Stabilization in spite of matched unmodelled dynamics and An equivalent definition of input-to-state stability

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Abstract. We consider nonlinear systems with input-to-output stable (IOS) unmodelled dynamics which are in the "range" of the input. Assuming the nominal system is globally asymptotically stabilizable and a nonlinear small gain condition is satisfied, we propose a first control law such that all solutions of the perturbed system are bounded and the state of the nominal system is captured by an arbitrarily small neighborhood of the origin. The design of this controller is based on a gain assignment result which allows us to prove our statement via a Small-Gain Theorem [JTP, Theorem 2.1]. However, this control law exhibits a high gain feature for all values. Since this may be undesirable, in a second stage we propose another controller with different characteristics in this respect. This controller requires more a priori knowledge on the unmodelled dynamics, as it is dynamic and incorporates a signal bounding the unmodelled effects. However this is only possible by restraining the IOS property into the exp-IOS property. Nevertheless we show that, in the case of input-to-state stability (ISS) — the output is the state itself —, ISS and exp-ISS are in fact equivalent properties.

Key Words. Nonlinear systems, Robust control, Uncertain systems, Gain assignment, Input-to-state stability.

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1 Introduction

Consider the system :

$$\begin{cases} \dot{x} &= f(x) + \sum_{i=1}^{p} g_i(x)\,[u_i + c_i(x,z,u)] \\ \dot{z} &= a(x,z,u) \end{cases} \tag{1}$$

where a and f are continuous vector fields, $G = (g_i)$ is a continuous "matrix field", and $c_1, \ldots, c_p$ are continuous functions. The x-subsystem represents, when $c = 0$, the nominal system. Its state x, taking values in R^n, is measured and the vector $u = (u_i)$, taking values in R^p, is its input. The z-subsystem represents the unmodelled dynamics, its state z, taking values in R^m, is unmeasured and the functions a and c are unknown.

The problem is to design a feedback law, with x as only input, guaranteeing boundedness of the solutions of the closed-loop system and regulating x around 0. To solve this problem we shall assume (see A1) that the nominal system is globally asymptotically stabilizable and that the z-subsystem has an appropriately "stable" input-output behavior (see A2 or A2').

In the terminology of linear systems, the perturbation introduced via c would be called stable and proper multiplicative perturbation. Its main characteristics are :

- The relative degree between u and any "generic" output function of x cannot be decreased by the presence of c.
- The so called matching assumption is met. Namely, if c were measured, we could completely annihilate its effects on the x-subsystem (see [I, Remark 4.6.2]). Here c is not assumed to be measured. Instead, we shall impose an amplitude limitation (see A2 or A2').
- The state x can be measured, and consequently, there is no inverse dynamics. This makes it theoretically possible to use "high gain" controllers. However, we know that, if other classes of "real life" unmodelled effects – input saturations, unmeasured noise, unmatched unmodelled dynamics, ... – are present, then "high gain" controllers may be unsuitable. For this reason, we shall propose two solutions to the problem stated above with a different high gain requirement.

The topic of stabilizing (nonlinear) systems with uncertainties has been attracting the attention of many authors for a long time, see for instance [BCL, C, CL, G, K]. While most of the work in this area focused on unmodelled static (time varying) uncertainties, less work has been done for systems with dynamic uncertainties. The recent work [KSK] has formulated very properly the problem of stabilizability for nonlinear systems with unmodelled dynamics. There also, the authors have proposed a solution for a specific class of systems with linear unmodelled dynamics at the input. Some related work in this area can also be found in [Q1, Q2], where the author has investigated the tracking problem for linear systems with unmodelled dynamic uncertainties.

Our problem generalizes the one stated and solved by Krstić, Sun and Kokotović in [KSK, Lemma 3.1] for x in R and functions c and a linear and not depending on x. The solution proposed by these authors incorporates, in the controller, a signal, called normalizing signal which captures the effect of the unmodelled dynamics. This concept of normalizing signal is nowadays widely used in linear adaptive control, and its extension to the nonlinear case has been suggested in [JP1, JP2, J]. Jiang, Mareels and Pomet have shown in [JMP] that the result of [KSK, Lemma 3.1] holds also with a static feedback law, without using the normalizing signal. For this, an appropriate change of coordinates of the unmodelled dynamics is made and the technique of propagating the ISS property through integrators proposed in [JTP] is applied. Based on the technique of gain assignment and the Small-Gain Theorem of [JTP], Krstić and Kokotović have obtained another solution in [KK], without normalizing signal for the system (1), allowing the functions c and a to be nonlinear and to depend on x but still imposing that x be in $\mathbb{R}$.

Here, we extend the work in [KK] to the general case when x is in $\mathbb{R}^n$. Our major assumptions are: (1) the nominal system is stabilizable, and (2) the unmodelled dynamics is input-to-output stable (IOS) with a small enough gain function. In the special case when there is no dynamic uncertainty presented in the system, that is, when the functions c_i's do not depend on z, the IOS condition reduces to the usual boundedness condition on the static uncertainties considered, for instance, in [C, K, Q2]. After stating our assumptions in section 2, we shall propose, in section 3, a first control law which solves the problem. It is a static feedback but, as already mentioned, it exhibits a high gain feature. This feature has been found usefull to solve some problems in robust control (see for instance [BCL] and [SK]) but it may also be undesirable in some other situations. This motivates our proposition of a second controller in section 4. Our two controllers are compared for a simplistic example in section 5. In section 6 we propose a framework allowing us to relax somehow the assumptions made in section 2. In fact, to prove that our second controller provides the closed-loop system with properties similar to the ones given by the first one, we need to restrain the class of unmodelled dynamics. Nevertheless in the case $c_i(x, z, u) = z$, i.e. the disturbance is the state of the unmodelled dynamics itself, we prove in section 7 that there is in fact no restriction.

2 Assumptions

We assume the nominal system is globally asymptotically stabilizable and more precisely :

A1 : We know a C^1 positive definite function V satisfying, for all x,

$$V(x) \geq \alpha_1(|x|) , \tag{2}$$

for some function α_1 of class[1] $\mathcal{K}_\infty$, and a C^0 feedback law $u_n(x)$ with $u_n(0) = 0$[2], such that the function[3] :

$$W(x) = -L_{[f+Gu_n]}V(x) \tag{3}$$

is also positive definite.

According to [S1], if there exists u_n satisfying Assumption A1, then the following feedback u_s also globally asymptotically stabilizes the nominal system :

$$u_{s_i}(x) = \begin{cases} -\dfrac{L_fV(x) + \sqrt{L_fV(x)^2 + \|L_GV(x)\|^4}}{\|L_GV(x)\|^2} L_{g_i}V(x) , & \text{if } L_GV(x) \neq 0 , \\ 0 , & \text{if } L_GV(x) = 0 , \end{cases} \tag{4}$$

[1] For the definitions of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions see [H].

[2] Assuming $u_n(0) = 0$ can be done without loss of generality as far the nominal system is concerned. Indeed, if $u_n(0) = u_0 \neq 0$, it is sufficient to replace f by $\hat{f} = f + Gu_0$ and u by $\hat{u} = u - u_0$.

[3] L_fV is the Lie derivative of V along f and L_GV is the row vector $(L_{g_i}V)$.

where $\|\cdot\|$ denotes the usual Euclidian norm of $\mathbb{R}^p$. With this control we get the following positive definite function :

$$W_s(x) \;=\; -L_{[f+Gu_s]}V(x) \;. \tag{5}$$

The interest of this particular feedback is that we have, for all x,

$$L_G V(x) \neq 0 \qquad \Longrightarrow \qquad L_G V(x) u_s(x) < 0 \;. \tag{6}$$

A2 : The z-subsystem of (1) with input (x, u) and output c is BIBS and IOpS. That is:

BIBS : For each initial condition $z(0)$ and each measurable essentially bounded function $(x, u) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \times \mathbb{R}^p$, the corresponding solution $z(t)$ is defined and bounded on $\mathbb{R}_{\geq 0}$.

IOpS : There exist a function β_c of class $\mathcal{KL}$, two functions γ_u and γ_x of class $\mathcal{K}$ and a positive real number c_0 such that, for each initial condition $z(0)$ and each measurable essentially bounded function $(x, u) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \times \mathbb{R}^p$, the corresponding solution satisfies, for all t in $\mathbb{R}_{\geq 0}$,[4]

$$|c(t)| \;\leq\; c_0 \;+\; \beta_c(|z(0)|, t) \;+\; \gamma_u(U(t)) \;+\; \gamma_x(\sup_{\tau \in [0,t)} \{|x(\tau)|\}) \;, \tag{7}$$

where :

$$U(t) = \sup_{\tau \in [0,t)} \{|u(\tau)|\} \;, \tag{8}$$

and for each vector v in $\mathbb{R}^p$, $|v|$ denotes $\max\{|v_1|, \ldots, |v_p|\}$, and similarly for vectors in $\mathbb{R}^n$.

With Assumption A2, we shall be able to get a control law whose design is based on the only fact that inequality (7) holds. However Krstić and Kokotović have noticed in [KK] that better performance can be obtained if one uses more a priori knowledge on c, namely that the last two terms in the right hand side of (7) can be evaluated on line and therefore used in the control law. Unfortunately such terms involve U which is the output of an infinite dimensional system with u as input. To overcome this difficulty, we remark that assumption A2 could apply to systems with dynamics involving mathematical objects more complex than the system :

$$\left\{ \begin{array}{rcl} \dot{z} &=& a(x, z, u) \;, \\ y &=& c(x, z, u) \;. \end{array} \right. \tag{9}$$

In particular, when the initial condition $z(0)$ is fixed, this system provides operators: $u \mapsto z$ and $u \mapsto y$ which are finite dimensional, the former being strictly proper, whereas, in (8), the operator: $u \mapsto U$ is only proper and infinite dimensional. From this, we conjecture that the restriction of A2 to systems in the particular form (9) should give a stronger property. These arguments lead us to restrain assumption A2 with replacing the infinite dimensional operator $\sup_{\tau \in [0,t)} \{\cdot\}$ by a first order one. This yields :

A2' The z-subsystem with input u and output c is BIBS and exp-IOpS. That is:

exp-IOpS : For some positive real number μ and some functions γ_{vx}, γ_{vu}, γ_{cx} and γ_c of class

[4] For the sake of simplicity, here and all along the paper we make the following abuse of notations: sup is to be taken as the essential supremum norm and "for all t" should be "for almost all t with respect to the Lebesgue measure".

$\mathcal{K}$, there exist a positive real number c_o, a function γ_{cu} of class $\mathcal{K}$ and a function β of class $\mathcal{KL}$, such that, for each initial condition $z(0)$ and for each measurable essentially bounded function $(x,u) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \times \mathbb{R}^p$, the corresponding solution satisfies, for all t in $\mathbb{R}_{\geq 0}$,

$$|c(t)| \;\leq\; c_0 \;+\; \beta(|z(0)|,t) \;+\; \gamma_{cu}(|u(t)|) \;+\; \gamma_{cx}(|x(t)|) \;+\; \gamma_c(r(t))\,, \tag{10}$$

where the function $r(t)$ satisfies the following equations :

$$\dot{r} \;=\; -\mu\, r \;+\; \gamma_{vu}(|u|) \;+\; \gamma_{vx}(|x|)\;, \qquad r(0) \;=\; 0\;. \tag{11}$$

The main difference between IOpS and exp-IOpS is that in (10), through r, an exponentially weighted L^1 norm is used instead of the L^∞ one expressed in U. Clearly, exp-IOpS implies IOpS with the gains of the relations $u \mapsto c$ and $x \mapsto c$ given by :

$$\gamma_u \;=\; \gamma_{cu} \;+\; \gamma_c \circ \tfrac{1}{\mu}\,\gamma_{vu} \quad , \qquad \gamma_x \;=\; \gamma_{cx} \;+\; \gamma_c \circ \tfrac{1}{\mu}\,\gamma_{vx} \tag{12}$$

respectively. But the previous arguments let us expect that the converse may be true. This will be proved in section 7 for the case when $c = z$.

Assumption A2' is strongly related with Assumption UEC (73) in [JP1]. From this relation, we note :

– [JP1, Lemma 1] is a helpful tool for selecting the real number μ and the functions γ_{vx}, γ_{vu}, γ_{cx} and γ_c.
– Equations in (11) provide us with r as a pseudo state for the stability analysis. In the proof of [JP1, Proposition 1], it is shown that it is well suited for the application of Lyapunov second method.
– To help the reader get a better grip on the meaning of this signal r, we refer to [P, Property 1].

Let us remark that Assumption A1 and A2 or A2' are not sufficient for guaranteeing the existence of a feedback law solving our problem. Consider the system :

$$\left\{\begin{array}{rcl} \dot{x} &=& x^2 \;-\; (u-\gamma(z))\, x\;, \\ \dot{z} &=& (u-z)\, z^2\;, \end{array}\right. \tag{13}$$

where γ is a smooth odd function satisfying :

$$\operatorname{sign}(r)\,[r-\gamma(r)] \;\leq\; M \qquad \forall\, r\, \in \mathbb{R}\;, \tag{14}$$

for some positive real number M. This condition says roughly that the function γ grows at least as much as the identity function. Assumption A1 holds, with :

$$V(x) \;=\; \tfrac{1}{2}\,x^2\,, \qquad u_n(x) \;=\; x \;+\; x^2\,, \tag{15}$$

and Assumption A2 holds also since the z-subsystem with input u and output $\gamma(z)$ is IOS with $\gamma_x \equiv 0$ and γ_u, any gain function of class $\mathcal{K}$ strictly greater than γ. However system (13) is not asymptotically controllable. Precisely, we prove in Appendix A that there is no control law $u(t)$ that can drive to zero the x-component of any solution starting from $(x_0, 1)$ with $x_0 > M\,\exp(1)$.

This example shows that it is in general impossible to solve the problem stated in section 1 if the function $\mathrm{Id} - \gamma_u$ is bounded.

3 First solution with a static feedback

Proposition 1 *Assume A1 and A2 hold. Under this condition, for any functions κ_u and κ_x of class $\mathcal{K}_\infty$, there exists a continuous feedback law $\omega(x)$ such that all the solutions of the closed-loop system (1) are bounded provided that we have :*

$$(\mathrm{Id} + \rho_2) \circ [\gamma_u \circ (\mathrm{Id} + \rho_1) \circ (\mathrm{Id} + \kappa_u) + \gamma_x \circ (\mathrm{Id} + \rho_1) \circ \kappa_x] \ \leq \ \mathrm{Id} \tag{16}$$

for some functions ρ_1 and ρ_2 of class $\mathcal{K}_\infty$. Moreover, for each closed-loop solution, we have :

$$\limsup_{t \to +\infty} |x(t)| \ \leq \ \kappa_x \circ (\mathrm{Id} + \rho_2^{-1})(c_0) \ . \tag{17}$$

Remark 2 : If, in (7), $c_0 = 0$, that is, when the z-subsystem of (1) is IOS with (x, u) as input and c as output, then it follows immediately from (17) that :

$$\lim_{t \to +\infty} |x(t)| \ = \ 0 \, . \tag{18}$$

If $c_0 \neq 0$, since κ_x can be chosen as an arbitrarily small function of class $\mathcal{K}_\infty$, (16) is mainly a condition on γ_u and (17) gives a practical convergence result. In fact, it can be shown that, for any given functions γ_u and γ_x of class $\mathcal{K}$, satisfying :

$$\mathrm{Id} - \gamma_u \ > \ \rho_0 \ , \tag{19}$$

for some function ρ_0 of class $\mathcal{K}_\infty$, we can find functions ρ_1, ρ_2, κ_u and κ_x of class $\mathcal{K}_\infty$ such that (16) holds. Note also that we do not claim stability in the proposition.

Proposition 1 will be established by showing first the existence of a continuous feedback law $\omega(x)$ assigning appropriate gains to the system :

$$\dot{x} \ = \ f(x) \ + \ G(x) \, [\omega(x) + c] \tag{20}$$

with c as input and $(x, \omega(x))$ as output. The conclusion will then follow from the Small-Gain Theorem [JTP, Theorem 2.1].

Lemma 3 (Strong Gain Assignment Theorem) *Assume A1 holds. Then for any functions κ_u and κ_x of class $\mathcal{K}_\infty$, there exists a continuous feedback law $\omega(x)$ and functions β_u and β_x of class $\mathcal{KL}$, such that, for each initial condition $x(0)$ and for each measurable essentially bounded function $c : \mathrm{R}_{\geq 0} \to \mathrm{R}^p$, the corresponding solutions of :*

$$\dot{x} \ = \ f(x) \ + \ G(x) \, [\omega(x) + c(t)] \tag{21}$$

satisfy, for all $0 \leq s \leq t$,

$$|\omega(x(t))| \ \ \leq \ \ \beta_u(|x(s)|, t - s) \ + \ (\mathrm{Id} + \kappa_u) \left(\sup_{\tau \in [s,t)} \{|c(\tau)|\} \right) \ , \tag{22}$$

$$|x(t)| \ \ \leq \ \ \beta_x(|x(s)|, t - s) \ + \ \kappa_x \left(\sup_{\tau \in [s,t)} \{|c(\tau)|\} \right) . \tag{23}$$

This result is to be compared with [JTP, Theorem 2.2]. We have here a stronger statement since not only any gain can be assigned to the relation $c \mapsto x$ but also we can limit the gain of the relation $c \mapsto \omega$.

Proof of Lemma 3. Let V and α_1 be the functions as in Assumption A1, and let $\dot{V}_{(21)}$ denote the function :

$$\dot{V}_{(21)}(x,t) \;=\; \frac{\partial V}{\partial x}(x)\left(f(x) + G(x)\left[\omega(x) + c(t)\right]\right) \ . \tag{24}$$

With assumption A1, we have :

$$\dot{V}_{(21)}(x,t) \;\leq\; -W(x) + L_G V(x)\omega(x) - L_G V(x) u_n(x) + L_G V(x) c(t) \ . \tag{25}$$

We restrict our attention to feedback laws ω of the form :

$$\omega_i(x) \;= -\mathrm{sign}(L_{g_i} V(x))\, \widehat{\omega}_i(x), \qquad \widehat{\omega}_i(x) \;\geq\; 0\,, \quad i = 1, 2, \ldots, p, \tag{26}$$

where the functions $\widehat{\omega}_i$ will be defined below. This yields :

$$\dot{V}_{(21)}(x,t) \;\leq\; -W(x) \;-\; \sum_{i=1}^{p} |L_{g_i} V(x)| \Big(\widehat{\omega}_i(x) \;-\; |u_{ni}(x)| \;-\; |c(t)|\Big) \ . \tag{27}$$

To define $\widehat{\omega}_i$, we let $\mathcal{S}$ be a function of class $\mathcal{K}_\infty$ such that, for all s and x, we have :

$$\kappa_u^{-1}(|u_n(x)|) \;\leq\; \mathcal{S}(V(x)) \quad , \qquad \kappa_x^{-1} \circ \alpha_1^{-1}(s) \;\leq\; \mathcal{S}(s) \ . \tag{28}$$

Such a function exists since V is positive definite and proper. Then we choose $\widehat{\omega}_i$ as :

$$\widehat{\omega}_i(x) \;=\; \theta_i(x)\, \overline{b}_i(x) \ , \tag{29}$$

where :

$$\overline{b}_i(x) \;=\; |u_{n_i}(x)| \;+\; \mathcal{S}(V(x)) \tag{30}$$

and θ_i is a function introduced to enforce continuity and defined as follows :
For each i, let :

$$\mathcal{B}_{0i} \;=\; \{x :\; L_{g_i} V(x) = 0,\; x \neq 0\} \ , \tag{31}$$

and :

$$\mathcal{B}_{1i} \;=\; \left\{x :\; |L_{g_i} V(x)| \Big(\mathcal{S}(V(x)) + |u_{n_i}(x)|\Big) \;\geq\; \frac{W(x)}{2p},\; x \neq 0\right\} \ . \tag{32}$$

Since W is positive definite, $\mathcal{B}_{0i}$ and $\mathcal{B}_{1i}$ are closed and disjoint subsets of $\mathrm{R}^n \setminus \{0\}$. It follows that we can define this function $\theta_i : \mathrm{R}^n \setminus \{0\} \to [0,1]$ as a continuous function satisfying (see the appendix for an explicit expression of such a function) :

$$\theta_i(x) \;=\; \left\{ \begin{array}{ll} 1\,, & \text{if} \quad x \in \mathcal{B}_{1i} \ , \\ 0\,, & \text{if} \quad x \in \mathcal{B}_{0i} \ . \end{array} \right. \tag{33}$$

To get a definition of θ_i on $\mathbb{R}^n$ we simply add $\theta_i(0) = 0$. Then, though θ_i may fail to be continuous at 0, the function $\widehat{\omega}$ is continuous on $\mathbb{R}^n$ since $\bar{b}_i(0) = 0$. Hence, from (27), we get:

$$\begin{aligned} \dot{V}_{(21)}(x,t) &\leq -W(x) - \sum_{i=1}^{p} |L_{g_i}V(x)| \left[\theta_i(x) \left(\mathcal{S}(V(x)) + |u_{n_i}(x)|\right) - |u_{ni}(x)| - |c(t)|\right] \\ &\leq -W(x) + \sum_{i=1}^{p} |L_{g_i}V(x)| \left(1 - \theta_i(x)\right) \left(\mathcal{S}(V(x)) + |u_{n_i}(x)|\right) \\ &\qquad - \left(\sum_{i=1}^{p} |L_{g_i}V(x)|\right) \left(\mathcal{S}(V(x)) - |c(t)|\right) \\ &\leq -\frac{W(x)}{2} - \left(\sum_{i=1}^{p} |L_{g_i}V(x)|\right) \left(\mathcal{S}(V(x)) - |c(t)|\right) \ . \end{aligned} \tag{34}$$

From this latter inequality, by using the fact that W is positive definite, V and $\mathcal{S}$ are positive definite and proper and following the same lines as in the Claims on p. 441 in [S2], we can show the existence of a function β_v of class $\mathcal{KL}$ such that, for all $0 \leq s \leq t$, we have :

$$V(x(t)) \ \leq \ \max\left\{\beta_v(V(x(s)), t-s) \, , \, \mathcal{S}^{-1}\left(\sup_{\tau\in[s,t)} \{|c(\tau)\}|\right)\right\} \ . \tag{35}$$

Inequality (23) follows readily with (28) and (2). Then, since we have :

$$|\omega(x)| \ \leq \ (\mathrm{Id} + \kappa_u) \circ \mathcal{S}(V(x)) \ , \tag{36}$$

the conclusion follows. ∎

Remark 4 : If instead of using u_n, we use u_s satisfying (6), the control law ω can be made simpler by modifying (29) and (30) so that :

$$\omega_i(x) \ = \ u_{s_i}(x) \ - \ \theta_i(x) \, \mathrm{sign}(L_{g_i}V(x)) \, \mathcal{S}(V(x)) \ , \tag{37}$$

and $\mathcal{B}_{1i}$ in (32) into :

$$\mathcal{B}_{1i} \ = \ \left\{x : \ |L_{g_i}V(x)| \, \mathcal{S}(V(x)) \ \geq \ \frac{W_s(x)}{2p}, \ x \neq 0\right\} \ . \tag{38}$$

Remark 5 : The control law ω can be made smooth if one allows the addition of arbitrarily small positive numbers to the right hand side of (22) and (23) (see [JTP]). More specifically, for any $\varepsilon_0 > 0$, one can always approximate each $\omega_i(x)$ by a smooth function $\widetilde{\omega}_i(x)$ so that, for all $x \in \mathbb{R}^n$, we have :

$$|\omega_i(x) - \widetilde{\omega}_i(x)| \ < \ \varepsilon_0 \ . \tag{39}$$

But, with such a choice of $\widetilde{\omega}_i$, it may fail to hold that :

$$L_{g_i}V(x) \, \widetilde{\omega}_i(x) \leq 0, \qquad \forall \, x \in \mathbb{R}^n \ . \tag{40}$$

To get a smooth feedback $\widetilde{\omega}_i$ satisfying restriction (40), we proceed as follows :
For each $m \in \{1, \ldots, m\}$, we let $\mathcal{B}_{2i}$ denote the open subset of R^n where $\omega_i(x) \neq 0$. We define :

$$\sigma_i(x) \;=\; \min\left\{\frac{|\omega_i(x)|}{2},\, \frac{\varepsilon_0}{2}\right\} \;. \tag{41}$$

so that $\sigma_i(x) > 0$ for all $x \in \mathcal{B}_{2i}$. Hence, there exists a function $\overline{\omega}_i(x)$ that is smooth on $\mathcal{B}_{2i}$ and such that :

$$|\overline{\omega}_i(x) - \omega_i(x)| \;<\; \sigma_i(x) \tag{42}$$

for all $x \in \mathcal{B}_{2i}$ (cf. [B, Theorem 4.8, p197]). One can then extends of the domain of $\overline{\omega}_i$ to R^n by letting $\overline{\omega}_i(x) = 0$ for $x \notin \mathcal{B}_{2i}$. Note then that $\overline{\omega}_i(x)$ is continuous everywhere, and, for all $x \in \mathrm{R}^n$,

$$\overline{\omega}_i(x)\,\omega_i(x) \;\geq\; 0 \;. \tag{43}$$

Now we let $\widetilde{\theta}_i(x) : \mathrm{R}^n \to [0,\, 1]$ be a smooth function satisfying the following :

$$\overline{\theta}_i(x) \;=\; \begin{cases} 0\,, & \text{if} \quad x \in \mathcal{B}_{3i}\,, \\ 1\,, & \text{if} \quad x \in \mathcal{B}_{4i}\,, \end{cases} \tag{44}$$

where the two sets $\mathcal{B}_{3i}$ and $\mathcal{B}_{4i}$ are defined by :

$$\mathcal{B}_{3i} = \{x \in \mathrm{R}^n : |\overline{\omega}_i(x)| \leq \varepsilon_0/4\}\;, \quad \mathcal{B}_{4i} = \{x \in \mathrm{R}^n : |\overline{\omega}_i(x)| \geq \varepsilon_0/2\}\;. \tag{45}$$

As before, such a smooth funciton exists because $\mathcal{B}_{3i}$ and $\mathcal{B}_{4i}$ are two disjoint closed subsets of R^n. Finally we let :

$$\widetilde{\omega}_i(x) \;=\; \overline{\theta}_i(x)\,\overline{\omega}_i(x) \;. \tag{46}$$

Then $\widetilde{\omega}_i$ is smooth everywhere, and, for all $x \in \mathrm{R}^n$,

$$\widetilde{\omega}_i(x)\,L_{g_i}V(x) \;\leq\; 0 \quad , \qquad |\widetilde{\omega}_i(x) - \omega_i(x)| \;<\; \varepsilon_0 \;. \tag{47}$$

Consequently, when the controls $\widetilde{\omega}_i$'s are used instead of the ω_i's, (34) becomes :

$$\dot{V}_{(21)}(x,t) \;\leq\; -\frac{W(x)}{2} \;-\; \left(\sum_{i=1}^{p} |L_{g_i}V(x)|\right)(\mathcal{S}(V(x)) - |c(t)| - \varepsilon_0)\;\;. \tag{48}$$

It follows that (22) and (23) are replaced by :

$$\begin{aligned} |\widetilde{\omega}(x(t))| \;&\leq\; \beta_u(|x(s)|, t-s) \;+\; (\mathrm{Id} + \kappa_u)\left(\sup_{\tau\in[s,t)}\{c(\tau)\} + \varepsilon_0\right) + p\varepsilon_0 \\ &\leq\; \beta_u(|x(s)|, t-s) \;+\; (\mathrm{Id} + \kappa_u)\left(\sup_{\tau\in[s,t)}\{\widetilde{c}(\tau)\}\right) \end{aligned} \tag{49}$$

$$|x(t)| \;\leq\; \beta_x(|x(s)|, t-s) \;+\; \kappa_x\left(\sup_{\tau\in[s,t)}\{\widetilde{c}(\tau)\}\right)\;, \tag{50}$$

where $\widetilde{c} = |c| + (p+1)\varepsilon_0$. ∎

Proof of Proposition 1. By applying Lemma 3 we get a continuous feedback law $\omega(x)$ which, when applied to (1), gives a closed-loop system which can be seen as the interconnection :

$$\dot{x} \;=\; f(x) \;+\; g(x)\left[\omega(x)+y_1\right] \quad , \quad y_1 \;=\; c(x,z,\omega(x))\,, \tag{51}$$

$$\dot{z} \;=\; a(y_{21},z,y_{22}) \quad , \quad y_{21} \;=\; x\,,\; y_{22} \;=\; \omega(x)\,, \tag{52}$$

where, from (7), (22) and (23),

$$|y_1(t)| \;\leq\; c_0 \;+\; \beta_c(|z(0)|,t) \;+\; \gamma_u\left(\sup_{\tau\in[0,t)}\{|y_{22}(\tau)|\}\right) \;+\; \gamma_x\left(\sup_{\tau\in[0,t)}\{|y_{21}(\tau)|\}\right)\;, \tag{53}$$

$$|y_{22}(t)| \;\leq\; \beta_u(|x(0)|,t) \;+\; (\mathrm{Id}+\kappa_u)\left(\sup_{\tau\in[0,t)}\{|y_1(\tau)|\}\right)\;, \tag{54}$$

$$|y_{21}(t)| \;\leq\; \beta_x(|x(0)|,t) \;+\; \kappa_x\left(\sup_{\tau\in[0,t)}\{|y_1(\tau)|\}\right)\;. \tag{55}$$

To conclude we could apply [JTP, Theorem 2.1] if :
– the function $\omega(x)$ were locally Lipschitz,
– we would have a one channel interconnection instead of the two channels given by y_{21} and y_{22} ,
Nevertheless, if the statement of this Theorem is not exactly appropriate, we can follow line by line its proof. First we can show with (16) that the outputs corresponding to any solutions are bounded on their maximal interval of definition. In particular, we have (see [JTP, (80)]) :

$$\begin{aligned}|y_1(t)| \;\leq\; c_0 \;+\; \beta_c(|z(0)|,t) \;+\; \gamma_u\left(\beta_u(|x(0)|,0)+(\mathrm{Id}+\kappa_u)\left(\sup_{\tau\in[0,t)}\{|y_1(\tau)|\}\right)\right)\\ +\;\gamma_x\left(\beta_x(|x(0)|,0)+\kappa_x\left(\sup_{\tau\in[0,t)}\{|c(\tau)|\}\right)\right)\;.\end{aligned} \tag{56}$$

With (16), this yields (see [JTP, (83)]) :

$$\begin{aligned}\sup_{\tau\in[0,t)}\{|y_1(\tau)|\} \;\leq\; (\mathrm{Id}+\rho_2^{-1})\Big(\beta_c\Big(|z(0)|,0\Big) \;+\; \gamma_u\circ(\mathrm{Id}+\rho_1^{-1})\Big(\beta_u(|x(0)|,0)\Big)\\ +\;\gamma_x\circ(\mathrm{Id}+\rho_1^{-1})\Big(\beta_x(|x(0)|,0)\Big) \;+\; c_0\Big)\,.\end{aligned} \tag{57}$$

With the BIBS property of both subsystems, this implies that all the solutions are defined and bounded on $\mathrm{R}_{\geq 0}$. This means that, for each $(x(t),z(t))$, there exists a positive real number s_∞ so that, for all t in $\mathrm{R}_{\geq 0}$, we have :

$$|(x(t),z(t))| \;\leq\; s_\infty\;. \tag{58}$$

Second, we get, for all t in $\mathrm{R}_{\geq 0}$ (see [JTP, (93)]),

$$\begin{aligned}|y_1(t)| \;\leq\; \Big[\beta_c(s_\infty,t/2)+\gamma_u\circ(\mathrm{Id}+\rho_1^{-1})\circ\beta_u(s_\infty,t/4)+\gamma_x\circ(\mathrm{Id}+\rho_1^{-1})\circ\beta_x(s_\infty,t/4)\Big]\\ +\;(\mathrm{Id}+\rho_2)^{-1}\left(\sup_{\tau\in[t/4,\infty)}\{|y_1(\tau)|\}\right)+c_0\,.\end{aligned} \tag{59}$$

So, with [JTP, Lemma A.1], for any function ρ_3 of class $\mathcal{K}_\infty$, we know the existence of a function $\widehat{\beta}$ of class $\mathcal{KL}$ such that we have, for all t in $\mathbb{R}_{\geq 0}$,

$$|y_1(t)| \;\leq\; \widehat{\beta}(s_\infty, t) \;+\; (\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3)(c_0)\ . \tag{60}$$

Since, with (58) and (52), (23) gives :

$$|x(t)| \;\leq\; \beta_x(s_\infty, t/2) \;+\; \kappa_x\left(\sup_{\tau\in[t/2,\infty)}\{|y_1(\tau)|\}\right)\ , \tag{61}$$

It follows readily that :

$$\limsup_{t\to\infty} |x(t)| \;\leq\; \kappa_x \circ (\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3)(c_0)\,, \tag{62}$$

for *any* function ρ_3 of class $\mathcal{K}_\infty$. But the solution $(x(t), z(t))$ is independent of ρ_3, this implies (17). □

4 Second solution with a dynamic feedback

The solution we have proposed in the previous section relies on the use of high gain. This fact is hidden in the choice of the function $\mathcal{S}$ which has to be sufficiently large and not only for small values. This may lead to problems if other robustness problems are considered. What is leading to high gain in the previous approach is the use of the matching assumption and a worst case design. By using more a priori knowledge on the unmodelled dynamics one may hope high gain to be involved in a different way. To this purpose, we incorporate assumption A2' in the following result.

Proposition 6 *Assume A1 holds with W a proper function[5], i.e. precisely :*

$$\alpha_3(V(x)) \;\leq\; \tfrac{1}{2}\, W(x)\,, \tag{63}$$

where α_3 is some function of class $\mathcal{K}_\infty$. Let us choose a real number μ, functions γ_{vu} and γ_c of class $\mathcal{K}$ and a function κ_1 of class $\mathcal{K}_\infty$ so that :

$$\gamma_{vu} \circ (\mathrm{Id} + \rho_4) \circ \kappa_1^{-1} \circ (\mathrm{Id} + \rho_4) \circ \gamma_c \;\leq\; \mu\,\mathrm{Id} \;-\; \rho_5 \tag{64}$$

for some functions ρ_4 and ρ_5 of class $\mathcal{K}_\infty$. We assume that, with such a choice, Assumption A2' holds with a function γ_{cu} satisfying :

$$\gamma_{cu} \;\leq\; \mathrm{Id} - \kappa_1\ . \tag{65}$$

Under these conditions, for any functions κ_2, κ_3 and κ_4 of class $\mathcal{K}_\infty$, there exists a continuous dynamic feedback law $\omega(x, r)$ with r given by (11) such that all the solutions of the closed-loop system are bounded and their x-components satisfy :

$$\limsup_{t\to+\infty} |x(t)| \;\leq\; \alpha_1^{-1} \circ (\mathrm{Id} + \kappa_3^{-1}) \circ \alpha_3^{-1} \circ (\mathrm{Id} + \kappa_4^{-1})\,(c_0\kappa_2(c_0))\ . \tag{66}$$

[5]With Assumption A1, we can always modify the function V to meet this requirement (see Proposition 13 for instance).

Remark 7 : When $c_0 = 0$, we get convergence of the x-component :

$$\lim_{t\to+\infty} |x(t)| \; = \; 0 \; . \tag{67}$$

When $c_0 \neq 0$, since κ_2, κ_3^{-1} and κ_4^{-1} can be chosen as arbitrarily small functions of class $\mathcal{K}_\infty$, (66) gives a practical convergence result.

Proof. First, we remark, with (11), that $r(t)$ is nonnegative for any t in $\mathrm{R}_{\geq 0}$. Then, we follow the same lines as for the proof of Lemma 3. Let $\dot{V}_{(1)}$ denote the function :

$$\dot{V}_{(1)}(x,u,t) \;=\; \frac{\partial V}{\partial x}(x)\Big(f(x) + G(x)(u + c(t))\Big), \tag{68}$$

and let ω be chosen of the form :

$$\omega_i \;= -\mathrm{sign}(L_{g_i}V(x))\,\widehat{\omega}_i \quad , \qquad \widehat{\omega}_i \;\geq\; 0\,, \quad i = 1,2,\ldots,p\;, \tag{69}$$

with functions $\widehat{\omega}_i$ to be defined below. With (10), we get :

$$\begin{aligned}\dot{V}_{(1)}(x,\omega,t) \;\leq\; -W(x) \;\; &- \;\; \sum_{i=1}^{p} |L_{g_i}V|\Big(\widehat{\omega}_i \;-\; |u_{ni}(x)|\Big) \\ &+ \;\; \sum_{i=1}^{p} |L_{g_i}V|\Big(c_0 \;+\; \beta(|z(0)|,t) \;+\; \gamma_{cu}(|\omega|) \;+\; \gamma_{cx}(|x|) \;+\; \gamma_c(r)\Big)\,. \end{aligned}\tag{70}$$

To define the functions $\widehat{\omega}_i$, let κ_2 be the function of class $\mathcal{K}_\infty$ chosen in Proposition 6. Let us also choose a function ℓ of class $\mathcal{K}$ and bounded by ℓ_∞. Since, for any positive real numbers a, b, we have :

$$ab \;\leq\; \frac{\kappa_2(b)\,b}{p} \;+\; \kappa_2^{-1}(pa)\,a \tag{71}$$

(consider two cases: $a \leq \kappa_2(b)/p$ and $a \geq \kappa_2(b)/p$), we obtain :

$$\dot{V}_{(1)}(x,\omega,t) \;\leq\; -W(x) \;-\; \sum_{i=1}^{p} |L_{g_i}V|\Big(\widehat{\omega}_i \;-\; \gamma_{cu}(|\omega|) \;-\; \bar{b}_i(x) \;-\; \gamma_c(r)\Big) \;+\; v_0(t)\kappa_2(v_0(t))\;, \tag{72}$$

where $v_0(t)$ is defined as :

$$v_0(t) \;=\; \beta(|z(0)|,\,t) \;+\; c_0\;, \tag{73}$$

and, for each i,

$$\bar{b}_i(x) \;=\; |u_{n_i}(x)| \;+\; \gamma_{cx}(|x|) \;+\; \kappa_2^{-1}(p|L_{g_i}V(x)|)\;. \tag{74}$$

Let :

$$\bar{b}(x) \;=\; \max_{j\in\{1,\ldots,p\}} \Big\{\bar{b}_j(x)\Big\}\;. \tag{75}$$

We define $\widehat{\omega}_i$ as :

$$\widehat{\omega}_i(x,r) \;=\; \theta_i(x,r)\,\kappa_1^{-1}\Big(\bar{b}(x) + \gamma_c(r)\Big)\;, \tag{76}$$

where κ_1 is chosen in Proposition 6. As in (29), the function $\theta_i: \mathbb{R}^n \times \mathbb{R} \to [0,\ 1]$ is introduced to enforce the continuity. It is defined as follows :
For each i, let :

$$\mathcal{B}_{0i} = \{(x,r):\ L_{g_i}V(x) = 0,\ (x,r) \neq (0,0)\}\ , \tag{77}$$

and :

$$\mathcal{B}_{1i} = \left\{(x,r):\ |L_{g_i}V(x)|\,\kappa_1^{-1}(\bar{b}(x) + \gamma_c(r)) \geq \frac{W(x)}{2p} + \frac{\ell(r)}{p},\ (x,r) \neq (0,0)\right\}\ . \tag{78}$$

Since both W and ℓ are positive definite, it follows that $\mathcal{B}_{0i}$ and $\mathcal{B}_{1i}$ are disjoint closed subsets of $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. Then, one can define a continuous function $\theta_i:\ \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\} \to [0,1]$ such that :

$$\theta_i(x,r) = \begin{cases} 1\,, & \text{if } \ x \in \mathcal{B}_{1i}\ ,\\ 0\,, & \text{if } \ x \in \mathcal{B}_{0i}\ . \end{cases} \tag{79}$$

To get a definition of θ_i on $\mathbb{R}^n \times \mathbb{R}$ we simply let $\theta_i(0,0) = 0$. Although θ_i may fail to be continuous at $(0,0)$, the functions $\widehat{\omega}_i$ and ω_i are continuous on $\mathbb{R}^n \times \mathbb{R}$ since $\bar{b}(0) = \gamma_c(0) = 0$.

Now, since condition (65) implies :

$$\begin{aligned} \widehat{\omega}_i(x,\,r) - \gamma_{cu}(|\omega(x,r)|) &\geq (-(1-\theta_i(x,r))\,\mathrm{Id} + \mathrm{Id} - \gamma_{cu}) \circ \kappa_1^{-1}(\bar{b}(x) + \gamma_c(r)) \\ &\geq -(1-\theta_i(x,r))\,\kappa_1^{-1}\left(\bar{b}(x) + \gamma_c(r)\right) + \bar{b}(x) + \gamma_c(r)\ , \end{aligned} \tag{80}$$

inequality (72) becomes, with (63), (76), (75) and the definition of θ_i,

$$\begin{aligned} \dot{V}_{(1)}(x,\omega,t) &\leq -W(x) + \sum_{i=1}^{p} |L_{g_i}V|(1-\theta_i(x,r))\kappa_1^{-1}(\bar{b}(x) + \gamma_c(r)) + v_0(t)\kappa_2(v_0(t)) \\ &\leq -\tfrac{1}{2}\,W(x) + v_0(t)\kappa_2(v_0(t)) + \ell(r) \\ &\leq -\alpha_3(V(x)) + v_0(t)\kappa_2(v_0(t)) + \ell(r)\ . \end{aligned} \tag{81}$$

On the other hand, with the control law given by (76), the equations (11) imply :

$$\dot{r} \leq -\mu\, r + \gamma_{vu} \circ \kappa_1^{-1}(\bar{b}(x) + \gamma_c(r)) + \gamma_{vx}(|x|)\ . \tag{82}$$

With condition (64), it follows :

$$\begin{aligned} \mu\, r - \gamma_{vu} \circ \kappa_1^{-1}(\bar{b}(x) + \gamma_c(r)) &\geq \mu\, r - \gamma_{vu}\Big(\kappa_1^{-1} \circ (\mathrm{Id} + \rho_4) \circ \gamma_c(r) + \kappa_1^{-1} \circ (\mathrm{Id} + \rho_4^{-1})(\bar{b}(x))\Big) \\ &\geq \mu\, r - \gamma_{vu} \circ (\mathrm{Id} + \rho_4) \circ \kappa_1^{-1} \circ (\mathrm{Id} + \rho_4) \circ \gamma_c(r) \\ &\qquad - \gamma_{vu} \circ (\mathrm{Id} + \rho_4^{-1}) \circ \kappa_1^{-1} \circ (\mathrm{Id} + \rho_4^{-1})(\bar{b}(x))) \\ &\geq \rho_5(r) - \gamma_{vu} \circ (\mathrm{Id} + \rho_4^{-1}) \circ \kappa_1^{-1} \circ (\mathrm{Id} + \rho_4^{-1})(\bar{b}(x))\,. \end{aligned} \tag{83}$$

Let ρ_6 be a function of class $\mathcal{K}_\infty$ satisfying :

$$\gamma_{vx}(|x|) + \gamma_{vu} \circ (\mathrm{Id} + \rho_4^{-1}) \circ \kappa_1^{-1} \circ (\mathrm{Id} + \rho_4^{-1})(\bar{b}(x)) \leq \rho_6(V(x))\ . \tag{84}$$

Such a function exists since (2) holds for all x. With (81), we have finally obtained the following system of differential inequalities :

$$\left\{ \begin{array}{rcl} \dot{V}_{(1)}(x,\omega,t) & \leq & -\alpha_3(V(x)) \ + \ v_0(t)\kappa_2(v_0(t)) \ + \ \ell(r) \\ & \leq & -\alpha_3(V(x)) \ + \ v_0(t)\kappa_2(v_0(t)) \ + \ \ell_\infty \ , \\ \dot{r} & \leq & -\rho_5(r) \ + \ \rho_6(V(x)) \ . \end{array} \right. \tag{85}$$

Now let $(x(t), r(t), z(t))$ be a solution of the closed-loop system (1),(11),(69),(76). Such a solution exists for any initial condition $(x(0), z(0))$ and has a right maximal interval of definition $[0, T)$. But since α_3 is of class $\mathcal{K}_\infty$, V is proper, v_0 and ℓ are bounded, (85) implies that $x(t)$ is bounded on $[0, T)$. This, with the fact that ρ_5 is of class $\mathcal{K}_\infty$, implies that $r(t)$ is also bounded on $[0, T)$. It follows that the control :

$$u(t) \ = \ \omega(x(t), r(t)) \tag{86}$$

is bounded on $[0, T)$. So with the BIBS property of the z subsystem, $z(t)$ is bounded on $[0, T)$. Hence, by contradiction, one shows that the solution is defined and bounded on $\mathbb{R}_{\geq 0}$, i.e., for all t in $\mathbb{R}_{\geq 0}$,

$$\|(V(x(t)), r(t), z(t))\| \ \leq \ s_\infty \ < \ +\infty \ . \tag{87}$$

Also, as in the proof of Proposition 1, by following the same lines as in the Claims on p. 441 in [S2], for any functions ρ_v and ρ_r of class $\mathcal{K}_\infty$, with :

$$\rho_v \ \leq \ \mathrm{Id} \ , \tag{88}$$

we can show the existence of class $\mathcal{KL}$ functions β_v and β_r such that, for all $0 \leq s \leq t$, we have :

$$V(x(t)) \ \ \leq \ \ \beta_v(V(x(s)), \, t-s) \ + \ \alpha_3^{-1} \circ (\mathrm{Id} + \rho_v) \left(\sup_{\tau \in [s,t)} \{ y_r(\tau) \} \right) \ , \tag{89}$$

$$r(t) \ \ \leq \ \ \beta_r(r(s), t-s) \ + \ \rho_5^{-1} \circ 2\rho_6 \left(\sup_{\tau \in [s,t)} \{ V(x(\tau)) \} \right) \ , \tag{90}$$

where we have introduced the function :

$$y_r(t) \ = \ v_0(t)\kappa_2(v_0(t)) \ + \ \ell(r(t)) \ . \tag{91}$$

But since :

$$\limsup_{t \to +\infty} y_r(t) \ \leq \ c_0\kappa_2(c_0) \ + \ \ell_\infty \ , \tag{92}$$

by taking $s = t/2$ in (89) and using (87), we conclude readily that :

$$\limsup_{t \to \infty} V(x(t)) \ \leq \ \alpha_3^{-1} \circ (\mathrm{Id} + \rho_v)(c_0\kappa_2(c_0) + \ell_\infty) \ , \tag{93}$$

The facts that function ρ_v is arbitrary and the solution $(x(t), r(t), z(t))$ is independent of ρ_v imply that

$$\limsup_{t \to \infty} V(x(t)) \ \leq \ \alpha_3^{-1}(c_0\kappa_2(c_0) + \ell_\infty) \ , \tag{94}$$

from which it follows :

$$\limsup_{t\to+\infty} |x(t)| \ \leq\ \alpha_1^{-1}\circ\alpha_3^{-1}\left(c_0\kappa_2(c_0)+\ell_\infty\right) \ . \tag{95}$$

To show that this bound can be improved, we proceed as follows. Let us choose the function ℓ not only of class $\mathcal{K}$ and bounded by ℓ_∞ but also satisfying :

$$\ell \ \leq\ \rho_7 \ , \tag{96}$$

where ρ_7 is of class $\mathcal{K}_\infty$ and defined by :

$$\rho_7 \ =\ (2(\mathrm{Id}+\kappa_4))^{-1}\circ\alpha_3\circ(\mathrm{Id}+\kappa_3)^{-1}\circ(2\rho_6)^{-1}\circ\left(2\rho_5^{-1}\right)^{-1} \ . \tag{97}$$

The constraint (96) can always be satisfied and the function ρ_7 depends only on known data. Indeed,
– the functions κ_3 and κ_4 are chosen in Proposition 6,
– the function α_3 is obtained, in order to meet (63), from the known function V and W,
– the functions ρ_4 and ρ_5 are obtained, in order to meet (64), from the chosen quantities μ, γ_{vu}, γ_c and κ_1,
– the function ρ_6 is obtained, in order to meet (84), from the known or chosen functions γ_{vu}, ρ_4, κ_1, u_n, γ_{cx}, κ_2, g, γ_{vx} and V.

Then, from (89), (91) and (90), we can consider the interconnection of a fictitious system with state x, input y_r and output $V(x)$ with a fictitious system with state r, input $V(x)$ and output $\ell(r)$. Although the systems are fictitious, the proof of the Small-Gain Theorem [JTP, Theorem 2.1] still applies. This can be seen by writing, in a way similar to (59),

$$\begin{aligned} V(x(t)) \ \leq\ & \beta_v(s_\infty,\, t/2) \ +\ \alpha_3^{-1}\circ(\mathrm{Id}+\rho_v)\circ(\mathrm{Id}+\kappa_4^{-1})\left(v_0(t/2)\kappa_2(v_0(t/2))\right) \\ & + \alpha_3^{-1}\circ(\mathrm{Id}+\rho_v)\circ(\mathrm{Id}+\kappa_4)\circ\ell(2\beta_r(s_\infty,t/4)) \\ & + \alpha_3^{-1}\circ(\mathrm{Id}+\rho_v)\circ(\mathrm{Id}+\kappa_4)\circ\ell\circ 2\rho_5^{-1}\circ 2\rho_6\left(\sup_{\tau\in[t/4,+\infty)}\{V(x(\tau))\}\right) \ . \end{aligned} \tag{98}$$

So, with (88) and (96), we can again apply [JTP, Lemma A.1] to conclude that :

$$\limsup_{t\to\infty} V(x(t)) \ \leq\ (\mathrm{Id}+\kappa_3^{-1})\circ(\mathrm{Id}+\rho_8)\circ\alpha_3^{-1}\circ(\mathrm{Id}+\rho_v)\circ(\mathrm{Id}+\kappa_4^{-1})(c_0\kappa_2(c_0)) \tag{99}$$

holds for any functions ρ_8 and ρ_v of class $\mathcal{K}_\infty$ with (88) satisfied. From this we get (66). □

5 Comparison between the static and dynamic feedback designs

Two common features of the designs we have proposed are that they require a similar small gain condition and, when $c_0 \neq 0$, they provide practical stability for the closed-loop system with a residual set which can be made smaller at the price of introducing a high gain feature for small values :

– Indeed, condition (16) of Proposition 1 and (64),(65) of Proposition 6 are approximately equivalent. In (16), since ρ_1, ρ_2 and κ_x are arbitrary, this condition can be interpreted as mainly requiring that the function $\mathrm{Id}-\gamma_u$ be bounded below by a function of class $\mathcal{K}_\infty$. Similarly in (64), since

ρ_4 and ρ_5 are arbitrary, this condition is mainly that the real number μ and the functions γ_{vu}, κ_1 and γ_c should be chosen such that the function $\kappa_1 - \left(\gamma_{cu} \circ \frac{1}{\mu}\gamma_{vu}\right)$ is bounded below by a function of class $\mathcal{K}_\infty$. Then, (65) is mainly requiring that the function $\mathrm{Id} - \left(\gamma_{cu} + \gamma_c \circ \frac{1}{\mu}\gamma_{vu}\right)$ is bounded below by a function of class $\mathcal{K}_\infty$. Our remark follows with (12).

- Concerning the size of the residual set, in the static case, i.e. in the context of Proposition 1, the solutions can be made to converge to a smaller neighborhood of the origin by choosing a smaller function κ_x. In the dynamic design, i.e. in the context of Proposition 6, one does so by choosing a smaller function κ_2. In both cases, this causes the "high gain" phenomenon at least for small values : in the static design, one needs to choose a bigger function $\mathcal{S}$ (see (28)), while in the dynamic design, the same problem occurs with the term $\kappa_2^{-1}(p|L_{g_i}V(x)|)$ in (74).

Two significant differences between the designs are that the dynamic design requires more a priori knowledge and that, when c_0 is known to be 0 and the unmodelled effect is more dynamic, and if we do not take into account the effects of the functions θ_i's presented in both designs, the dynamic design is less demanding high gain than the static design :

- While in the static design, one only needs to know that Assumptions A1 and A2 hold, in the dynamic design one needs the knowledge of the real number μ and the functions γ_{vx}, γ_{vu}, γ_{cx} and γ_c so that Assumption A2' holds.
- In the static design, we cannot use the a priori knowledge $c_0 = 0$. Indeed, in any case, the function $\mathcal{S}$, involved in $\bar{b}_i$ defined in (30), needs to satisfy the "high gain" inequality (28). But, in the dynamic design, the function κ_2, involved in $\bar{b}_i$ defined in (74), is completely arbitrary. For instance, we can take $\kappa_2 = p\,\mathrm{Id}$ in (76) and (74). This yields :

$$\bar{b}_i(x) \;=\; |u_{n_i}(x)| \;+\; \gamma_{cx}(|x|) \;+\; |L_{g_i}V(x)| \;, \tag{100}$$

  where every term is a "raw" data of the problem. When there is no static unmodelled effect, that is, when $\gamma_{cu} = 0$, the gain function κ_1^{-1} in (76) can be taken as the identity function, so the high gain feature is only caused by the function $\theta(x,r)$ used to make the feedback smooth. This difference between the two designs follows from the fact that the static one is definitely a worst case design using very little a priori knowledge.

To better understand the difference beween our two designs, we consider the following system:

$$\dot{x} = x^2 + u + c(z,u) \tag{101}$$

where the function c is assumed to satisfy :

$$|c(z,u)| \;\le\; c_u\,|u| \;+\; c_z\,|z| \tag{102}$$

and the unmodelled dynamics are given by :

$$\dot{z} \;=\; -\delta\, z \;+\; u \;, \tag{103}$$

for some $\delta > 0$.
Assumption A1 is satisfied with $V(x) = x^2$ by taking :

$$u_n(x) \;=\; -x\,|1+x| \;. \tag{104}$$

Assumption A2 is satisfied with :

$$c_0 \;=\; 0 \quad , \qquad \gamma_u(s) \;=\; \left(c_u + \frac{c_z}{\delta}\right) s \quad , \qquad \gamma_x(s) \;=\; 0 \; . \tag{105}$$

Finally, by choosing :

$$\gamma_{vx}(s) \;=\; \gamma_{cx}(s) \;=\; 0 \quad , \qquad \gamma_{vu}(s) = s \quad , \qquad \gamma_c(s) \;=\; h\, s \; , \tag{106}$$

Assumption A2' is satisfied with :

$$c_0 \;=\; 0 \quad , \qquad \gamma_u(s) \;=\; c_u\, s \; , \tag{107}$$

if we have :

$$\mu \;\leq\; \delta \quad , \qquad h \;\geq\; c_z \; . \tag{108}$$

Our static feedback is :

$$u(x) \;=\; -x \left(|1+x| + \frac{1}{k}(1+|x|)\right) \; , \tag{109}$$

with the parameter k to be chosen. It is obtained by taking :

$$\mathcal{S}(s) \;=\; \frac{1}{k}\left(\sqrt{s} + s\right) \; . \tag{110}$$

To obtain boundedness of the solutions and global attractivity of the origin, it is sufficient for the system to meet :

$$c_u \;+\; \frac{c_z}{\delta} \;<\; 1 \tag{111}$$

and, for the controller parameter k to meet :

$$k \;<\; \frac{1 - (c_u + \frac{c_z}{\delta})}{(c_u + \frac{c_z}{\delta})} \; . \tag{112}$$

This shows that an upperbound for $c_u + c_z/\delta$ is needed for the design.

Our dynamic feedback is, if continuity is not enforced,

$$\dot{r} \;=\; -\mu\, r \;+\; |u(x,r)| \quad , \qquad u(x,r) \;=\; -\frac{1}{k_1}\left(x\,|1+x| + x + h\, r\, \mathrm{sign}(x)\right) \; , \tag{113}$$

with the parameters μ, k_1 and h to be chosen. It is obtained by taking :

$$\kappa_1(s) \;=\; k_1\, s \quad , \qquad \kappa_2(s) \;=\; s \; , \tag{114}$$

where, according to (64) and (65), k_1 should satisfy :

$$\frac{h}{k_1} \;<\; \mu \quad , \qquad c_u \;\leq\; 1 \;-\; k_1 \; . \tag{115}$$

To obtain boundedness of the solutions and global attractivity of the origin, it is sufficient for the controller parameters (μ, h, k_1) to meet :

$$\mu \;\leq\; \delta \quad , \qquad h \;\geq\; c_z \quad , \qquad \frac{h}{\mu} \;<\; k_1 \;\leq\; 1 \;-\; c_u \; . \tag{116}$$

This shows that a lower bound for δ and upperbounds for c_u and c_z are needed. Also the system must satisfy :

$$c_u + \frac{c_z}{\delta} < 1 . \tag{117}$$

We conclude that the restrictions on the system are the same with both controllers. But the implementation of the dynamic controller requires more a priori knowledge.

Concerning the gains $1/k_1$, used in the dynamic feedback, and $1/k$, used in the static feedback, we see, with (116) and (112), that their need to be high depends on c_u and $\frac{c_z}{\delta}$. Notice, however, that the high gain occurs in different ways in the two methods: when the unmodelled effect is more static, the gain $1/k$ in the static feedback is lower than the gain $1/k_1$ used in the dynamic feedback; when the unmodelled effect is more dynamic, the gain $1/k_1$ is lower than the gain $1/k$. This can be observed in two extreme cases when $c_z = 0$ and when $c_u = 0$. When $c_z = 0$, that is, when the unmodelled effect is purely static, the static gain $1/k = c_u/(1 - c_u)$, and the dynamic gain is $1/k_1 = 1/(1 - c_u)$. If c_u gets close to 1, both $1/k$ and $1/k_1$ become high gain, but clearly, $1/k$ is lower than $1/k_1$. This suggests that when the unmodelled effect is more static, the static feedback is more suitable than the dynamic one. When $c_u = 0$, that is, when the unmodelled effect is purely dynamic, the static gain is $1/k = c_1/(1 - c_1)$, where $c_1 = c_z/\delta$, and the dynamic gain $1/k_1$ can be taken as any number between 1 and μ/h. When c_1 gets close to 1, the static gain k becomes a high gain, while the dynamic gain $1/k_1$ remains to be bounded. With more detailed analysis, it can be shown that if c_u is bounded away from 1, then the dynamic unmodelled effect will cause the high gain in the static design, while the gain used in the dynamic design remains bounded as long as c_u remains bounded. This is what should be expected, because the dynamic feedback was introduced mainly to deal with the dynamic unmodelled effect. But to be able to carry out the dynamic design, one needs more data on how the unmodelled dynamics affects the system.

Finally, since in this example x is in R and the functions a and c are linear, we can compare our second design method with the one proposed in [KSK]. This method leads to the dynamic state feedback :

$$\dot{r} = -\mu r + |u(x,r)| \quad , \qquad u(x,r) = -x\,|1+x| - k\,x\,(1 + hr + |x||1+x|) \ , \tag{118}$$

with the parameters μ, h and k to be chosen. It guarantees boundedness of the solutions and convergence of x to the set

$$\left\{ x \,:\, |x| \leq \frac{\max\{c_u, c_z/h\}}{k(1-c_u)} \right\}$$

provided that the controller parameters h and μ satisfy :

$$\frac{c_z}{1-c_u} < \mu \leq \delta \quad , \qquad \frac{h}{\mu} < \frac{1-c_u}{c_u} . \tag{119}$$

Hence the system should be such that :

$$c_u + \frac{c_z}{\delta} < 1 . \tag{120}$$

This shows that a lower bound for δ and an upperbound for c_u are needed for the design. So, in this case, as opposed to our second design method, c_z plays no explicit role in the control design. But the convergence of the solutions to the origin is not guaranteed without a further restriction.

An interesting topic for future research would be to find a way to combine the two designs leading to a dynamic controller retaining the advantages of both.

6 Extension to *one-sided* IOpS and *one-sided* exp-IOpS

In the case of linear systems, designs of static state feedback providing infinite gain margin are known. From Propositions 1 or 6, we get that u can be changed into $k\,u$ with k in $[\varepsilon, 2-\varepsilon]$, with a chosen $\varepsilon > 0$. Nevertheless the property that k can be in $[\varepsilon, +\infty[$ is recovered by noting that, in fact, our results still hold if, in A2, IOpS is replaced by *one-sided* IOpS, and in A2', exp-IOpS is replaced by *one-sided* exp-IOpS where :

one-sided IOpS : is the same as IOpS except that (7) is replaced by :

$$\max_{i\in\{1,\dots,p\}}\{c_{\mathrm{sided}_i}(t)\} \;\le\; c_0 \;+\; \beta_c(|z(0)|,t) \;+\; \gamma_u(U(t)) \;+\; \gamma_x(\sup_{\tau\in[0,t)}\{|x(\tau)|\})\;, \tag{121}$$

where :

$$c_{\mathrm{sided}_i}(t) \;=\; \max\Big\{-\mathrm{sign}(u_i(t))\,c_i(x(t),z(t),u(t))\,,\,0\Big\}\;. \tag{122}$$

one-sided exp-IOpS : is the same as exp-IOpS except that (10) is replaced by :

$$\max_{i\in\{1,\dots,p\}}\{c_{\mathrm{sided}_i}(t)\} \;\le\; c_0 + \beta(|z(0)|,t) + \gamma_{cu}(|u(t)|) + \gamma_{cx}(|x(t)|) + \gamma_c(r(t))\;. \tag{123}$$

Such a fact can be proved by following exactly the same lines as for the *two-sided* case with, in particular, the fact that Lemma 3 still holds if in (22) and (23), we replace $c(\tau)$ by $c_+(\tau)$ defined as :

$$c_+(\tau) \;=\; \max_{i\in\{1,\dots,p\}}\{c_{+i}(t)\}\;, \tag{124}$$

where :

$$c_{+i}(t) \;=\; \max\Big\{c_i(t)\mathrm{sign}\left(L_{g_i}V(x(t))\right)\,,\,0\Big\}\;. \tag{125}$$

With such *one-sided* properties, we see that if u is changed into into ku then c is given by :

$$c(x,z,u) \;=\; (k-1)\,u\;. \tag{126}$$

It follows that we have, for all i,

$$c_{\mathrm{sided}_i} \;=\; \max\{1-k,0\}\,|u_i|\;. \tag{127}$$

In this case, we get :

$$\gamma_x \;=\; \gamma_{cx} \;=\; \gamma_c \;=\; 0 \quad,\qquad \gamma_u(s) \;=\; \gamma_{cu}(s) \;=\; \max\{1-k,0\}\,s\;. \tag{128}$$

So, given $\varepsilon > 0$, we can design our controller so that we can allow $k \in [\varepsilon, +\infty)$.

7 On the equivalence of the IOS and exp-IOS properties

Let us study now the relation between IOS and exp-IOS properties. We have already mentioned that exp-IOS implies IOS. For finite dimensional observable linear systems, the converse is true. Indeed, in this case IOS implies that the eigen-values of any appropriate realization have strictly negative real part. For nonlinear systems, we replace observability by the strong unboundedness observability (SUO) property introduced in [JTP], i.e.

SUO : There exist a function β^o of class $\mathcal{KL}$, a function γ^o of class $\mathcal{K}$ and a nonnegative real number d^o such that, for each initial condition $z(0)$ and each measurable function $(x,u) : [0,T) \to \mathrm{R}^n \times \mathrm{R}^p$, with $0 < T \leq \infty$, the corresponding solution $z(t)$, right maximally defined on $[0,T')$, with T' in $(0,T]$, satisfies, for all t in $[0,T')$,

$$|z(t)| \;\leq\; \beta^o(|z(0)|,t) \;+\; \gamma^o\left(\sup_{\tau\in[0,t)} \{|(x(\tau),u(\tau),c(x(\tau),z(\tau),u(\tau))|\}\right) \;+\; d^o \ . \tag{129}$$

Indeed, by following exactly the same arguments as in the proof of [JTP, Proposition 3.1], we see that IOS and SUO, with $d^o = 0$, imply ISS, and if in addition $c(0,0,0) = 0$, then exp-ISS implies exp-IOS. Therefore if ISS and exp-ISS are equivalent properties, IOS and exp-IOS are also equivalent properties under the extra assumptions SUO, with $d^o = 0$, and $c(0,0,0) = 0$.

To study this equivalence of ISS and exp-ISS, let us consider the following system :

$$\dot{x} = f(x,u) \ , \tag{130}$$

with x in R^n and $f : \mathrm{R}^n \times \mathrm{R}^p \to \mathrm{R}^n$ a locally Lipschitz function satisfying $f(0,0) = 0$. We assume this system is ISS, i.e.

ISS : There exist a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for each initial condition $x(0)$ and each measurable essentially bounded function $u : \mathrm{R}_{\geq 0} \to \mathrm{R}^p$, the corresponding solution $x(t)$ satisfies, for all t in $\mathrm{R}_{\geq 0}$,

$$|x(t)| \;\leq\; \beta(|x(0)|,t) \;+\; \gamma(U(t)) \tag{131}$$

where $U(t)$ is defined in (8).

In this context, the exp-ISS property is :

exp-ISS : Given $\mu > 0$, there exist a function β of class $\mathcal{KL}$, a function γ_c of class $\mathcal{K}$ which is C^1 on $\mathrm{R}_{>0}$, and a function γ_v of class $\mathcal{K}$ which is C^1 on $\mathrm{R}_{\geq 0}$, such that, for each initial condition $z(0)$ and each measurable essentially bounded function $u : \mathrm{R}_{\geq 0} \to \mathrm{R}^p$, the corresponding solution $x(t)$ satisfies, for all t in $\mathrm{R}_{\geq 0}$,

$$|x(t)| \;\leq\; \beta(|x(0),t) \;+\; \gamma_c(r(t)) \ , \tag{132}$$

where $r(t)$ is the solution of the following initial value problem :

$$\dot{r} \;=\; -\mu\, r \;+\; \gamma_v(|u|) \ , \qquad r(0) \;=\; 0 \ . \tag{133}$$

Clearly exp-ISS implies ISS with γ in (131) given by $\gamma_c \circ \frac{1}{\mu}\gamma_v$. The converse is also true. Precisely, we have :

Proposition 8 *System (130) is ISS if and only if it is* exp-*ISS. This result still holds if we impose that γ_c be concave and γ_v be convex.*

To establish this statement, we need to recall the definition of ISS-Lyapunov functions introduced in [SW].

Definition 9 A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an *ISS-Lyapunov function* for system (130) if there exist functions α_1 and α_2 of class $\mathcal{K}_\infty$ and α_3 and χ of class $\mathcal{K}$ such that, for all x,

$$\alpha_1(|x|) \;\leq\; V(x) \;\leq\; \alpha_2(|x|) \tag{134}$$

and

$$|x| \;\geq\; \chi(|u|) \qquad \Longrightarrow \qquad \frac{\partial V}{\partial x}(x) f(x,u) \;\leq\; -\alpha_3(|x|) \; . \tag{135}$$

One of the main results in [SW] provides the following Lyapunov characterization of ISS :

Lemma 10 *The system (130) is ISS if and only if it admits an ISS-Lyapunov function.*

In fact, the property for a system to have an ISS-Lyapunov function can be strengthened as follows :

Lemma 11 *If a system admits an ISS-Lyapunov function V satisfying (134) and (135), then, for any $\mu > 0$, there exists a C^1 function $\mathcal{V}$ and functions $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ of class $\mathcal{K}_\infty$ such that, for all x,*

$$\tilde{\alpha}_1(|x|) \;\leq\; \mathcal{V}(x) \;\leq\; \tilde{\alpha}_2(|x|) \tag{136}$$

and

$$|x| \;\geq\; \chi(|u|) \qquad \Longrightarrow \qquad \frac{\partial \mathcal{V}}{\partial x}(x)\, f(x,u) \;\leq\; -\mu \mathcal{V}(x) \; . \tag{137}$$

Note that if $\mathcal{V}$ is smooth, then $\mathcal{V}$ is again an ISS-Lyapunov function with an associated function α_3 equal to $-\mu\mathcal{V}$.

Proof. First of all, observe that, by renaming by α_3 the function $\frac{2}{\mu}\alpha_3 \circ \alpha_1^{-1}$ which is still a function of class $\mathcal{K}$, (135) becomes :

$$|x| \;\geq\; \chi(|u|) \qquad \Longrightarrow \qquad \frac{2}{\mu}\frac{\partial V}{\partial x}(x)\, f(x,u) \;\leq\; -\alpha_3\left(V(x)\right) \; . \tag{138}$$

Now consider a C^1 function a of class $\mathcal{K}$ with the property[6] :

$$a(\tau) \leq \min\{\tau, \alpha_3(\tau)\} \quad , \qquad a'(0) = 0 \; . \tag{139}$$

For instance, we can take:

$$a(\tau) = \frac{2}{\pi}\int_0^\tau \frac{\min\{s, \alpha_3(s)\}}{1+s^2} ds \; . \tag{140}$$

[6] a' denotes the first derivative of the real function a.

Then, let ρ be the function defined as :

$$\left\{\begin{array}{rcl}\rho(\tau) & = & \exp\left(\int_1^\tau \frac{2ds}{a(s)}\right) \quad , \quad \forall \tau \in \mathrm{R}_{>0} \ , \\ \rho(0) & = & 0 \ . \end{array}\right. \tag{141}$$

This function is continuous on $\mathrm{R}_{>0}$. And, since the integral inside the exponential function diverges to $-\infty$ as τ tends to 0, and diverges to $+\infty$ as τ tends to $+\infty$, one sees that ρ is of class $\mathcal{K}_\infty$. Furthermore, we have the following :

Lemma 12 *The function ρ can be extended as a C^1 function on $\mathrm{R}_{\geq 0}$.*

Before proving this Lemma, we remark that the function $\mathcal{V}$, defined as :

$$\mathcal{V} \ = \ \rho \circ V \ , \tag{142}$$

allows us to prove Lemma 11. Indeed, in this case, (136) holds with :

$$\tilde{\alpha}_1 = \rho \circ \alpha_1 \ , \quad \tilde{\alpha}_2 = \rho \circ \alpha_2 \ . \tag{143}$$

And we get :

$$|x| \ \geq \ \chi(|u|) \qquad \Longrightarrow \qquad \frac{\partial \mathcal{V}}{\partial x}(x) f(x,u) \ = \ \frac{2}{a\left(V(x)\right)} \mathcal{V}(x) \frac{\partial V}{\partial x}(x) f(x,u) \ \leq \ -\mu \mathcal{V}(x) \ . \tag{144}$$

Proof of Lemma 12. Clearly ρ is a C^2 function on $\mathrm{R}_{>0}$. So it is enough to show :

$$\rho'(0) \ = \ 0 \quad , \qquad \lim_{\tau \to 0^+} \rho'(\tau) \ = \ 0 \ . \tag{145}$$

First note that, for τ small enough, we have the estimation :

$$\rho(\tau) \ = \ \exp\left(-\int_\tau^1 \frac{2ds}{a(s)}\right) \ \leq \ \exp\left(-\int_\tau^1 \frac{2ds}{s}\right) \ = \ \exp(\ln \tau^2) \ = \ \tau^2 \ . \tag{146}$$

It follows that $\rho'(0)$ exists and :

$$\rho'(0) \ = \ 0 \ . \tag{147}$$

To show the second point of (145), we proceed as follows :
For $\tau \neq 0$, we get readily :

$$\rho'(\tau) \ = \ \frac{2}{a(\tau)} \rho(\tau) \quad , \qquad \rho''(\tau) \ = \ \left(\frac{4}{a^2(\tau)} - \frac{2a'(\tau)}{a^2(\tau)}\right) \rho(\tau) \ . \tag{148}$$

Since $a'(0)$ is 0, it follows that there exists some strictly positive real number δ such that :

$$0 \ < \ \tau \ < \ \delta \qquad \Longrightarrow \qquad 0 \ < \ a'(\tau) \ < \ 1 \ . \tag{149}$$

We conclude:

- The function ρ' is positive and strictly increasing on $(0, \delta)$. This implies that $\lim_{\tau\to 0^+} \rho'(\tau)$ exists and is non negative.
- The function ρ'' is bounded below by $\dfrac{\rho'(\tau)}{a(\tau)}$ on $(0, \delta)$.

Now to get a contradiction, we assume that $\lim_{\tau\to 0^+} \rho'(\tau)$ is strictly positive. In this case, there exists some strictly positive real number c such that :

$$\tau \in (0, \delta) \implies \rho'(\tau) \geq c \,, \tag{150}$$

$$\implies \rho''(\tau) \geq \frac{c}{a(\tau)} \geq \frac{c}{\tau} \,. \tag{151}$$

But, with

$$\rho'(\tau) = \rho'(\delta) - \int_\tau^\delta \rho''(s)\, ds \,, \tag{152}$$

this implies :

$$\lim_{\tau\to 0^+} \rho'(\tau) = -\infty \,. \tag{153}$$

This contradicts the fact that ρ' is positive on $\mathrm{R}_{>0}$. So ρ' must be continuous on $\mathrm{R}_{\geq 0}$. ∎

In proving Lemma 11, we have also reestablished the following statement which can be found for example in [LL, Theorem 3.6.10] but is rarely used :

Proposition 13 *If a system $\dot{x} = f(x)$ admits a C^1 Lyapunov function V, that is, there exist functions α_1 and α_2 of class $\mathcal{K}_\infty$ and α_3 of class $\mathcal{K}$, such that we have, for all x,*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad , \qquad \frac{\partial V}{\partial x}(x)\, f(x) \leq -\alpha_3(|x|) \,, \tag{154}$$

then, for each $\mu > 0$, the system also admits a C^1 Lyapunov function $\mathcal{V}$ satisfying, for all x,

$$\tilde{\alpha}_1(|x|) \leq \mathcal{V}(x) \leq \tilde{\alpha}_2(|x|) \quad , \qquad \frac{\partial \mathcal{V}}{\partial x}(x)\, f(x) \leq -\mu\, \mathcal{V}(x) \,, \tag{155}$$

for some functions $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ of class $\mathcal{K}_\infty$.

We are now ready to prove Proposition 8.

Proof of Proposition 8. We know already that exp-ISS implies ISS. We now show that ISS implies exp-ISS. Assume that system (130) is ISS. Then by Lemma 11, there exists some C^1 function $\mathcal{V}$ satisfying (136) and (137). We define on $\mathrm{R}_{\geq 0}$ the function γ_v as follows :

$$\gamma_v(s) = s + \max_{|x|\leq\chi(|u|),\, |u|\leq s} \left\{ \frac{\partial \mathcal{V}}{\partial x}(x) f(x,u) + \mu \mathcal{V}(x) \right\} \,. \tag{156}$$

It is of class $\mathcal{K}_\infty$ and, from (137), we get readily (see also [SW] for more detailed reasoning), for all (x, u),

$$\frac{\partial \mathcal{V}}{\partial x}(x)\, f(x,u) \leq -\mu\mathcal{V}(x) + \gamma_v(|u|) \,. \tag{157}$$

Now pick any measurable essentially bounded function $u : \mathrm{R}_{\geq 0} \to \mathrm{R}^p$ and any initial condition $x(0)$ in R^n. By (157), the corresponding solution $x(t)$ satisfies, for all t in $\mathrm{R}_{\geq 0}$,

$$\dot{\overbrace{\mathcal{V}(x(t))}} \;\leq\; -\mu\,\mathcal{V}(x(t)) \;+\; \gamma_v(|u(t)|)\ . \tag{158}$$

It follows :

$$\mathcal{V}(x(t)) \;\leq\; \exp(-\mu t)\,\mathcal{V}(x(0)) \;+\; r(t)\ , \tag{159}$$

where $r(t)$, defined here as :

$$r(t) \;=\; \int_0^t \exp(-\mu[t-s])\,\gamma_v(|u(s)|)\,ds\ , \tag{160}$$

is the unique solution of the initial value problem (133). With (136), we have obtained :

$$|x(t)| \;\leq\; \tilde{\alpha}_1^{-1}\left(\exp(-\mu t)\,\mathcal{V}(x(0)) \;+\; r(t)\right) \;\leq\; \beta(s,t) \;+\; \gamma_c(r(t))\ , \tag{161}$$

where :

$$\beta(s,t) \;=\; \tilde{\alpha}_1^{-1}(2\exp(-\mu t)\tilde{\alpha}_2(s))\,, \qquad\qquad \gamma_c(s) \;=\; \tilde{\alpha}_1^{-1}(2s)\,, \tag{162}$$

To complete the proof of the proposition, we need to show that γ_c and γ_v can be restricted to be concave and convex respectively, with the desired continuous differentiability. To this purpose, we need the following :

Lemma 14 *For any function γ of class $\mathcal{K}$, there exist a convex function γ_v, of class $\mathcal{K}$ and C^1 on $\mathrm{R}_{\geq 0}$, and a concave function γ_c, of class $\mathcal{K}$ and C^1 on $\mathrm{R}_{>0}$, such that :*

$$\gamma_c \circ \gamma_v \;\geq\; \gamma\ . \tag{163}$$

Proof. Let $[0,S)$, (where $S \leq +\infty$), be the image by γ of $\mathrm{R}_{\geq 0}$ and let :

$$s_0 \;=\; \min\{1,\ S/2\}\ . \tag{164}$$

We define :

$$\gamma_c^{-1}(s) \;=\; \left\{ \begin{array}{ll} \displaystyle\int_0^s \gamma^{-1}(\tau)d\tau\ , & \forall\, s \,\leq\, s_0\ , \\ \gamma_c^{-1}(s_0) \;+\; (s-s_0)\,\gamma^{-1}(s_0)\ , & \forall\, s_0 \,<\, s\ . \end{array} \right. \tag{165}$$

Since γ^{-1} is increasing and continuous on $[0,s_0]$, the function γ_c^{-1} is convex, of class $\mathcal{K}$ and C^1 on $\mathrm{R}_{\geq 0}$. So the function γ_c is concave, of class $\mathcal{K}$ and C^1 on $\mathrm{R}_{>0}$. Also we have :

$$\gamma_c^{-1}(s) \;\leq\; s\,\gamma^{-1}(s) \;\leq\; \gamma^{-1}(s) \qquad \forall\, s \,\leq\, s_0\ . \tag{166}$$

This implies :

$$\gamma_c(s) \;\geq\; \gamma(s) \qquad \forall\, s \,\leq\, \gamma^{-1}(s_0)\ . \tag{167}$$

Now we define a function γ_v as :

$$\gamma_v(s) \;=\; \frac{\gamma^{-1}(s_0)}{s_0}\int_0^{2s}\gamma(\tau)d\tau \;+\; s\ . \tag{168}$$

This function γ_v is convex, of class $\mathcal{K}$ and C^1 on $\mathrm{R}_{\geq 0}$ and we have :

$$\gamma_v(s) \ \geq \ \frac{\gamma^{-1}(s_0)}{s_0}\, s\,\gamma(s) \ + \ s\,. \tag{169}$$

Then, for $s \leq \gamma^{-1}(s_0)$, we have, with (167) and (169),

$$\gamma_c(\gamma_v(s)) \ \geq \ \gamma_c(\tfrac{\gamma^{-1}(s_0)}{s_0} s\gamma(s) + s) \ \geq \ \gamma(s)\ . \tag{170}$$

And, for $s \geq \gamma^{-1}(s_0)$, we have, with (166) and (169),

$$\gamma_v(s) \ \geq \ \gamma^{-1}(s_0) \ \geq \ \gamma_c^{-1}(s_0)\,. \tag{171}$$

So, in this case, we can use the second definition in (165) to evaluate $\gamma_c(\gamma_v(s))$. With (169), this yields :

$$\begin{aligned}\gamma_c(\gamma_v(s)) \ &\geq \ \frac{(\frac{\gamma^{-1}(s_0)}{s_0} s\gamma(s) + s) + \gamma^{-1}(s_0)s_0 - \gamma_c^{-1}(s_0)}{\gamma^{-1}(s_0)} \\ &\geq \ \frac{\frac{\gamma^{-1}(s_0)}{s_0} s\gamma(s)}{\gamma^{-1}(s_0)} \geq \gamma(s)\ . \end{aligned}\tag{172}$$

■

Lemma 15 *For any functions γ_2 and γ_3 of class $\mathcal{K}$, there exist a convex function γ_v, of class $\mathcal{K}$ and C^1 on $\mathrm{R}_{\geq 0}$, and a concave function γ_c, of class $\mathcal{K}$ and C^1 on $\mathrm{R}_{>0}$, such that :*

$$\gamma_2\left(\int_0^t \exp(-\mu(t-\tau))\gamma_3(|u(\tau)|)d\tau\right) \ \leq \ \gamma_c\left(\int_0^t \exp(-\mu(t-\tau))\gamma_v(|u(\tau)|)d\tau\right)\ . \tag{173}$$

Proof. From Lemma 14, we know the existence of functions γ_{v3} and γ_{c3} with the desired properties so that :

$$\gamma_3 \ \leq \ \gamma_{c3}\circ\gamma_{v3}\ . \tag{174}$$

Let

$$f(t) \ = \ \frac{1-\exp(-\mu t)}{\mu} \ \leq \ \frac{1}{\mu}\ . \tag{175}$$

Then, with Jensen's inequality and concavity, we get :

$$\begin{aligned}\int_0^t \exp(-\mu(t-\tau))\gamma_3(|u(\tau)|)d\tau \ &\leq \ f(t)\,\gamma_{c3}\left(\tfrac{1}{f(t)}\int_0^t \exp(-\mu(t-\tau))\gamma_{v3}(|u(\tau)|)d\tau\right) \\ &\leq \ \tfrac{1}{\mu}\gamma_{c3}\left(\mu\int_0^t \exp(-\mu(t-\tau))\gamma_{v3}(|u(\tau)|)d\tau\right)\ . \end{aligned}\tag{176}$$

But again there exist functions γ_{v2} and γ_{c2} with the desired properties so that :

$$\gamma_2\circ\tfrac{1}{\mu}\gamma_{c3} \ \leq \ \gamma_{c2}\circ\gamma_{v2}\ . \tag{177}$$

So we get :

$$\begin{aligned}\gamma_2\left(\int_0^t \exp(-\mu(t-\tau))\gamma_3(|u(\tau)|)d\tau\right) &\leq \gamma_2\circ\tfrac{1}{\mu}\gamma_{c3}\left(\mu\int_0^t \exp(-\mu(t-\tau))\gamma_{v3}(|u(\tau)|)d\tau\right)\\ &\leq \gamma_{c2}\circ\gamma_{v2}\left(\mu\int_0^t \exp(-\mu(t-\tau))\gamma_{v3}(|u(\tau)|)d\tau\right)\\ &\leq \gamma_{c2}\circ\mu\, f(t)\gamma_{v2}\left(\tfrac{1}{f(t)}\int_0^t \exp(-\mu(t-\tau))\gamma_{v3}(|u(\tau)|)d\tau\right)\\ &\leq \gamma_{c2}\left(\mu\int_0^t \exp(-\mu(t-\tau))\gamma_{v2}\circ\gamma_{v3}(|u(\tau)|)d\tau\right)\ . \qquad (178)\end{aligned}$$

Hence, we can take :

$$\gamma_c(s) \;=\; \gamma_{c2}(s) \quad , \qquad \gamma_v(s) \;=\; \gamma_{v2}\circ\gamma_{v3}(s)\ . \qquad (179)$$

■

Proof of Proposition 8 (Continued). From (161), one gets :

$$|x(t)| \;\leq\; \beta(|x(0)|,t) + \gamma_c\left(\int_0^t \exp(-\mu(t-\tau))\gamma_v(|u(\tau)|)d\tau\right)\ . \qquad (180)$$

By Lemma 15, there exist a concave function $\tilde{\gamma}_c$ of class $\mathcal{K}$ and a convex function $\tilde{\gamma}_v$ of class $\mathcal{K}$ with all the desired properties such that :

$$\gamma_c\left(\int_0^t \exp(-\mu(t-\tau))\gamma_v(|u(\tau)|)d\tau\right) \;\leq\; \tilde{\gamma}_c\left(\int_0^t \exp(-\mu(t-\tau))\tilde{\gamma}_v(|u(\tau)|)d\tau\right)\ . \qquad (181)$$

The conclusion of Proposition 8 follows readily. □

The advantage of the exp-ISS is that it allows to replace the L_∞ norm with a memory fading L_1 norm in the ISS estimation. However, one may worry if the exp-ISS will lead to more conservative results. Our objective of the following example is to show that this is not necessarily the case if some care is taken in choosing the real number μ the functions γ_c and γ_v.

Consider the system :

$$\dot{x} \;=\; -ax^3 \;+\; \gamma_0(|u|)\ , \qquad a>0\ , \qquad (182)$$

where γ_0 is a function of class $\mathcal{K}$. This system is ISS and its gain function γ can be taken as any function of class $\mathcal{K}$ satisfying :

$$\gamma \;>\; \left(\frac{\gamma_0}{a}\right)^{1/3}\ . \qquad (183)$$

To get an estimation on γ_v and γ_c, we let, for each integer k strictly larger than $3a$ and $\mu/2$,

$$V_k(x) = \alpha_k(|x|)\ , \qquad (184)$$

where, for each k, α_k is a C^1 function of class $\mathcal{K}_\infty$ defined as :

$$\alpha_k(s) \;=\; \left\{ \begin{array}{ll} \exp\left(\frac{k}{a}\left[1-\frac{1}{s^2}\right]\right) \;, & \text{if} \quad 0 \leq s \leq 1 \;, \\ s^{2k/a} \;, & \text{if} \quad s > 1 \;. \end{array} \right. \tag{185}$$

Then, for $|x|$ in $(0,1]$, we have :

$$\begin{aligned} \dot{V}_{k\,(182)}(x) \;&=\; -\mu\, V_k(x) \;-\; (2k-\mu)\, V_k(x) \;+\; \frac{2kV_k(x)}{a|x|^3}\,\gamma_0(|u|) \\ &\leq\; -\mu\, V_k(x) \;+\; \max_{|x|\leq\chi_k(|u|)} \left\{\frac{2kV_k(x)}{a|x|^3}\gamma_0(|u|)\right\} \\ &\leq\; -\mu\, V_k(x) \;+\; (2k-\mu)\, V_k(\chi_k(|u|))\,, \end{aligned} \tag{186}$$

where χ_k is a function of class $\mathcal{K}$ defined as :

$$\chi_k(s) \;=\; \left(\frac{2k}{2k-\mu}\right)^{1/3} \gamma(s) \;>\; \left[\frac{2k\,\gamma_0(s)}{(2k-\mu)a}\right]^{1/3} \;, \tag{187}$$

with γ given in (183). To get (186), we used the fact that $V_k(x)/|x|^3$ is an increasing function in $|x|$ for all k strictly larger than $3a$.
When $|x|$ is in $(1,+\infty)$, by applying the same arguments, we have :

$$\begin{aligned} \dot{V}_{k,(182)} \;&=\; -\mu\, x^2\, V_k(x) \;-\; (2k-\mu)\, x^2\, V_k(x) \;+\; \frac{2kx^2V_k(x)}{a|x|^3}\,\gamma_0(|u|) \\ &\leq\; -\mu\, V_k(x) \;+\; (2k-\mu)\, \chi_k^2(|u|)\, V_k(\chi_k(|u|)) \;. \end{aligned} \tag{188}$$

Thus, for any solution $x(t)$ of (182), one has :

$$V_k(x(t)) \;=\; V_k(x(0))\, \exp(-\mu t) \;+\; r(t) \;, \tag{189}$$

with $r(t)$ solution of :

$$\dot{r} \;=\; -\mu\, r \;+\; \gamma_{vk}(|u|) \quad , \qquad r(0) \;=\; 0 \;, \tag{190}$$

where, for each k,

$$\gamma_{vk}(s) \;=\; \left\{ \begin{array}{ll} (2k-\mu)\, V_k(\chi_k(s)) \;, & \text{if} \quad \chi_k(s) \leq 1 \;, \\ (2k-\mu)\, \chi_k^2(s)\, V_k(\chi_k(s)) \;, & \text{if} \quad \chi_k(s) > 1 \;. \end{array} \right. \tag{191}$$

From (189), one gets :

$$|x(t)| \;\leq\; \beta_k(|x_0|, t) \;+\; \gamma_{ck}(r(t)) \;, \tag{192}$$

for some function β_k of class $\mathcal{KL}$ and with γ_{ck} given as :

$$\gamma_{ck}(s) \;=\; \alpha_k^{-1}\left(\frac{2k}{2k-\mu}s\right) \;. \tag{193}$$

Let us now prove that, as k is going to $+\infty$, the function $\gamma_{ck} \circ \frac{1}{\mu}\gamma_{vk}$ approaches γ. When $\chi_k(s)$ is in $(1, +\infty)$, $V_k(\chi_k(s))$ is strictly larger than 1. Thus we have :

$$\begin{aligned} \gamma_{ck} \circ \frac{1}{\mu}\gamma_{vk}(s) &\leq \alpha_k^{-1}\left(\frac{2k}{\mu}\,\chi_k^2(s)\,V_k(\chi_k(s))\right) \\ &\leq \left(\frac{2k}{\mu}\right)^{a/2k} \chi_k(s)^{(1+a/k)} \\ &\leq \left(\frac{2k}{\mu}\right)^{a/2k} \left(\frac{2k}{2k-\mu}\right)^{(1/3)\,(1+a/k)} \gamma(s)^{(1+a/k)} . \end{aligned} \tag{194}$$

When $\chi_k(s)$ is in $[0,1]$ but $\dfrac{2k}{\mu}\,[V_k(\chi_k(s))]$ is still strictly larger than 1, we have :

$$\gamma_{ck} \circ \frac{1}{\mu}\gamma_{vk}(s) \leq \alpha_k^{-1}\left(\left(\frac{2k}{\mu}\right) V_k(\chi_k(s))\right) \tag{195}$$

$$\leq \left(\frac{2k}{\mu}\right)^{a/2k} \sqrt{\exp\left(1 - \frac{1}{\chi_k^2(s)}\right)}$$

$$\leq \left(\frac{2k}{\mu}\right)^{a/2k} \chi_k(s) \tag{196}$$

$$\leq \left(\frac{2k}{\mu}\right)^{a/2k} \left(\frac{2k}{2k-\mu}\right)^{1/3} \gamma(s) . \tag{197}$$

When both $\chi_k(s)$ and $\dfrac{2k}{\mu}\,[V_k(\chi_k(s))]$ are in $[0,1]$, we have :

$$\gamma_{ck} \circ \frac{1}{\mu}\gamma_{vk}(s) \leq \alpha_k^{-1}\left(\frac{2k}{\mu} V_k(\chi_k(s))\right) \tag{198}$$

$$\leq \frac{1}{\sqrt{1 - \frac{a}{k}\ln\left(\frac{2k}{\mu} V_k(\chi_k(s))\right)}} \tag{199}$$

$$\leq \frac{1}{\sqrt{1 - \frac{a}{k}\ln\left(\frac{2k}{\mu}\right) - \frac{a}{k}\ln(\exp(\frac{k}{a}(1 - \frac{1}{\chi_k^2(s)})))}} \tag{200}$$

$$\leq \frac{\chi_k(s)}{\sqrt{1 - \frac{a}{k}\chi_k(s)^2\ln\left(\frac{2k}{\mu}\right)}} \tag{201}$$

$$\leq \frac{1}{\sqrt{1 - \frac{a}{k}\ln\left(\frac{2k}{\mu}\right)}} \left(\frac{2k}{2k-\mu}\right)^{1/3} \gamma(s) . \tag{202}$$

Combining (194), (197) and (202), we see that, for any strictly positive real number ε, there exists some integer K such that, for any $k \geq K$,

$$\gamma_{ck} \circ \frac{1}{\mu}\gamma_{vk}(s) \;\leq\; \begin{cases} (1+\varepsilon)\gamma(s) \;, & \text{if} \quad \gamma(s) \leq 1 \;, \\ (1+\varepsilon)\,(\gamma(s))^{1+\varepsilon} \;, & \text{if} \quad \gamma(s) > 1 \;. \end{cases} \tag{203}$$

With (203), we conclude that, for the system (182), see that, by working with the exp-ISS gain function instead of the ISS gain function, we can get results which are as equally conservative as we want on any compact set.

8 Conclusion

Consider the system :

$$\begin{cases} \dot{x} &= f(x) \;+\; \displaystyle\sum_{i=1}^{p} g_i(x)\,[u_i + c_i(x,z,u)] \;, \\ \dot{z} &= a(x,z,u)\,. \end{cases} \tag{204}$$

Under the following conditions

– the system :

$$\dot{x} \;=\; f(x) \;+\; \sum_{i=1}^{p} g_i(x)\,u_i \tag{205}$$

is globally asymptotically stabilizable by a feedback law $(u_{ni}(x))$ (see A1),

– the system :

$$\begin{cases} \dot{z} &= a(x,z,u) \\ y_i &= c_i(x,z,u) \end{cases} \tag{206}$$

has appropriate input-to-state and input-to-output properties (see A2 or A2'),

we have shown how to modify the feedback u_n into a static or a dynamic feedback in order to guarantee that all the solutions of (204) are bounded and their x-components are captured by an arbitrarily small neighborhood of the origin. This result belongs to the broad class of results known on uncertain systems (see [C] for a survey and [Q1, Q2, KSK, KK, JMP] for some recent developments).

The modifications we have proposed for the control law u_n are based on Lyapunov design and gain assignment techniques as introduced in [JTP]. The analysis of the properties of the closed loop system is based on the application of the Small-Gain Theorem [JTP, Theorem 2.1]. The assumptions on the z-subsystem are written in terms of the notion of input-to-output stability (IOS) introduced in [JTP] which is an extension of the notion of input-to-state stability (ISS) as introduced by Sontag in [S2].

To carry out our design of a dynamic feedback, we have been led to introduce a new notion of ISS systems called exp-ISS. We have shown that for finite dimensional systems the two notions are equivalent. For this we have used the link between the ISS property and the existence of an appropriate Lyapunov function which has been established in [LSW, SW].

An important feature of the system (204) is that the unmodelled effects are in the "range" of the input. This is the well known matching assumption. By using arguments similar to those used for propagating the ISS property through integrators in [JTP, Corollary 2.3], this matching assumption can be relaxed for systems and uncertainties having a recurrent so called feedback structure.

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Appendices

A On the non existence of a stabilizing feedback for (13).

We prove that there is no control law $u(t)$ that can drive to zero the x-component of any solution of (13), starting from $(x_0, 1)$ with $x_0 > M \exp(1)$. To do this, let us assume that such a control exists. By the uniqueness property, the corresponding solution $(x(t), z(t))$ of (13) remains in $\mathbf{R}^2_{>0}$ for all positive time. Moreover, since

$$x(t) \;=\; \exp\left(\int_0^t (x(s) - u(s) + \gamma(z(s)))\, ds\right) x_0 \;, \qquad \forall\, t \geq 0, \tag{207}$$

we have, necessarily :

$$\lim_{t\to\infty} \int_0^t (u(s) - \gamma(z(s)) - x(s)) ds \;=\; +\infty \;. \tag{208}$$

On the other hand, we have :

$$\frac{dz}{z^2} = (u(t) - z(t))\, dt \;. \tag{209}$$

With (14) and the fact that $z(t)$ is strictly positive, (209) yields :

$$-\frac{1}{z(t)} + 1 = \int_0^t (u(s) - z(s)) ds \;\geq\; \int_0^t (u(s) - M - \gamma(z(s))) ds \,, \tag{210}$$

and :

$$\frac{1}{z(t)} \;\leq\; 1 \;-\; \int_0^t (u(s) - \gamma(z(s)) - x(s)) ds \;-\; \int_0^t x(s) ds \;+\; M\, t \;. \tag{211}$$

Now let us define :

$$t_1 = \inf\left\{ t \geq 0 : \;\int_0^t (u(s) - \gamma(z(s)) - x(s)) ds \geq 1 \right\} \;. \tag{212}$$

This real number is well defined (see (208)) and is positive. By continuity, we get :

$$x(t_1) \;=\; x_0 \exp(-1) \;, \tag{213}$$

$$\begin{aligned} \frac{1}{z(t_1)} \;&\leq\; 1 \;-\; \int_0^{t_1} (u(s) - \gamma(z(s)) - x(s)) ds \;-\; \int_0^{t_1} x(s) ds \;+\; M\, t_1 \\ &\leq\; -\frac{x_0\, t_1}{\exp(1)} \;+\; M\, t_1 \;. \end{aligned} \tag{214}$$

By the choice of x_0, this yields that $\frac{1}{z(t_1)} < 0$ which contradicts the fact that $z(t) > 0$ for all positive t.

B An explicit expression for θ_i.

Working within the context of the proof of Lemma 3, we propose here an explicit expression for the function θ_i.

First of all, we define a function $\hat{\theta}_i$ as :

$$\hat{\theta}_i(x) \;=\; \begin{cases} \dfrac{\sqrt{\left(\frac{W(x)}{2p}\right)^2 + 3\left(|L_{g_i}V(x)|\,\varphi_i(x)\right)^2} - \frac{W(x)}{2p}}{|L_{g_i}V(x)|\,\varphi_i(x)} \;, & \text{if } \; |L_{g_i}V(x)| \neq 0 \;, \\ 0 \;, & \text{if } \; |L_{g_i}V(x)| = 0 \;, \end{cases} \tag{215}$$

where :

$$\varphi_i(x) = \mathcal{S}(V(x)) + |u_{n_i}(x)| \;. \tag{216}$$

According to the arguments in the proof of [S1, Theorem 1], this function is continuous on $\mathbb{R}^n \setminus \{0\}$. Moreover, we have :

$$x \in \mathcal{B}_{1i} \quad \Longrightarrow \quad \left(\frac{W(x)}{2p}\right)^2 + 3\left(|L_{g_i}V(x)|\,\varphi_i(x)\right)^2 \;\geq\; \left(\frac{W(x)}{2p} \;+\; |L_{g_i}V(x)|\,\varphi(x)\right)^2 \;. \tag{217}$$

It follows that $\hat{\theta}_i(x)$ is larger than 1 on $\mathcal{B}_{1i}$. This allows us to define θ_i on $\mathbb{R}^n \backslash \{0\}$ as :

$$\theta_i(x) \;=\; \operatorname{sat}\!\left(\hat{\theta}_i(x)\right) \;, \tag{218}$$

where sat : $\mathbb{R}_{\geq 0} \to [0,1]$ is the saturation function :

$$\operatorname{sat}(r) \;=\; \begin{cases} r \;, & \text{if } \; r \in [0,1] \;, \\ 1 \;, & \text{if } \; r > 1 \;. \end{cases} \tag{219}$$