and all $\lambda \in \Lambda$.

(b) The solution $x^\lambda(\cdot)$ in part (a) is of class $\mathcal{C}^\omega$ if u is of class $\mathcal{C}^\omega$, and $x^\lambda(\cdot)$ is of class $\mathcal{C}^{k+1}$ if u is of class $\mathcal{C}^k$.

(c) There holds the following regularity condition: There exists some open dense set Λ_1 of Λ such that for $\lambda \in \Lambda_1$, there exists some set Ω_λ of analytic functions which is dense in $\mathcal{C}^\infty[0, T_\lambda]$ (with respect to Whitney topology) such that for any $u \in \mathcal{V}_{T_\lambda} \cap \Omega_\lambda^m$, there exists some $\mathcal{C}^\omega$ solution $x^\lambda(\cdot)$ as in (a) so that

$$q(x^\lambda(\cdot)) \;\neq\; 0$$

If $\mathbf{F}_{\underline{\mathbf{c}}}$ can be realized by a singular polynomial system with

$$q(x) \;=\; 1 \quad \textit{for all } x \in \mathbb{R}^n,$$

we say that $\mathbf{F}_{\underline{\mathbf{c}}}$ is realizable by a polynomial system, and if in addition, the vector fields $g_0, \ldots, g_m$ are linear in x, then we say that $\mathbf{F}_{\underline{\mathbf{c}}}$ is realizable by a bilinear system. □For an analytic family of power series $\underline{\mathbf{c}}$, we associated with it an observation algebra $\tilde{\mathcal{A}}_1(\underline{\mathbf{c}})$ defined as the $\mathbb{R}$-algebra generated by the elements of $\tilde{\mathcal{F}}_1(\underline{\mathbf{c}})$ and an observation field $\tilde{\mathcal{Q}}_1(\underline{\mathbf{c}})$ defined as the quotient field of $\tilde{\mathcal{A}}_1(\underline{\mathbf{c}})$. Note here that the analyticity of the family implies that the quotient field of $\tilde{\mathcal{A}}_1(\underline{\mathbf{c}})$ is well defined.

By using the same techniques used in §5, we get the following conclusion:

Theorem 7.9. *Let $\underline{\mathbf{c}}$ be an analytic family of power series. Then*

(a) The family of i/o operators $\mathbf{F}_{\underline{\mathbf{c}}}$ is realizable by a polynomial system if $\tilde{\mathcal{A}}_1(\underline{\mathbf{c}})$ is a finitely generated $\mathbb{R}$-algebra.

(b) The family of i/o operators $\mathbf{F}_{\underline{\mathbf{c}}}$ is realizable by a singular polynomial system if $\tilde{\mathcal{Q}}_1(\underline{\mathbf{c}})$ is a finitely generated field extension of $\mathbb{R}$. ∎

Combining all the results in this section, one sees that the existence of i/o equations implies realizability. On the other hand, if $\mathbf{F}_{\underline{\mathbf{c}}}$ is realizable by some singular polynomial system, then by using approximation arguments, one can show that $\mathbf{F}_c$ must satisfy some algebraic i/o equation. Hence we have

Theorem 7.10. *Assume that $\underline{\mathbf{c}}$ is an analytic families of series. Then*

(a) The following statements are equivalent:
- *(i) $\mathbf{F}_{\underline{\mathbf{c}}}$ satisfies an algebraic i/o equation;*
- *(ii) $\mathbf{F}_{\underline{\mathbf{c}}}$ satisfies a rational i/o equation;*
- *(iii) $\mathbf{F}_{\underline{\mathbf{c}}}$ is realizable by a singular polynomial system.*

(b) $\mathbf{F}_{\underline{\mathbf{c}}}$ is realizable by a polynomial system if $\mathbf{F}_{\underline{\mathbf{c}}}$ satisfies a recursive i/o equation. ∎

Remark 7.1. In the proofs of part (a) of Theorems 7.9 and 7.9, one needs not to assume that $\tilde{\mathcal{A}}_1(c)$ and $\tilde{\mathcal{A}}_2(c)$ are integral domains. Hence part (b) of Theorem 7.10 also holds for continuous families, that is, for continuous families of operators, existence of recursive i/o equation implies realizability by polynomial systems. □

8. Closing Remarks. We envision our results being used as follows (the idea is very similar to that employed in the discrete case, and explored in some detail in [5]). If there are reasons to believe that the system producing the observed data is well-posed, then an equation E may be fit to the data. We are *assured* that there is then a realization of the type to be considered, and we then try to find this realization. We are still very far from having constructive techniques for obtaining realizations; this is a major topic for further research involving symbolic computation. The following example illustrates the type of construction suggested by the proofs.

Consider the input/output equation

$$uy'' = y^2u^2 + y'u' \tag{43}$$

and assume that it is "well-posed" in the sense mentioned above, that is, that there is a Fliess operator $y = F_c[u]$ for which every pair $(u, F_c[u])$ satisfies the equation. Then we know that F_c can be realized by some polynomial state space system

$$x' = f(x) + g(x)u, \tag{44}$$

$$y = h(x) \; . \tag{45}$$

with some fixed initial state. We try to deduce now what $f,\ g,\ h$ should be. We have

$$\begin{aligned} y' &= L_f h(x) + L_g h(x)u, \\ y'' &= L_f^2 h(x) + (L_f L_g h(x) + L_g L_f h(x))\, u + L_g^2 h(x) u^2 + L_g h(x) u'. \end{aligned}$$

Substituting y, y', y'' into equation (43) we get the following formulas:

$$L_f h = 0, \tag{46}$$

$$L_f L_g h + L_g L_f h = h^2, \tag{47}$$

$$L_g^2 h = 0. \tag{48}$$

Formulas (46) and (47) suggest that $L_f^2 h = 0$ and $L_f L_g h = h^2$. Now let

$$z_1 = h(x), \;\; z_2 = L_g h(x).$$

Then along any trajectory $x(t)$ of (44),

$$\begin{aligned} z_1'(t) &= L_f h(x(t)) + L_g h(x(t)) u(t) = z_2(t) u(t), \\ z_2'(t) &= L_f L_g h(x(t)) + L_g^2 h(x(t)) u(t) = z_1(t)^2. \end{aligned}$$

Hence, F_c can be realized by the following polynomial system

$$\begin{aligned} z_1' &= z_2 u, \\ z_2' &= z_1^2, \\ y &= z_1, \end{aligned}$$

where the choice of initial state will depend on additional data (such as the knowledge of $y(0)$ and $y'(0)$ for some nonzero control).

Of course, for practical applications, it is not clear when one would be justified in assuming well-posedness. But we take the position that postulating well-posedness is a far weaker assumption than assuming that the data was produced by a linear system, an assumption that itself underlies most applications of control theory.

Sometimes one imposes a "causality" constraint on i/o equations, requiring that the highest derivative of u be of lower order than derivatives of y. However, it is easy to see (cf. [28]) that for i/o behaviors described by generating series, an equation of the type (1) always leads to an equation in which the highest order of derivative of inputs is lower than the highest order of derivative of outputs, i.e., an equation of the type

$$E\left(u(t), u'(t), u''(t), \ldots, u^{(r-1)}(t), y(t), y'(t), y''(t), \ldots, y^{(r)}(t)\right) \;=\; 0 \; .$$

This is essentially a consequence of the fact that an i/o operator induced by a generating series must be causal in the sense that the k-th order derivatives of outputs do not depend on the k-th order derivatives of inputs.

Though nonsingular systems are to be preferred, but we do not yet know if there is always a realization of that type (for nonrecursive equations). However, the analytic results in [29] can be applied to prove that about every singular point of the realization obtained here there is another system, locally defined in terms of analytic functions, that realizes (locally) the desired behavior. The picture that emerges then is that, at least, one can cover the possibly singular part with local analytic realizations. In a computer simulation, this would be achieved by passing to a subroutine to deal with trajectories near this set.

As a final remark, we explain how this work relates to alternative foundations for systems theory recently proposed by various authors. One may consider the *behavior* $w(\cdot) = (u(\cdot), y(\cdot))$ associated to an input/output description. It has been proposed by [31] that one should formulate systems modeling without a priori distinctions between input and output signals. In these terms, an input/output equation takes the form

$$E\left(w(t), w'(t), w''(t), \ldots, w^{(r)}(t)\right) \;=\; 0\,. \tag{49}$$

One of the central questions in [31] and related work is that of in some sense partitioning an abstract behavior $w(\cdot)$ into "inputs" and "outputs." Once this task is achived, however, and provided that one may assume a suitable structure –in our case, the existence of a Fliess-operator relationship between inputs and outputs,– it is still important to be able to relate an abstract equation such as (49) to realizability, and this is precisely what our result does. Similarly, the work [9] *defined* realizability by the requirement that outputs be differentiably dependent on inputs, in other words, that an equation such as (1) hold; we showed that this is basically the same as realizability in the more classical sense.

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