Generalized Input/Output Equations and Nonlinear Realizability*

Yuan Wang

Mathematics Department, Florida Atlantic University, Boca Raton, FL 33431

(407)367-3317, E-mail: ywang@control.math.fau.edu

Abstract

This work studies various types of input/output representations for analytic input/output operators. It is shown that if an operator satisfies an integro-differential input/output equation or an integral one, then it is locally realizable by analytic state space systems. This generalizes the results previously obtained for differential input/output equations to integral and integro-differential equations.

1 Introduction

In the previous work (Wang & Sontag 1992a, Wang & Sontag 1992b), it was shown that if an input/output operator satisfies a possibly high-order differential input/output equation, then it is locally realizable by an analytic state space system (a set of controlled first order equations). Besides its intrinsic mathematical interest, this result is relevant in identification theory, since high-order equations involving only observed variables can be in principle determined from experimental data.

In practice, however, the signals of physical systems are often affected by noise, which

*This research was supported in part by NSF Grants DMS-9108250 and DMS-9403924

Keywords: Generating series, local realization of input/output operators, input/output equations.

makes numerical identification, especially under high frequency noise, a very difficult task for differential input/output equations. It is in principle better to work with integrals of the data instead. Indeed, in linear systems identification practice, it is often the case that one prefilters measured signals prior to estimation, so as to eliminate noise. Various filtering procedures have been suggested in the nonlinear case as well. For instance, in the paper (Pearson 1988) and other related work, the author discussed the "projected variable filter" method. To avoid taking derivatives of the input/output signals, Pearson (1988) proposed to base the parameter identification problem on the use of equations of the following type:

$$\sum_{i=1}^{k} h_i(\theta) w_i(t) = 0, \tag{1}$$

where θ stands for the system parameters to be identified, each $w_i(t)$ is an iterated integral of a function of (u, y), and the h_i's are functions (which have certain special properties discussed in the paper). This method and others like it lead to a natural and nontrivial question : once that the value of θ is estimated, can one find a state space realization for (1)? This motivates us to study the relations between state space realizability and the existence of integral equations, or the existence of even more general types of integro-differential equations, for i/o operators.

We start the paper by introducing generalized rational i/o equations for i/o operators. Roughly speaking, an operator satisfies a generalized rational i/o equation if the k-th derivative of the output, for some k, can be expressed rationally in terms of lower-order derivatives as well as integrals of inputs and outputs. The main result of the paper says that if an operator admits such an equation, then it is locally realizable by an analytic state space system. In particular, if an operator satisfies an integral i/o equation, then it is locally realizable. This generalizes the results in (Wang & Sontag 1992*a*, Wang & Sontag 1992*b*) for differential i/o equations to integral, or integro-differential equations. In Section 4, we focus on integral equations, and the same result is established for analytic integral i/o equations. It is also indicated that in general the existence of differential i/o equations and the existence of integral ones are not equivalent, and examples are provided to illustrate this fact.

The formalism of this work is based on the generating series suggested by Fliess. The operators induced by generating series form a very general class of causal operators. For definitions and properties of such operators, we refer the readers to (Fliess 1983, Fliess & C.Reutenauer 1982, Isidori 1989, Wang & Sontag 1992*a*, Wang & Sontag 1992*b*), see also the appendix. An important property of such an operator is that for any analytic input function, the output function is also analytic. For general terminology about control systems, see (Sontag 1990).

2 Generalized Rational Input/Output Equations

In this section we introduce the so called generalized rational input/output equations for i/o operators.

Let $m, T > 0$ be given. Consider analytic i/o operators defined on $\mathcal{V}_T$, the set of measurable functions $u : [0,\, T] \to \mathrm{I\!R}^m$ with $\|u\|_\infty \leq 1$ (see (Wang & Sontag 1992*a*) or the Appendix for the detailed definitions of such i/o operators and related notions). To define generalized rational i/o equations for such i/o operators, we consider the set of operators mapping $\mathcal{C}^\omega[0,\, T] \times (\mathcal{C}^\omega[0,\, T])^m$ to $\mathcal{C}^\omega[0,\, T]$, where $\mathcal{C}^\omega[0, T]$ denotes the set of analytic functions defined on $[0,\, T]$. We define "+" and "·" for such operators by the usual pointwise operations, i.e., for any two such operators $P_1,\ P_2$, we have:

$$(P_1 + P_2)\,(\varphi,\, \psi)(t) = P_1(\varphi,\, \psi)(t) + P_2(\varphi,\, \psi)(t),$$

$$(P_1 \cdot P_2)\,(\varphi,\, \psi)(t) = P_1(\varphi,\, \psi)(t) \cdot P_2(\varphi,\, \psi)(t),$$

for all $\varphi \in \mathcal{C}^\omega[0, T]$, $\psi \in (\mathcal{C}^\omega[0,T])^m$. The integral $\int P$ of an operator P is defined in the following way:

$$\left(\int P\right)(\varphi,\, \psi)(t) = \int_0^t P(\varphi,\, \psi)(s)\, ds, \;\; \text{for } 0 \le t \le T.$$

In this work, we are interested in a special class $\mathcal{P}$ of *differential integral operators*(DIO's). To define $\mathcal{P}$, we first need to define $\mathcal{P}_k$ for each fixed nonnegative integer k. For a given k, $\mathcal{P}_k$ is defined to be the smallest set of DIO's satisfying the following properties:

a. the derivation operators

$$\alpha_i : (\varphi,\, \psi) \mapsto \varphi^{(i)} \qquad \text{and} \qquad \beta_{ij} : (\varphi,\; (\psi_1,\, \psi_2\, \ldots, \psi_m)) \mapsto \psi_j^{(i)}$$

are in $\mathcal{P}_k$, for each $0 \le i \le k$ and $1 \le j \le m$;

b. $\int P \in \mathcal{P}_k$ if $P \in \mathcal{P}_k$,

c. $P_1 + P_2,\; P_1 \cdot P_2 \in P_k$ if $P_1, P_2 \in \mathcal{P}_k$, and $cP \in P_k$ for any $c \in \mathbb{R}$ if $P \in P_k$,

d. the "constant" DIO : $(\varphi, \psi) \mapsto 1\; \in \mathcal{P}_k$.

Note that $\mathcal{P}_k$ is in fact the smallest $\mathbb{R}$-algebra containing

$$\alpha_0,\, \alpha_1,\, \ldots, \alpha_k,\, \beta_0,\, \beta_2,\, \ldots,\, \beta_k$$

that satisfies property b., where $\beta_i = (\beta_{i1}, \ldots, \beta_{im})$ for each i. For instance, when $m = 1$, the operator $P(\varphi, \psi) = \int_0^t \int_0^s \varphi(s)\psi(\tau)\, d\tau\, ds \in \mathcal{P}_0$, and $Q(\varphi, \psi) = \varphi'''(t) \cdot \int_0^t \psi(s)\, ds\; \in \mathcal{P}_3$.

We define $\mathcal{P}$ to be the union of all $\mathcal{P}_k$'s, that is,

$$\mathcal{P} \;=\; \bigcup_{k=0}^{\infty} \mathcal{P}_k\,.$$

Let F be an analytic i/o operator defined on $\mathcal{V}_T$. We say that the operator F satisfies a *generalized rational i/o equation* if there exist some integer k, some $T > 0$, and two operators $P_0, P_1 \in \mathcal{P}_{k-1}$, such that for every analytic i/o pair of $(u,\, y)$ of F it holds that

$$P_0(u,\, y)(t) \cdot y^{(k)}(t) = P_1(u,\, y)(t)\,, \tag{2}$$

for all $t \in [0, T]$, and moreover, the following nondegeneracy property holds:

$$P_0(u,\, y) \neq 0$$

for some analytic i/o pair $(u,\, y)$ of F.

Roughly speaking, the i/o operator F satisfies an equation of this type if the k-th derivative of the output, for some k, can be expressed rationally in terms of lower-order derivatives as well as integrals of inputs and outputs.

Remark 2.1 The i/o operators defined by generating series are (strictly) causal, that is, for any i/o pair (u, y) of any i/o operator F, $y(t)$ depends only on the restriction of u on $[0,\, t)$ (see the appendix for more detailed discussions). Consequently, such an operator can never admit an i/o equation where the highest order of the derivatives of the input signals is higher than or equal to the highest order of the derivatives of the output signals such as $y'(t) = u'(t)$. $\square$

Remark 2.2 Generalized rational i/o equations form a very general class of i/o equations. Some special cases are as follows:

1. *Algebraic differential equations:* An i/o operator F satisfies an algebraic differential i/o

equation if the following holds for all $\mathcal{C}^\omega$ i/o pairs (u, y) of F:

$$E(u(t), u'(t), \ldots, u^{(k)}(t), y(t), y'(t), \ldots, y^{(k)}(t)) = 0, \tag{3}$$

for all $t \in [0,\, T]$, where E is a polynomial. If this is the case, then F also satisfies a rational i/o equation of the following type:

$$\begin{aligned} & E_0\left(u, u', \ldots, u^{(k)}, y, y', \ldots, y^{(k)}\right)\, y^{(k+1)} \\ & \qquad = E_1(u, u', \ldots, u^{(k)}, y, y', \ldots, y^{(k)})\,. \end{aligned}$$

which is a generalized rational i/o equation (note that $u^{(k+1)}$ does not appear in E_1 because of the causality property of F, see (Wang & Sontag 1992a, Lemma 4.4) for a detailed construction of E_0 and E_1).

2. *Algebraic integral equations:* An operator F satisfies an algebraic integral i/o equation if the following holds for all $\mathcal{C}^\omega$ i/o pairs (u, y) of F:

$$E\left(u(t),\, U_1(t),\, \ldots, U_k(t),\, y(t),\, y_1(t),\, \ldots,\, y_k(t)\right) = 0\,, \tag{4}$$

for all $t \in [0,\, T]$, where E is a polynomial, and

$$U_1(t) \;=\; \int_0^t u(s)\, ds, \quad U_i(t) = \int_0^t U_{i-1}(s)\, ds, \quad i \geq 2\,, \tag{5}$$

$$y_1(t) \;=\; \int_0^t y(s)\, ds, \quad y_i(t) = \int_0^t y_{i-1}(s)\, ds, \quad i \geq 2\,. \tag{6}$$

To show that F also admits a generalized rational i/o equation, we first assume that $\dfrac{\partial E}{\partial y}$ is a nonzero polynomial and $\deg_y E$ is as small as possible. Differentiating both sides of equation (4), we get

$$\frac{\partial E}{\partial y}\left(u, U_1\, \ldots, U_k,\, y,\, y_1,\, \ldots, y_k\right) y' = Q\left(u',\, u,\, U_1,\, \ldots,\, U_k,\; y,\; y_1, \ldots, y_k\right) . \tag{7}$$

Again, by causality, one can show that Q does not depend on u'. Thus, equation (7) is again a generalized rational i/o equation. If $\frac{\partial E}{\partial y} \equiv 0$, pick up the first r so that $\frac{\partial E}{\partial y_r} \not\equiv 0$. Differentiating enough times, we are in the previous situation.

3. More generally, given an equation $P(u,\, y) = 0$ with $P \in \mathcal{P}$, in practice one can often transform it into form (2) after repeated differentiation. ∎

3 Main Results

We say that an i/o operator F is *locally realizable* by an analytic system if there exist some analytic manifold $\mathcal{M}$, some $x_0 \in \mathcal{M}$, $(m+1)$ analytic vector fields

$$g_0,\, g_1,\, \ldots, g_m$$

on $\mathcal{M}$, an analytic function $h: \mathcal{M} \to \mathbb{R}$, and some $\tau > 0$ such that for each i/o pair (u, y) of F with $\|u\|_\infty < 1$, it holds that

$$y(t) = h(x(t))$$

for $0 \le t \le \tau$, where $x(\cdot)$ is the solution of the equation

$$x' \;=\; g_0(x) \,+\, \sum_{j=1}^{m} g_j(x)\, u_j \;,\quad x(0) \;=\; x_0 \,.$$

The following is the main theorem of this work.

Theorem 1 *If an operator F satisfies a generalized rational i/o equation, then it is locally realizable by an analytic system.*

3.1 Proof of Theorem 1

In order to study local realizability by analytic systems, we first introduce an auxiliary notion, weak realizability. An i/o operator F is said to be *weakly realizable* if there exist integers n and r, polynomial functions $q : \mathbb{R}^n \times \mathbb{R}^{m(r+1)} \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$, a polynomial vector function $f : \mathbb{R}^n \times \mathbb{R}^{m(r+1)} \to \mathbb{R}^n$, and a continuous function $\alpha : \mathbb{R}^{m(r+1)} \to \mathbb{R}^n$, such that

(a) for every analytic i/o pair (u, y) with $y = F[u]$, it holds that

$$y(t) = h(\varphi(t)) \quad \text{for all } t\,,$$

where $\varphi(\cdot)$ is an analytic solution of the differential equation

$$q(x, u, u', \ldots, u^{(r)})x' = f(x, u, u', \ldots, u^{(r)})\,; \tag{8}$$

with $\varphi(0) = \alpha(u(0), u'(0), \ldots, u^{(r)}(0))$;

(b) the following *regularity* condition holds: there exists some set Ω which is dense in $\mathcal{C}^\omega([0, T])$ (with respect to the Whitney topology) such that for any $u \in \mathcal{V}_T \cap \Omega^m$, there exists some $\mathcal{C}^\omega$ solution $\varphi(\cdot)$ as in (a) so that

$$q(\varphi(t), u(t), u'(t), \ldots, u^{(r)}(t)) \not\equiv 0\,. \tag{9}$$

By using the same argument used in the proof of Lemma 6.1 in (Wang & Sontag 1992*a*) which is based on elimination theory, one can show the following:

Lemma 3.1 If an i/o operator F is weakly realizable, then it satisfies an algebraic differential i/o equation, that is, there exist some $k \geq 0$ and some polynomial function E such that

$$E(u(t),\, u'(t),\, \ldots, u^{(k)}(t),\, y(t),\, y'(t),\, \ldots,\, y^{(k)}(t)) \,=\, 0$$

for all $t \in [0, T]$, for all C^ω i/o pairs of F.

In the following we give a detailed proof of the lemma.

Proof. Let F be an i/o operator defined on $\mathcal{V}_T$. Assume that an i/o operator F is weakly realizable by a system as in (8). We must prove that F satisfies some differential i/o equation

$$E(u(t),\, u'(t),\, \ldots,\, u^{(k)}(t),\, y(t),\, y'(t),\, \ldots,\, y^{(k)}(t)) \,=\, 0\,,$$

for some polynomial E.

First we let

$$\mathcal{W} := \left\{(x,\, \mu) \in \mathbb{R}^n \times \mathbb{R}^{m(r+1)} :\; q(x,\, \mu) \neq 0\right\}\,.$$

Take $x \in \mathbb{R}^n$, and assume that there exists some $\bar{\mu}$ such that $(x, \bar{\mu}) \in \mathcal{W}$. Then for any analytic input function u with $u^{(i)}(0) = \bar{\mu}_i$ $(i = 0,\, \ldots, r)$, there exists some $\delta > 0$ such that the solution $\varphi(t,\, x,\, u)$ of system (8) with the initial state x and the input u is uniquely defined on $[0, \delta)$, and

$$q\left(\varphi(t, x, u),\, u(t),\, u'(t),\, \ldots,\, u^{(r)}(t)\right) \neq 0$$

for any $t \in [0, \delta)$. Let $y_x(t) = h(\varphi(t, x, u))$ for such x and u. Then

$$y_x(0),\, y_x'(0),\, \ldots, y_x^{(n)}(0) \tag{10}$$

are rational functions of x over the field K, the field obtained by adjoining μ_i ($i = 0, 1, \ldots, r + n - 1$) to $\mathbb{R}$. Since the transcendence degree of $K(x)$ over K is n, the $(n+1)$ functions in (10) are algebraically dependent over K; i.e., there exists some nontrivial polynomial L over K such that

$$L(y_x(0), y_x'(0), \ldots, y_x^{(n)}(0)) = 0 .$$

Clearing the denominators in the coefficients of L (which are rational functions in μ), one obtains a polynomial E such that

$$E(\mu_0, \mu_1, \ldots, \mu_{r+n-1}, y_x(0), y_x'(0), \ldots, y_x^{(n)}(0)) = 0$$

holds for all $(x, \mu) \in \mathcal{W}$. Observe that E is a nontrivial polynomial since L is nontrivial.

Take $(x, \mu) \in \mathcal{W}$. Let u be an analytic input function with $u^{(i)}(0) = \mu_i$, $0 \le i \le r$. Then there exists some $\delta > 0$ such that

$$\left(\varphi(t, x, u(t)), u(t), u'(t), \ldots, u^{(r)}(t)\right) \in \mathcal{W}, \quad \forall t \in [0, \delta) .$$

For any $t_1 \in [0, \delta)$, let $x_1 = \varphi(t_1, x, u)$, and let

$$\nu = \left(u(t_1), u'(t_1), \ldots, u^{(n+r-1)}(t_1)\right) .$$

Then $(x_1, \nu_0, \ldots, \nu_r) \in \mathcal{W}$, and hence,

$$E\left(\nu_0, \nu_1, \ldots, \nu_{r+n-1}, y_{x_1}(0), y_{x_1}'(0), \ldots, y_{x_1}^{(n)}(0)\right) = 0 .$$

This is exactly:

$$E\left(u(t_1),\, u'(t_1),\, \ldots,\, u^{(r+n-1)}(t_1),\, y_x(t_1),\, y_x'(t_1),\, \ldots,\, y_x^{(n)}(t_1)\right) \;=\; 0\,.$$

This shows that if $(x,\,\mu) \in \mathcal{W}$, then for any analytic u with $u^{(i)}(0) = \mu_i,\ 0 \le i \le r$, there exists some $\delta > 0$ such that

$$E\left(u(t),\, u'(t),\, \ldots, u^{(n+r-1)}(t),\, h(\varphi(t,\,x,\,u)),\, \frac{d}{dt}h(\varphi(t,\,x,\,u)),\, \ldots,\, \frac{d^n}{dt^n}h(\varphi(t,\,x,\,u))\right) \;=\; 0$$

for all $0 \le t < \delta$. In particular, if μ is such that $q(\alpha(\mu),\,\mu) \neq 0$, then for any analytic u with $u^{(i)}(0) = \mu_i$, there exists some $\delta > 0$, such that

$$E(u(t),\, \ldots,\, u^{(r+n-1)}(t),\, y(t),\, \ldots,\, y^{(n)}(t)) \;=\; 0 \tag{11}$$

for all $t \in [0, \delta)$. By analyticity, it follows that (11) holds for all $t \in [0, T]$.

Finally, we show how to overcome the restriction $q(\alpha(\mu),\,\mu) \neq 0$. By the regularity condition of (8), one knows that there exists a dense set Ω of analytic inputs, so that for each $u \in \mathcal{V}_T \cap \Omega^m$, there exists some $\mathcal{C}^\omega$ solution φ such that (9) holds. It then follows from analyticity that for every such u, there exists some $\tau > 0$ such that

$$q(\varphi(t),\, u(t), \ldots, u^{(r)}(t)) \neq 0\,,$$

for $t \in (0,\,\tau)$. According to the previous argument, we see that (u, y) satisfies (11) for $t \in (0, \tau)$. Again, using analyticity, one shows that (u, y) satisfies (11) for all $t \in [0,\,T]$.

Since Ω is dense in $\mathcal{C}^\omega$ controls with respect to the Whitney topology, it follows from the continuity of the operator F that (11) holds for all $\mathcal{C}^\omega$ i/o pairs of F, see (Wang & Sontag 1992*b*).

Finally, by the causality of the operator F, one sees that (11) yields an i/o equation of the following type:

$$E_1(u(t),\, u'(t),\, \ldots,\, u^{(n-1)}(t),\, y(t),\, y'(t),\, \ldots,\, y^{(n)}(t)) \;=\; 0\,,$$

where E_1 is some polynomial. (See (Wang 1990) for the details on the existence of such E_1). ∎

Lemma 3.1 says that if F is weakly realizable, then F satisfies a differential i/o equation in the usual sense, cf. (Wang & Sontag 1992*a*). By (Wang & Sontag 1992*b*, Theorem 3), F is locally realizable by an analytic system. Hence, we get the following:

Corollary 3.2 If F is weakly realizable, then F is locally realizable by an analytic state space system. ∎

The following Lemma connects the existence of generalized rational i/o equations to weak realizability.

Lemma 3.3 If an operator F satisfies a generalized rational i/o equation, then F is weakly realizable.

Proof. We shall say that a DIO P is a *simple monomial of (iterated) integrals* if P has the following form:

$$P(u,\, y)(t) = \int_0^t \xi_1(s_1) \int_0^{s_1} \xi_2(s_2) \cdots \int_0^{s_{r-1}} \xi_r(s_r)\, ds,$$

where each ξ_i is a monomial in $u, u', \ldots, y, y', \ldots$ (possibly a constant). Assume now that F satisfies a generalized rational i/o equation. Then there are polynomial functions E_0 and E_1

such that the following holds for every i/o pair of F:

$$
\begin{aligned}
&E_0\left(u(t),\, u'(t),\, \ldots,\, u^{(k-1)}(t),\, y(t),\, \ldots,\, y^{(k-1)}(t),\, Q_1(u,\, y)(t),\, \ldots,\, Q_\sigma(u,\, y)(t)\right) y^{(k)}(t) \\
&\qquad = E_1\left(u(t),\, u'(t),\, \ldots,\, u^{(k-1)}(t),\, y(t),\, \ldots,\, y^{(k-1)}(t),\, Q_1(u,\, y)(t),\, \ldots,\, Q_\sigma(u,\, y)(t)\right)\,,
\end{aligned}
$$

where $Q_1,\, \ldots,\, Q_\sigma \in \mathcal{P}_{k-1}$ are simple monomials of integrals, and there exists some analytic i/o pair $(u,\, y)$ of F such that

$$
E_0\left(u(t),\, u'(t),\, \ldots,\, u^{(k-1)}(t),\, y(t),\, \ldots,\, y^{(k-1)}(t),\, Q_1(u,\, y)(t),\, \ldots,\, Q_\sigma(u,\, y)(t)\right) \not\equiv 0\,. \tag{12}
$$

Assume that

$$
Q_i(u,\, y)(t) \;=\; \int_0^t \xi_{i1}(s_1) \int_0^{s_1} \xi_{i2}(s_2) \cdots \int_0^{s_{r_i-1}} \xi_{ir_i}(s_{r_i})\, ds\,,
$$

for $i = 1, \ldots, \sigma$. Note then that each ξ_{ij} is a monomial in $u^{(p)}$ and $y^{(l)}$ with $0 \le p,\, l \le k-1$. Formally denoting $x_{i1} = Q_i$, $x_{i2} = \int_0^t \xi_{i2}(s_2) \cdots \int_0^{s_{r_i-1}} \xi_{ir_i}(s_{r_i})\, ds$ and so forth, then one has the following equations:

$$
x'_{i1} \;=\; \xi_{i1} x_{i2},\; x'_{i2} \;=\; \xi_{i2} x_{i3}\,,\; \ldots,\; x'_{ir_i} \;=\; \xi_{ir_i}\,, \tag{13}
$$

with $x_{il}(0) = 0$ for all l. Let $x_i = (x_{i1},\, x_{i2},\, \ldots, x_{ir_i})$. Then it follows from the fact that $Q_i \in \mathcal{P}_{k-1}$ that Eq. (13) can be written as:

$$
x'_i \;=\; f_i(x_i,\, u,\, u',\, \ldots, u^{(k-1)},\, y,\, y',\, \ldots,\, y^{(k-1)})\,, \tag{14}
$$

for some polynomial vector function f_i. Now let $z_i = y^{(i)}$ for $0 \le i \le k-1$, and let $x = (x_1, x_2, \ldots x_r)$. Then Eq. (12) implies the following equations:

$$\begin{aligned} x' &= f(x, u, u', \ldots, u^{(k-1)}, z)\,, \\ z_0' &= z_1,\; z_1' = z_2,\; \ldots,\; z_{k-2}' = z_{k-1}, \\ Q_0(x, z, u, u', \ldots, u^{(k-1)})z_{k-1}' &= Q_1(x, z, u, u', \ldots, u^{(k-1)})\,, \\ x(0) &= 0, \quad z(0) = (y(0), y'(0), \ldots, y^{(k-1)}(0))\,, \end{aligned} \tag{15}$$

where $Q_0,\ Q_1$ are given by

$$Q_i(x, z, u, u', \ldots, u^{(k-1)}) = E_i(u, u', \ldots, u^{(k-1)}, z, x)\,, \qquad i = 1, 2\,.$$

Since $z_i(0) = y^{(i)}(0)$ for each i, it follows that the mapping $(u(0), u'(0), \ldots, u^{(i)}(0)) \mapsto z_i(0)$ is a polynomial function (cf (31) in the Appendix), thus, the mapping

$$\alpha : (u(0), u'(0), \ldots, u^{(i)}(0)) \mapsto (x(0), z(0))$$

is continuous.

By the nondegenracy condition (12), one knows that there exists at least one analytic ω such that

$$Q_0(x(t), z(t), \omega(t), \omega(t)', \ldots, \omega^{(k-1)}(t)) \not\equiv 0\,. \tag{16}$$

Using the same argument as in the proof of (Wang & Sontag 1992a, Theorem 5.2), one can show that there exists some open dense set Ω in $\mathcal{C}^\omega$ such that (16) holds for every $u \in \mathcal{V}_T \subset \Omega^m$. Hence, (15) is a weak realization of F. ∎

Finally, Theorem 1 follows from Corollary 3.2 and Lemma 3.3.

4 Integral I/O Equations

As we indicated earlier in Remark 2.2, if an operator F satisfies an algebraic integral i/o equation of type (4), then it satisfies a generalized rational i/o equation. Thus we conclude, in particular:

Corollary 4.1 If an operator F satisfies an algebraic integral i/o equation, then F is locally realizable. ∎

In contrast to the case of differential i/o equations (in which existence of an algebraic *differential* i/o equation is equivalent to realizability by a "rational" system, cf. (Wang & Sontag 1992*a*)), it is generally not true that every operator which is realizable by a polynomial system satisfies an algebraic *integral* i/o equation. As an illustration of tis matter of fact, consider the following example.

Example 4.2 Let F be the operator given by

$$y(t) = F[u](t) = \exp\left(\int_0^t u(s)\,ds\right). \tag{17}$$

To be more precise, the series that defines the operator is given by

$$c = 1 + \eta_1 + \eta_1\eta_1 + \eta_1\eta_1\eta_1 + \cdots .$$

The operator F is realized by the system:

$$x' = xu, \quad y = x, \quad x_0 = 1. \tag{18}$$

Assume that F satisfies an integral i/o equation, which we write as

$$P(u(t),\, U_1(t),\, \ldots,\, U_r(t),\, y(t),\, \ldots,\, y_r(t)) \;=\; 0\,. \tag{19}$$

By the causality of the operator, one sees that P can be chosen independently of u. Thus one may assume that Eq. (19) is of the following type:

$$P(U_1(t),\, \ldots,\, U_r(t),\, y(t),\, \ldots,\, y_r(t)) \;=\; 0\,. \tag{20}$$

For each $i = 1, 2, \ldots, r$, let $x_i = y_{r-i+1}$ and $z_i = U_{r-i+1}$. Then x, z satisfy the following equations:

$$\begin{aligned}
&x_1' = x_2,\ \ldots,\ x_{r-1}' = x_r,\ x_r' = \exp(z_r),\\
&z_1' = z_2,\ \ldots,\ z_{r-1}' = z_r,\ z_r' = u\\
&x(0) = 0,\ z(0) = 0.
\end{aligned} \tag{21}$$

A straightforward computation shows that system (21) satisfies the accessibility rank condition at 0, and therefore, the system is accessible in a neighborhood of 0, (see e.g. (Sussmann 1985). This means that there exist open sets $\mathcal{W}_1$ and $\mathcal{W}_2$ in $\mathbb{R}^r$ such that, for any point $(\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_r) \in \mathcal{W}_1 \times \mathcal{W}_2$, there is some control ω and some $t > 0$ so that, for the solution of system (21) corresponding to the control ω, we have

$$(x_1(t),\ \ldots,\ x_r(t),\ z_1(t),\ \ldots,\ z_r(t)) = (\nu_1,\ \ldots,\ \nu_r,\ \mu_1,\ \ldots,\ \mu_r),$$

which, together with (20), implies that

$$P(\mu_1, \ldots, \mu_r, \exp(\mu_1), \nu_1, \ldots, \nu_r) = 0 \tag{22}$$

for any $(\nu, \mu) \in \mathcal{W}_1 \times \mathcal{W}_2$. This is impossible since μ_1 and $\exp(\mu_1)$ are algebraically independent over $\mathbb{R}$. Thus no equation like (19) exists. □

Note that even though F does not admit any integral equation, F does satisfy the *differential* i/o equation

$$y' = yu.$$

This illustrates the fact that the existence of differential equations and the existence of integral equations are not equivalent, unless much stronger conditions (such as linearity of the equations, as for linear systems) are assumed.

4.1 Analytic Integral I/O Equations

In (Wang & Sontag 1992*b*), it was shown that if an operator satisfies an analytic differential i/o equation, then it is locally realizable by an analytic state space system. In this section, we will study the analogue of this result for integral i/o equations.

Definition 4.3 We say that an operator F satisfies an analytic integral equation if there exist some integer $k > 0$, some $T > 0$, and a nontrivial analytic function A defined on $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1}$ such that for each analytic i/o pair (u, y), it holds that

$$A(u(t), U_1(t), \ldots, U_k(t), y(t), y_1(t), \ldots, y_k(t)) = 0 \tag{23}$$

for all $t \in [0, T]$. □

In the previous example, though the operator F defined by (17) does not satisfy any algebraic integral i/o equation, it does satisfy an analytic integral equation:

$$y - \exp\left(\int_0^t u(s)\,ds\right) = 0\,.$$

The following is a generalization of Corollary 4.1 to the analytic equations.

Proposition 4.4 If an operator F satisfies an analytic integral i/o equation, then it is locally realizable by an analytic state space system.

As in the case of differential i/o equations, to prove the result, one needs to employ some basic properties of observation spaces and observation fields associated with F, cf. (Wang & Sontag 1989, Wang & Sontag 1992a, Wang & Sontag 1992b). For the convenience of reference, we will put the needed materials in the Appendix.

In the following, we will use F_c to denote an i/o operator defined by the generating series c.

Assume that an operator F_c satisfies an analytic integral i/o equation (23). For each $j = 1, 2, \ldots, m$ and each $i \geq 1$, we let d_{ij} be the monomial defined by $\eta_0^{i-1}\eta_j$. Observe that each d_{ij} defines an operator:

$$F_{d_{ij}}[u](t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} U_j(s_i)\,ds\,.$$

We use d_i (F_{d_i}, respectively) to denote $(d_{i1},\, d_{i2},\, \ldots, d_{im})$ $((F_{d_{i1}}, \ldots, F_{d_{im}})$, respectively).

Also, for each i, let $e_i = \eta_0^i c$, then e_i is such a series that the operator F_{d_i} defined by d_i is given by

$$F_{e_i}[u](t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} F_c[u](s_i)\,ds\,.$$

Hence, Eq. (23) is equivalent to

$$A\left(u(t),\, F_{d_1}[u](t),\, \ldots,\, F_{d_k}[u](t),\, F_c[u](t),\, F_{e_1}[u](t),\, \ldots, F_{e_k}[u](t)\right) \;=\; 0\,.$$

We now define the *generalized observation space* $\tilde{\mathcal{F}}_1(c)$ by:

$$\tilde{\mathcal{F}}_1(c) \;=\; \mathrm{span}_{\mathrm{I\!R}}\Big\{\{d_1,\, d_2,\, \ldots, d_k,\, e_1,\, e_2,\, \ldots, e_k\}\bigcup\{d:\; d \in \mathcal{F}_1(c)\}\Big\}\,.$$

(See the Appendix for the definition of $\mathcal{F}_1(c)$.) The generalized observation algebra $\tilde{\mathcal{A}}_1(c)$ associated with c is the $\mathrm{I\!R}$-algebra generated by the elements of $\mathcal{F}_1(c)$, under the shuffle product (see (Wang & Sontag 1992*a*)), and the generalized observation field $\tilde{\mathcal{Q}}_1(c)$ is the quotient field of $\tilde{\mathcal{A}}_1(c)$.

Corresponding to the second type of observation space $\mathcal{F}_2(c)$ studied in (Wang & Sontag 1992*a*), we define the second type of generalized observation space $\tilde{\mathcal{F}}_2$ by:

$$\tilde{\mathcal{F}}_2(c) \;=\; \mathrm{span}_{\mathrm{I\!R}}\Big\{\{d_1,\, d_2,\, \ldots, d_k,\, e_1,\, e_2,\, \ldots, e_k\}\bigcup\{d:\; d \in \mathcal{F}_2(c)\}\Big\}\,.$$

(See again the Appendix for the definition of $\mathcal{F}_2(c)$.) The generalized observation algebra $\tilde{\mathcal{A}}_2(c)$ is the algebra generated by the elements of $\tilde{\mathcal{F}}_2(c)$, and the generalized observation field $\mathcal{Q}_2(c)$ is the quotient field of $\tilde{\mathcal{A}}_2(c)$. See also (Wang & Sontag 1992*b*), or the Appendix, for the meaning of meromorphically finitely generated field extension.

Following exactly the same steps of the proof of Theorem 2 in (Wang & Sontag 1992*b*), one proves the following:

Lemma 4.5 If an operator F_c satisfies an analytic integral i/o equation, then the field $\mathcal{Q}_2(c)$ is meromophically finitely generated. $\square$

We now return to the proof of Proposition 4.4.

Proof of Proposition 4.4: According to (Wang & Sontag 1989, Theorem 1) (see also the Appendix), one knows that $\tilde{\mathcal{F}}_1(c) = \tilde{\mathcal{F}}_2(c)$, hence, $\tilde{\mathcal{Q}}_1(c) = \tilde{\mathcal{Q}}_2(c)$. This implies that if F_c satisfies an analytic integral i/o equation, then $\tilde{\mathcal{Q}}_1(c)$ is meromorphically finitely generated. Without loss of generality, one may assume that the following are generators of $\tilde{\mathcal{Q}}_1(c)$:

$$d_1,\, d_2,\, \ldots,\, d_k,\, e_1,\, e_2,\, \ldots, e_k,\, c_1,\, c_2,\, \ldots,\, c_n,$$

where $c_1,\, c_2,\, \ldots,\, c_n \in \mathcal{F}_1(c)$ and $c_1 = c$. We now let $z_i(t) = F_{d_{k-i+1}}[u](t)$, $w_i = F_{e_{k-i+1}}[u](t)$, $x_i(t) \,=\, F_{c_i}[u](t)$. It is clear that

$$z_i'(t) \;=\; z_{i+1}(t), \qquad \text{for } 1 \le i \le k-1, \quad z_k'(t) \;=\; u(t)\,, \qquad z(0) = 0\,, \tag{24}$$

$$w_i'(t) \;=\; w_{i+1}(t), \quad \text{for } 1 \le i \le k-1, \quad w_k'(t) \;=\; x_0(t)\,, \quad w(0) = 0. \tag{25}$$

Following the same steps as in the proof of Theorem 1 in (Wang & Sontag 1992*b*), one sees that there exists some analytic vector functions $g_0,\, g_1,\, \ldots, g_m$, and an analytic function q satisfying

$$q(x(t),\, z(t),\, w(t)) \;\neq\; 0$$

for some analytic control $v(\cdot)$ so that the following holds for each input function u:

$$q(x(t),\, z(t),\, w(t))x'(t) \;=\; g_0(x(t), z(t), w(t)) \;+\; \sum_{i=1}^{m} g_i(x(t), z(t), w(t))u_i(t)\,, \tag{26}$$

and $x(0) = x_0$ for some $x_0 \in \mathbb{R}^n$. Let $\hat{x} = (x,\, w,\, z)$. Combining (24), (25) and (26), one gets:

$$
\begin{aligned}
q(\hat{x}(t))\hat{x}'(t) &= G_0\left(\hat{x}(t)\right) + \sum_{i=1}^{m} G_i\left(\hat{x}(t)\right) u_i(t) \\
\hat{x}(0) &= (x_0,\, 0,\, 0) \\
y(t) &= F_c[u](t) \,=\, \hat{x}_1(t)\,,
\end{aligned}
$$

for some analytic vector fields $G_0, G_1, \ldots, G_m$. Then the the proof of (Wang & Sontag 1992*b*, Theorem 1(b)) shows that the operator F_c is locally realizable by an analytic state space system. ∎

4.2 Integral I/O Equations vs. Differential I/O Equations

According to Lemmas 3.1 and 3.3, if an operator F satisfies an algebraic integral i/o equation, then it also satisfies a differential i/o equation, and Example 4.2 shows that an operator may fail to satisfy any algebraic integral i/o equation though it admits an algebraic differential i/o equation. However, in the analytic case, an operator could satisfy an analytic integral i/o equation even if it does *not* satisfy any analytic differential i/o equation. The following example is based on a construction from W. Respondek.

Example 4.6 Let F be the operator defined by the following initialized state space system:

$$
\begin{aligned}
x_1' &= u_1\,, \\
x_2' &= u_2\,, \\
x_3' &= u_3\,, \\
h(x) &= e^{x_1} \sum_{k=0}^{\infty} a_k f_k(x_2)\, \frac{x_3^k}{k!}\;,
\end{aligned}
$$

with initial state $x(0) = 0$. The functions f_k and coefficients a_k are defined via

$$f_k(x) \;=\; \underbrace{\exp(\exp(\cdots(\exp(x))\cdots))}_{k}$$

for $k \geq 1$, and $f_0(x) = 1$, and $a_k = (f_k(1))^{-1}$, $k = 0, 1, \ldots$.

According to (Isidori 1989, Theorem III-1.5), the initialized system defines an operator F_c for some c (in fact c is determined by the Lie derivatives of h along the directions of the vector fields of the system). Apparently, $F_c[u]$ is defined for $0 \leq t \leq 1$ for all u for which $\|u_2\|_\infty \leq 1$. It was shown in (Wang 1990) that the operator F does not satisfy any analytic i/o equation. However, it does satisfy the following analytic integral i/o equation:

$$y \;=\; e^{U_1} \sum_{k=0}^{\infty} a_k f_k(U_2)\, \frac{U_3^k}{k!}\ .$$

This again shows that, in general, the existence of integral i/o equations and the existence of differential i/o equations are not equivalent. □

Acknowledgment: The author wishes to thank E. D. Sontag for many very helpful and valuable discussions. The author would also like to thank an anonymous reviewer for many valuable and detailed remarks.

References

Fliess, M. (1983), 'Réalisation locale des systèmes non linéaires, algèbres de Lie filtrées transitives et séries génératrices non commutatives', *Invent. Math.* **71**, 521–537.

Fliess, M. & C.Reutenauer (1982), Une application de l'algèbre differentielle aux systèmes réguliers (ou bilinéaires), *in* A. Bensoussan & J. L. Lions, eds, 'Analysis and Optimization of Systems, Lect. Notes. Control Informat. Sci.', Springer-Verlag, Berlin, pp. 99–107.

Isidori, A. (1989), *Nonlinear Control Systems*, second edn, Springer-Verlag, Berlin.

Pearson, A. E. (1988), Least squares parameter identification of nonlinear differential i/o models, *in* 'Proceedings of the 27th IEEE Conference on Decision and Control', Austin, Texas, pp. 1831–1835.

Sontag, E. D. (1990), *Mathematical Control Theory, Deterministic Finite Dimensional Systems*, Springer-Verlag, New York.

Sussmann, H. J. (1985), Lie brackets and real analyticity in control theory, *in* C. Olech, B. Jakubczyk & J. Zabczyk, eds, 'Mathematical Control Theory', PWN-Polish Scientific Publishers, Warszawa, pp. 515–542. Banach Center Publications, Vol 14.

Wang, Y. (1990), Algebraic Differential Equations and Nonlinear Control Systems, PhD thesis, Rutgers, the State University of New Jersey.

Wang, Y. & Sontag, E. D. (1989), 'On two definitions of observation spaces', *Systems and Control Letters* **13**, 279–289.

Wang, Y. & Sontag, E. D. (1992*a*), 'Algebraic differential equations and rational control systems', *SIAM J. Control and Optimization* **30**(6), 1126–1149.

Wang, Y. & Sontag, E. D. (1992*b*), 'Generating series and nonlinear systems: Analytic aspects, local realizability and i/o representations', *Forum Mathematicum* **4**, 299–322.

Appendix

For the convenience of reference, we provide some background materials on i/o operators defined by generating series in this appendix. For the detailed study of such operators, we refer the reader to (Fliess 1983, Fliess & C.Reutenauer 1982, Isidori 1989, Wang & Sontag 1992a, Wang & Sontag 1992b).

A.1 Some Continuity Properties of I/O Operators

Let P be the set of $m+1$ noncommutative variables $\{\eta_0, \eta_1, \ldots, \eta_m)$ and let

$$P^* = \{w = \eta_{i_1}\eta_{i_2}\cdots\eta_{i_r} : \ 0 \le i_s \le m, \text{ for } s \le r,\, r \ge 0\}\,.$$

We use ϕ to denote w if $w = \eta_{i_1}\cdots\eta_{i_r}$ with $r = 0$. A generating series in the variables $\eta_0, \ldots, \eta_m$ is a formal power series

$$c = \sum_{w \in P^*} \langle c,\ w\rangle w\,,$$

where $\langle c,\ w\rangle \in \mathbb{R}$ for all w.

A generating series c is said to be *convergent* if there exist $K, M \ge 0$ such that

$$|\langle c,\ w\rangle| \le KM^k k! \quad \text{if } |w| = k\,, \tag{27}$$

where $|w|$ is the length of w, i.e., $|w| = k$ if $w = \eta_{i_1}\eta_{i_2}\cdots\eta_{i_k}$.

For each $T > 0$, consider the set $\mathcal{U}_T$ of all essentially bounded measurable functions $u : [0,T] \to \mathbb{R}^m$ with $\|u\|_\infty := \max\{\|u_i\|_\infty : 1 \le i \le m\} < 1$. For each $w \in P^*$, we define $V_w : \mathcal{U}_T \to \mathcal{C}[0,\, T]$ inductively by $V_\phi = 1$ and

$$V_w[u](t) = \int_0^t u_{i_1}(s) V_{w'}(s)\, ds\,, \tag{28}$$

if $w = \eta_i w'$ for some η_i. Assume now that c is a convergent series and let K and M be as in (27). Then the series of functions

$$F_c[u](t) := \sum_{w \in P^*} \langle c,\ w \rangle V_w[u](t)$$

is uniformly and absolutely convergent on $[0,\ T]$ for any T such that

$$T < \frac{1}{Mm + M}, \tag{29}$$

and for any $u \in \mathcal{V}_T$ (recall that $\mathcal{V}_T$ is the set of measurable functions $u : [0,\ T] \to \mathbb{R}^m$ with $\|u\|_\infty \leq 1$). We say that T is admissible to a convergent series c if T satisfies (29). Thus, each convergent series c defines an i/o operator F_c on $\mathcal{V}_T$ if T is admissible to c. Such an operator is (strictly) causal in the sense that $y(t)$ only depends on u restricted to $[0,\ t)$. This follows from the fact that each V_w is an integral operator, and consequently, $V_w[u](t)$ only depends on u restricted on $[0,\ t)$.

Lemma A.7 Let F_c be an i/o operator and T is admissible to c. Then $F_c[u]$ is analytic on $[0,\ T]$ if $u \in \mathcal{V}_T$ is analytic. $\square$

To each monomial $w_0 \in P^*$, the shift operator $c \mapsto w_0^{-1}c$ is defined by

$$\langle w_0^{-1}c,\ w \rangle = \langle c,\ w_0 w \rangle$$

for each $w \in P^*$. It then follows from (28) that

$$\frac{d}{dt} F_c[u](t) = F_{\eta_0^{-1}c}[u](t) + \sum_{j=1}^{m} u_j(t)\, F_{\eta_j^{-1}c}[u](t). \tag{30}$$

More generally, for each $k \geq 0$, if u is $\mathcal{C}^k$, then $F_c[u]$ is $\mathcal{C}^{k+1}$, and:

$$\frac{d^k}{dt^k} F_c[u](t) \;=\; F_{c_k(u(t),\ldots,u_{k-1}(t))}[u](t)$$

for some $c_k(\mu_0, \ldots, \mu_{k-1})$. See (Wang & Sontag 1989) for the precise definition of c_k. Roughly speaking, for each k, $c_k(\mu_0, \ldots, \mu_{k-1})$ is a polynomial in the μ_i's whose coefficients are series of the form $w^{-1}c$. In particular,

$$\left.\frac{d^k}{dt^k}\right|_{t=0} F_c[u](t) \;=\; \langle c_k(u(0), \ldots, u^{(k-1)}(0)),\; \phi\rangle \tag{31}$$

is a polynomial in $u(0), \ldots, u^{(k-1)}(0)$ whose coefficients are real numbers.

We say that (u, y) is a $\mathcal{C}^k$ ($\mathcal{C}^\omega$, respectively) i/o pair of F_c if $y = F_c[u]$ and u is $\mathcal{C}^k$ ($\mathcal{C}^\omega$, respectively).

A.2 Observation Spaces and Observation Fields

For any given power series c, we define the *observation space* $\mathcal{F}_1(c)$ as the $\mathbb{R}$-space spanned by all the series $w^{-1}c$, the *observation algebra* $\mathcal{A}_1(c)$ is the $\mathbb{R}$-algebra generated by the elements of $\mathcal{F}_1(c)$, under the shuffle product (see (Wang & Sontag 1992*a*)), and the *observation field* $\mathcal{Q}_1(c)$ is the quotient field of $\mathcal{A}_1(c)$. Note that $\mathcal{Q}_1(c)$ is always defined since $\mathcal{A}_1(c)$ is an integral domain, see (Wang & Sontag 1992*a*).

For any convergent series c, we say that the observation field $\mathcal{Q}_1(c)$ is a *meromorphically finitely generated field extension of* $\mathbb{R}$ if there exist an integer n and

$$c_1,\, c_2,\, \ldots,\, c_n \in \mathcal{A}_1(c)$$

such that for each element d in $\mathcal{Q}_1(c)$, there exist some analytic functions φ_0 and φ_1 defined on $\mathbb{R}^n$ such that

$$\varphi_0\left(F_{c_1}[u](t), \ldots, F_{c_n}[u](t)\right) F_d[u](t) = \varphi_1\left(F_{c_1}[u](t), \ldots, F_{c_n}[u](t)\right)$$

for all $u \in \mathcal{V}_T$, $t \in [0, T]$ and for any T admissible for c, and,

$$\varphi_0\left(F_{c_1}[u], \ldots, F_{c_n}[u]\right) \neq 0$$

for some $u \in \mathcal{U}_T$, and some T admissible for c. If this is the case, we call $c_1, \ldots, c_n$ the generators of the field, or, we say that the field is generated by $c_1, \ldots, c_n$.

While the finiteness properties of $\mathcal{F}_1(c)$ and $\mathcal{Q}_1(c)$ are related to the realizability of F_c (cf (Wang & Sontag 1992b, Theorem 1)), the finiteness properties of the following type of observation spaces and fields are related to the existence of i/o equations.

The observation space $\mathcal{F}_2(c)$ of the second type is defined to be the $\mathbb{R}$-space spanned by $c_n(\mu_0, \ldots, \mu_{n-1})$ for all n and all μ. The observation algebra $\mathcal{A}_2(c)$ is defined to be the $\mathbb{R}$-algebra generated by the elements of $\mathcal{F}_2(c)$, and the observation field $\mathcal{Q}_2(c)$ is the quotient field of $\mathcal{A}_2(c)$.

The following is one of the main results in (Wang & Sontag 1989):

Lemma A.8 Let c be a convergent series. Then $\mathcal{F}_1(c) = \mathcal{F}_2(c)$. $\square$