Generalized Input/Output Equations and Nonlinear Realizability*

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Abstract

This work studies various types of input/output representations for analytic input/output operators. It is shown that if an operator satisfies an integro-differential input/output equation or an integral one, then it is locally realizable by analytic state space systems. This generalizes the results previously obtained for differential input/output equations to integral and integro-differential equations.

1 Introduction

In the previous work (Wang & Sontag 1992a, Wang & Sontag 1992b), it was shown that if an input/output operator satisfies a possibly high-order differential input/output equation, then it is locally realizable by an analytic state space system (a set of controlled first order equations). Besides its intrinsic mathematical interest, this result is relevant in identification theory, since high-order equations involving only observed variables can be in principle determined from experimental data.

In practice, however, the signals of physical systems are often affected by noise, which

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*This research was supported in part by NSF Grants DMS-9108250 and DMS-9403924
Keywords: Generating series, local realization of input/output operators, input/output equations.
makes numerical identification, especially under high frequency noise, a very difficult task for
differential input/output equations. It is in principle better to work with integrals of the data
instead. Indeed, in linear systems identification practice, it is often the case that one prefilters
measured signals prior to estimation, so as to eliminate noise. Various filtering procedures
have been suggested in the nonlinear case as well. For instance, in the paper (Pearson 1988)
and other related work, the author discussed the “projected variable filter” method. To avoid
taking derivatives of the input/output signals, Pearson (1988) proposed to base the parameter
identification problem on the use of equations of the following type:

\[ \sum_{i=1}^{k} h_i(\theta) w_i(t) = 0, \]  

(1)

where \( \theta \) stands for the system parameters to be identified, each \( w_i(t) \) is an iterated integral of a
function of \((u, y)\), and the \( h_i \)'s are functions (which have certain special properties discussed in
the paper). This method and others like it lead to a natural and nontrivial question: once that
the value of \( \theta \) is estimated, can one find a state space realization for (1)? This motivates us to
study the relations between state space realizability and the existence of integral equations, or
the existence of even more general types of integro-differential equations, for i/o operators.

We start the paper by introducing generalized rational i/o equations for i/o operators.
Roughly speaking, an operator satisfies a generalized rational i/o equation if the \( k \)-th derivative
of the output, for some \( k \), can be expressed rationally in terms of lower-order derivatives as well
as integrals of inputs and outputs. The main result of the paper says that if an operator admits
such an equation, then it is locally realizable by an analytic state space system. In particular,
if an operator satisfies an integral i/o equation, then it is locally realizable. This generalizes
the results in (Wang & Sontag 1992a, Wang & Sontag 1992b) for differential i/o equations to
integral, or integro-differential equations. In Section 4, we focus on integral equations, and the
same result is established for analytic integral i/o equations. It is also indicated that in general
the existence of differential i/o equations and the existence of integral ones are not equivalent,
and examples are provided to illustrate this fact.

The formalism of this work is based on the generating series suggested by Fliess. The oper-
ators induced by generating series form a very general class of causal operators. For definitions
and properties of such operators, we refer the readers to (Fliess 1983, Fliess & C.Reutenauer
important property of such an operator is that for any analytic input function, the output
function is also analytic. For general terminology about control systems, see (Sontag 1990).

2 Generalized Rational Input/Output Equations

In this section we introduce the so called generalized rational input/output equations for i/o
operators.

Let $m, T > 0$ be given. Consider analytic i/o operators defined on $V_T$, the set of measurable
functions $u : [0, T] \to \mathbb{R}^m$ with $\|u\|_\infty \leq 1$ (see (Wang & Sontag 1992a) or the Appendix for the
detailed definitions of such i/o operators and related notions). To define generalized rational i/o
equations for such i/o operators, we consider the set of operators mapping $C^\omega [0, T] \times (C^\omega [0, T])^m$
to $C^\omega [0, T]$, where $C^\omega [0, T]$ denotes the set of analytic functions defined on $[0, T]$. We define “+”
and “.” for such operators by the usual pointwise operations, i.e., for any two such operators
$P_1, P_2$, we have:

$$(P_1 + P_2) (\varphi, \psi)(t) = P_1(\varphi, \psi)(t) + P_2(\varphi, \psi)(t),$$

$$(P_1 \cdot P_2) (\varphi, \psi)(t) = P_1(\varphi, \psi)(t) \cdot P_2(\varphi, \psi)(t),$$
for all \( \varphi \in C^\omega[0, T] \), \( \psi \in (C^\omega[0, T])^m \). The integral \( \int P \) of an operator \( P \) is defined in the following way:

\[
\left( \int P \right)(\varphi, \psi)(t) = \int_0^t P(\varphi, \psi)(s) \, ds, \quad \text{for } 0 \leq t \leq T.
\]

In this work, we are interested in a special class \( \mathcal{P} \) of \textit{differential integral operators} (DIO’s). To define \( \mathcal{P} \), we first need to define \( \mathcal{P}_k \) for each fixed nonnegative integer \( k \). For a given \( k \), \( \mathcal{P}_k \) is defined to be the smallest set of DIO’s satisfying the following properties:

a. the derivation operators

\[
\alpha_i : (\varphi, \psi) \mapsto \varphi^{(i)} \quad \text{and} \quad \beta_{ij} : (\varphi, (\psi_1, \psi_2, \ldots, \psi_m)) \mapsto \psi_j^{(i)}
\]

are in \( \mathcal{P}_k \), for each \( 0 \leq i \leq k \) and \( 1 \leq j \leq m \); 

b. \( \int P \in \mathcal{P}_k \) if \( P \in \mathcal{P}_k \),

c. \( P_1 + P_2, \ P_1 \cdot P_2 \in \mathcal{P}_k \) if \( P_1, P_2 \in \mathcal{P}_k \), and \( cP \in \mathcal{P}_k \) for any \( c \in \mathbb{R} \) if \( P \in \mathcal{P}_k \),

d. the “constant” DIO : \( (\varphi, \psi) \mapsto 1 \) \( \in \mathcal{P}_k \).

Note that \( \mathcal{P}_k \) is in fact the smallest \( \mathbb{R} \)-algebra containing

\[
\alpha_0, \alpha_1, \ldots, \alpha_k, \beta_0, \beta_2, \ldots, \beta_k
\]

that satisfies property b., where \( \beta_i = (\beta_{i1}, \ldots, \beta_{im}) \) for each \( i \). For instance, when \( m = 1 \), the operator \( P(\varphi, \psi) = \int_0^t \int_0^s \varphi(\tau) \psi(s) \, d\tau \, ds \in \mathcal{P}_0 \), and \( Q(\varphi, \psi) = \varphi'''(t) \cdot \int_0^t \psi(s) \, ds \in \mathcal{P}_3 \).

We define \( \mathcal{P} \) to be the union of all \( \mathcal{P}_k \)’s, that is,

\[
\mathcal{P} = \bigcup_{k=0}^{\infty} \mathcal{P}_k.
\]
Let $F$ be an analytic i/o operator defined on $\mathcal{V}_T$. We say that the operator $F$ satisfies a \textit{generalized rational i/o equation} if there exist some integer $k$, some $T > 0$, and two operators $P_0, P_1 \in \mathcal{P}_{k-1}$, such that for every analytic i/o pair of $(u, y)$ of $F$ it holds that

$$P_0(u, y)(t) \cdot y^{(k)}(t) = P_1(u, y)(t),$$

for all $t \in [0, T]$, and moreover, the following nondegeneracy property holds:

$$P_0(u, y) \neq 0$$

for some analytic i/o pair $(u, y)$ of $F$.

Roughly speaking, the i/o operator $F$ satisfies an equation of this type if the $k$-th derivative of the output, for some $k$, can be expressed rationally in terms of lower-order derivatives as well as integrals of inputs and outputs.

\textbf{Remark 2.1} The i/o operators defined by generating series are (strictly) causal, that is, for any i/o pair $(u, y)$ of any i/o operator $F$, $y(t)$ depends only on the restriction of $u$ on $[0, t)$ (see the appendix for more detailed discussions). Consequently, such an operator can never admit an i/o equation where the highest order of the derivatives of the input signals is higher than or equal to the highest order of the derivatives of the output signals such as $y'(t) = u'(t)$. \hfill \Box

\textbf{Remark 2.2} Generalized rational i/o equations form a very general class of i/o equations. Some special cases are as follows:

1. \textit{Algebraic differential equations}: An i/o operator $F$ satisfies an algebraic differential i/o
equation if the following holds for all \( C \) i/o pairs \((u,y)\) of \( F \):

\[
E(u(t), u'(t), \ldots, u^{(k)}(t), y(t), y'(t), \ldots, y^{(k)}(t)) = 0,
\]

for all \( t \in [0, T] \), where \( E \) is a polynomial. If this is the case, then \( F \) also satisfies a rational i/o equation of the following type:

\[
E_0 \left( u, u', \ldots, u^{(k)}; y, y', \ldots, y^{(k)} \right) y^{(k+1)} = E_1(u, u', \ldots, u^{(k)}; y, y', \ldots, y^{(k)}),
\]

which is a generalized rational i/o equation (note that \( u^{(k+1)} \) does not appear in \( E_1 \) because of the causality property of \( F \), see (Wang & Sontag 1992a, Lemma 4.4) for a detailed construction of \( E_0 \) and \( E_1 \)).

2. **Algebraic integral equations**: An operator \( F \) satisfies an algebraic integral i/o equation if the following holds for all \( C \) i/o pairs \((u,y)\) of \( F \):

\[
E(u(t), U_1(t), \ldots, U_k(t), y(t), y_1(t), \ldots, y_k(t)) = 0,
\]

for all \( t \in [0, T] \), where \( E \) is a polynomial, and

\[
U_i(t) = \int_0^t u(s) \, ds, \quad U_i(t) = \int_0^t U_{i-1}(s) \, ds, \quad i \geq 2,
\]

\[
y_i(t) = \int_0^t y(s) \, ds, \quad y_i(t) = \int_0^t y_{i-1}(s) \, ds, \quad i \geq 2.
\]

To show that \( F \) also admits a generalized rational i/o equation, we first assume that \( \frac{\partial E}{\partial y} \) is a nonzero polynomial and \( \deg_y E \) is as small as possible. Differentiating both sides of
equation (4), we get

$$\frac{\partial E}{\partial y}(u, U_1, \ldots, U_k, y, y_1, \ldots, y_k) y' = Q(u', u, U_1, \ldots, U_k, y, y_1, \ldots, y_k).$$  (7)

Again, by causality, one can show that $Q$ does not depend on $u'$. Thus, equation (7) is again a generalized rational i/o equation. If $\frac{\partial E}{\partial y} \equiv 0$, pick up the first $r$ so that $\frac{\partial E}{\partial y^r} \neq 0$. Differentiating enough times, we are in the previous situation.

3. More generally, given an equation $P(u, y) = 0$ with $P \in \mathcal{P}$, in practice one can often transform it into form (2) after repeated differentiation.

\[\] 3 Main Results

We say that an i/o operator $F$ is \textit{locally realizable} by an analytic system if there exist some analytic manifold $\mathcal{M}$, some $x_0 \in \mathcal{M}$, $(m + 1)$ analytic vector fields

$$g_0, g_1, \ldots, g_m$$

on $\mathcal{M}$, an analytic function $h : \mathcal{M} \rightarrow \mathbb{R}$, and some $\tau > 0$ such that for each i/o pair $(u, y)$ of $F$ with $\|u\|_{\infty} < 1$, it holds that

$$y(t) = h(x(t))$$

for $0 \leq t \leq \tau$, where $x(\cdot)$ is the solution of the equation

$$x' = g_0(x) + \sum_{j=1}^{m} g_j(x) u_j, \; x(0) = x_0.$$  

The following is the main theorem of this work.
Theorem 1 If an operator $F$ satisfies a generalized rational i/o equation, then it is locally realizable by an analytic system.

3.1 Proof of Theorem 1

In order to study local realizability by analytic systems, we first introduce an auxiliary notion, weak realizability. An i/o operator $F$ is said to be weakly realizable if there exist integers $n$ and $r$, polynomial functions $q: \mathbb{R}^n \times \mathbb{R}^{m(r+1)} \to \mathbb{R}$ and $h: \mathbb{R}^n \to \mathbb{R}$, a polynomial vector function $f: \mathbb{R}^n \times \mathbb{R}^{m(r+1)} \to \mathbb{R}^n$, and a continuous function $\alpha: \mathbb{R}^{m(r+1)} \to \mathbb{R}^n$, such that

(a) for every analytic i/o pair $(u, y)$ with $y = F[u]$, it holds that

$$y(t) = h(\varphi(t)) \text{ for all } t,$$

where $\varphi(\cdot)$ is an analytic solution of the differential equation

$$q(x, u, u', \ldots, u^{(r)})x' = f(x, u, u', \ldots, u^{(r)});$$

with $\varphi(0) = \alpha(u(0), u'(0), \ldots, u^{(r)}(0));$

(b) the following regularity condition holds: there exists some set $\Omega$ which is dense in $C^\omega([0, T])$ (with respect to the Whitney topology) such that for any $u \in \mathcal{V}_T \cap \Omega^m$, there exists some $C^\omega$ solution $\varphi(\cdot)$ as in (a) so that

$$q(\varphi(t), u(t), u'(t), \ldots, u^{(r)}(t)) \not= 0.$$
Lemma 3.1 If an i/o operator $F$ is weakly realizable, then it satisfies an algebraic differential i/o equation, that is, there exist some $k \geq 0$ and some polynomial function $E$ such that

$$E(u(t), u'(t), \ldots, u^{(k)}(t), y(t), y'(t), \ldots, y^{(k)}(t)) = 0$$

for all $t \in [0, T]$, for all $C^\omega$ i/o pairs of $F$.

In the following we give a detailed proof of the lemma.

Proof. Let $F$ be an i/o operator defined on $\mathcal{V}_T$. Assume that an i/o operator $F$ is weakly realizable by a system as in (8). We must prove that $F$ satisfies some differential i/o equation

$$E(u(t), u'(t), \ldots, u^{(k)}(t), y(t), y'(t), \ldots, y^{(k)}(t)) = 0,$$

for some polynomial $E$.

First we let

$$\mathcal{W} := \{ (x, \mu) \in \mathbb{R}^n \times \mathbb{R}^{m(r+1)} : q(x, \mu) \neq 0 \}.$$ 

Take $x \in \mathbb{R}^n$, and assume that there exists some $\bar{\mu}$ such that $(x, \bar{\mu}) \in \mathcal{W}$. Then for any analytic input function $u$ with $u^{(i)}(0) = \bar{\mu}_i$ ($i = 0, \ldots, r$), there exists some $\delta > 0$ such that the solution $\varphi(t, x, u)$ of system (8) with the initial state $x$ and the input $u$ is uniquely defined on $[0, \delta)$, and

$$q \left( \varphi(t, x, u), u(t), u'(t), \ldots, u^{(r)}(t) \right) \neq 0$$

for any $t \in [0, \delta)$. Let $y_x(t) = h(\varphi(t, x, u))$ for such $x$ and $u$. Then

$$y_x(0), y_x'(0), \ldots, y_x^{(n)}(0)$$

(10)
are rational functions of $x$ over the field $K$, the field obtained by adjoining $\mu_i$ ($i = 0, 1, \ldots, r + n - 1$) to $\mathbb{R}$. Since the transcendence degree of $K(x)$ over $K$ is $n$, the $(n + 1)$ functions in (10) are algebraically dependent over $K$; i.e., there exists some nontrivial polynomial $L$ over $K$ such that

$$L(y_x(0), y'_x(0), \ldots, y_x^{(n)}(0)) = 0.$$  

Clearing the denominators in the coefficients of $L$ (which are rational functions in $\mu$), one obtains a polynomial $E$ such that

$$E(\mu_0, \mu_1, \ldots, \mu_{r+n-1}, y_x(0), y'_x(0), \ldots, y_x^{(n)}(0)) = 0$$

holds for all $(x, \mu) \in W$. Observe that $E$ is a nontrivial polynomial since $L$ is nontrivial.

Take $(x, \mu) \in W$. Let $u$ be an analytic input function with $u^{(i)}(0) = \mu_i$, $0 \leq i \leq r$. Then there exists some $\delta > 0$ such that

$$\left(\varphi(t, x, u(t)), u(t), u'(t), \ldots, u^{(r)}(t)\right) \in W, \quad \forall t \in [0, \delta).$$

For any $t_1 \in [0, \delta)$, let $x_1 = \varphi(t_1, x, u)$, and let

$$\nu = \left(u(t_1), u'(t_1), \ldots, u^{(n+r-1)}(t_1)\right).$$

Then $(x_1, \nu_0, \ldots, \nu_r) \in W$, and hence,

$$E\left(\nu_0, \nu_1, \ldots, \nu_{r+n-1}, y_{x_1}(0), y'_{x_1}(0), \ldots, y^{(n)}_{x_1}(0)\right) = 0.$$
This is exactly:

\[ E\left( u(t_1), u'(t_1), \ldots, u^{(r+n-1)}(t_1), y_x(t_1), y'_x(t_1), \ldots, y^{(n)}_x(t_1) \right) = 0. \]

This shows that if \((x, \mu) \in \mathcal{W}\), then for any analytic \(u\) with \(u^{(i)}(0) = \mu_i\), \(0 \leq i \leq r\), there exists some \(\delta > 0\) such that

\[ E\left( u(t), u'(t), \ldots, u^{(n+r-1)}(t), h(\varphi(t, x, u)), \frac{d}{dt}h(\varphi(t, x, u)), \ldots, \frac{d^n}{dt^n}h(\varphi(t, x, u)) \right) = 0 \]

for all \(0 \leq t < \delta\). In particular, if \(\mu\) is such that \(q(\alpha(\mu), \mu) \neq 0\), then for any analytic \(u\) with \(u^{(i)}(0) = \mu_i\), there exists some \(\delta > 0\), such that

\[ E(u(t), \ldots, u^{(r+n-1)}(t), y(t), \ldots, y^{(n)}(t)) = 0 \quad (11) \]

for all \(t \in [0, \delta]\). By analyticity, it follows that (11) holds for all \(t \in [0, T]\).

Finally, we show how to overcome the restriction \(q(\alpha(\mu), \mu) \neq 0\). By the regularity condition of (8), one knows that there exists a dense set \(\Omega\) of analytic inputs, so that for each \(u \in \mathcal{V}_T \cap \Omega^m\), there exists some \(C^\omega\) solution \(\varphi\) such that (9) holds. It then follows from analyticity that for every such \(u\), there exists some \(\tau > 0\) such that

\[ q(\varphi(t), u(t), \ldots, u^{(r)}(t)) \neq 0, \]

for \(t \in (0, \tau)\). According to the previous argument, we see that \((u, y)\) satisfies (11) for \(t \in (0, \tau)\). Again, using analyticity, one shows that \((u, y)\) satisfies (11) for all \(t \in [0, T]\).
Since $\Omega$ is dense in $C^\omega$ controls with respect to the Whitney topology, it follows from the continuity of the operator $F$ that (11) holds for all $C^\omega$ i/o pairs of $F$, see (Wang & Sontag 1992b).

Finally, by the causality of the operator $F$, one sees that (11) yields an i/o equation of the following type:

$$E_1(u(t), u'(t), \ldots, u^{(n-1)}(t), y(t), y'(t), \ldots, y^{(n)}(t)) = 0,$$

where $E_1$ is some polynomial. (See (Wang 1990) for the details on the existence of such $E_1$).

Lemma 3.1 says that if $F$ is weakly realizable, then $F$ satisfies a differential i/o equation in the usual sense, cf. (Wang & Sontag 1992a). By (Wang & Sontag 1992b, Theorem 3), $F$ is locally realizable by an analytic system. Hence, we get the following:

**Corollary 3.2** If $F$ is weakly realizable, then $F$ is locally realizable by an analytic state space system.

The following Lemma connects the existence of generalized rational i/o equations to weak realizability.

**Lemma 3.3** If an operator $F$ satisfies a generalized rational i/o equation, then $F$ is weakly realizable.

**Proof.** We shall say that a DIO $P$ is a **simple monomial of (iterated) integrals** if $P$ has the following form:

$$P(u, y)(t) = \int_0^t \xi_1(s_1) \int_0^{s_1} \xi_2(s_2) \cdots \int_0^{s_{r-1}} \xi_r(s_r) \, ds,$$

where each $\xi_i$ is a monomial in $u, u', \ldots, y, y', \ldots$ (possibly a constant). Assume now that $F$ satisfies a generalized rational i/o equation. Then there are polynomial functions $E_0$ and $E_1$
such that the following holds for every i/o pair of $F$:

$$E_0 \left( u(t), u'(t), \ldots, u^{(k-1)}(t), y(t), \ldots, y^{(k-1)}(t), Q_1(u, y)(t), \ldots, Q_{\sigma}(u, y)(t) \right) y^{(k)}(t)$$

$$= E_1 \left( u(t), u'(t), \ldots, u^{(k-1)}(t), y(t), \ldots, y^{(k-1)}(t), Q_1(u, y)(t), \ldots, Q_{\sigma}(u, y)(t) \right),$$

where $Q_1, \ldots, Q_{\sigma} \in \mathcal{P}_{k-1}$ are simple monomials of integrals, and there exists some analytic i/o pair $(u, y)$ of $F$ such that

$$E_0 \left( u(t), u'(t), \ldots, u^{(k-1)}(t), y(t), \ldots, y^{(k-1)}(t), Q_1(u, y)(t), \ldots, Q_{\sigma}(u, y)(t) \right) \neq 0. \quad (12)$$

Assume that

$$Q_i(u, y)(t) = \int_0^t \xi_{i1}(s_1) \int_0^{s_1} \xi_{i2}(s_2) \cdots \int_0^{s_{r_i}-1} \xi_{ir_i}(s_{r_i}) \, ds,$$

for $i = 1, \ldots, \sigma$. Note then that each $\xi_{ij}$ is a monomial in $u^{(p)}$ and $y^{(l)}$ with $0 \leq p, l \leq k - 1$. Formally denoting $x_{i1} = Q_i$, $x_{i2} = \int_0^t \xi_{i2}(s_2) \cdots \int_0^{s_{r_i}-1} \xi_{ir_i}(s_{r_i}) \, ds$ and so forth, then one has the following equations:

$$x'_{i1} = \xi_{i1}x_{i2}, \quad x'_{i2} = \xi_{i2}x_{i3}, \quad \ldots, \quad x'_{ir_i} = \xi_{ir_i}, \quad (13)$$

with $x_{il}(0) = 0$ for all $l$. Let $x_i = (x_{i1}, x_{i2}, \ldots, x_{ir_i})$. Then it follows from the fact that $Q_i \in \mathcal{P}_{k-1}$ that Eq. (13) can be written as:

$$x'_i = f_i(x_i, u, u', \ldots, u^{(k-1)}, y, y', \ldots, y^{(k-1)}), \quad (14)$$
for some polynomial vector function \( f \). Now let \( z_i = y^{(i)} \) for \( 0 \leq i \leq k - 1 \), and let \( x = (x_1, x_2, \ldots, x_r) \). Then Eq. (12) implies the following equations:

\[
\begin{align*}
    x' &= f(x, u, u', \ldots, u^{(k-1)}, z), \\
    z_0' &= z_1, \\
    z_1' &= z_2, \\
    \ldots \\
    z_{k-2}' &= z_{k-1}, \\
    Q_0(x, z, u, u', \ldots, u^{(k-1)})z_{k-1}' &= Q_1(x, z, u, u', \ldots, u^{(k-1)}), \\
    x(0) &= 0, \\
    z(0) &= (y(0), y'(0), \ldots, y^{(k-1)}(0)),
\end{align*}
\]

(15)

where \( Q_0, Q_1 \) are given by

\[
Q_i(x, z, u, u', \ldots, u^{(k-1)}) = E_i(u, u', \ldots, u^{(k-1)}, z, x), \quad i = 1, 2.
\]

Since \( z_i(0) = y^{(i)}(0) \) for each \( i \), it follows that the mapping \((u(0), u'(0), \ldots, u^{(i)}(0)) \mapsto z_i(0)\) is a polynomial function (cf (31) in the Appendix), thus, the mapping

\[
\alpha : (u(0), u'(0), \ldots, u^{(i)}(0)) \mapsto (x(0), z(0))
\]

is continuous.

By the nondegeneracy condition (12), one knows that there exists at least one analytic \( \omega \) such that

\[
Q_0(x(t), z(t), \omega(t), \omega(t)', \ldots, \omega^{(k-1)}(t)) \neq 0.
\]

(16)

Using the same argument as in the proof of (Wang & Sontag 1992a, Theorem 5.2), one can show that there exists some open dense set \( \Omega \) in \( C^\omega \) such that (16) holds for every \( u \in \mathcal{V}_T \subset \Omega^m \). Hence, (15) is a weak realization of \( F \).  

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Finally, Theorem 1 follows from Corollary 3.2 and Lemma 3.3.

4 Integral I/O Equations

As we indicated earlier in Remark 2.2, if an operator $F$ satisfies an algebraic integral i/o equation of type (4), then it satisfies a generalized rational i/o equation. Thus we conclude, in particular:

**Corollary 4.1** If an operator $F$ satisfies an algebraic integral i/o equation, then $F$ is locally realizable.

In contrast to the case of differential i/o equations (in which existence of an algebraic differential i/o equation is equivalent to realizability by a “rational” system, cf. (Wang & Sontag 1992a)), it is generally not true that every operator which is realizable by a polynomial system satisfies an algebraic integral i/o equation. As an illustration of this matter of fact, consider the following example.

**Example 4.2** Let $F$ be the operator given by

$$y(t) = F[u](t) = \exp \left( \int_0^t u(s) \, ds \right).$$

To be more precise, the series that defines the operator is given by

$$c = 1 + \eta_1 + \eta_1 \eta_1 + \eta_1 \eta_1 \eta_1 + \cdots.$$ 

The operator $F$ is realized by the system:

$$x' = xu, \quad y = x, \quad x_0 = 1.$$
Assume that $F$ satisfies an integral i/o equation, which we write as

$$P(u(t), U_1(t), \ldots, U_r(t), y(t), \ldots, y_r(t)) = 0.$$  \hfill (19)

By the causality of the operator, one sees that $P$ can be chosen independently of $u$. Thus one may assume that Eq. (19) is of the following type:

$$P(U_1(t), \ldots, U_r(t), y(t), \ldots, y_r(t)) = 0.$$  \hfill (20)

For each $i = 1, 2, \ldots, r$, let $x_i = y_{r-i+1}$ and $z_i = U_{r-i+1}$. Then $x, z$ satisfy the following equations:

\begin{align*}
  x'_1 &= x_2, \ldots, x'_{r-1} = x_r, \ x'_r = \exp(z_r), \\
  z'_1 &= z_2, \ldots, z'_{r-1} = z_r, \ z'_r = u
\end{align*}

\begin{align*}
  x(0) &= 0, \ z(0) = 0. \quad \hfill (21)
\end{align*}

A straightforward computation shows that system (21) satisfies the accessibility rank condition at 0, and therefore, the system is accessible in a neighborhood of 0, (see e.g. (Sussmann 1985). This means that there exist open sets $W_1$ and $W_2$ in $\mathbb{R}^r$ such that, for any point $(\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_r) \in W_1 \times W_2$, there is some control $\omega$ and some $t > 0$ so that, for the solution of system (21) corresponding to the control $\omega$, we have

\begin{align*}
  (x_1(t), \ldots, x_r(t), z_1(t), \ldots, z_r(t)) &= (\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_r).
\end{align*}
which, together with (20), implies that

\[ P(\mu_1, \ldots, \mu_r, \exp(\mu_1), \nu_1, \ldots, \nu_r) = 0 \]  

(22)

for any \((\nu, \mu) \in W_1 \times W_2\). This is impossible since \(\mu_1\) and \(\exp(\mu_1)\) are algebraically independent over \(\mathbb{R}\). Thus no equation like (19) exists.

Note that even though \(F\) does not admit any integral equation, \(F\) does satisfy the differential i/o equation

\[ y' = yu. \]

This illustrates the fact that the existence of differential equations and the existence of integral equations are not equivalent, unless much stronger conditions (such as linearity of the equations, as for linear systems) are assumed.

### 4.1 Analytic Integral I/O Equations

In (Wang & Sontag 1992b), it was shown that if an operator satisfies an analytic differential i/o equation, then it is locally realizable by an analytic state space system. In this section, we will study the analogue of this result for integral i/o equations.

**Definition 4.3** We say that an operator \(F\) satisfies an analytic integral equation if there exist some integer \(k > 0\), some \(T > 0\), and a nontrivial analytic function \(A\) defined on \(\mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1}\) such that for each analytic i/o pair \((u, y)\), it holds that

\[ A(u(t), U_1(t), \ldots, U_k(t), y(t), y_1(t), \ldots, y_k(t)) = 0 \]  

(23)

for all \(t \in [0, T]\).
In the previous example, though the operator $F$ defined by (17) does not satisfy any algebraic integral i/o equation, it does satisfy an analytic integral equation:

$$y - \exp\left(\int_0^t u(s) \, ds\right) = 0.$$  

The following is a generalization of Corollary 4.1 to the analytic equations.

**Proposition 4.4** If an operator $F$ satisfies an analytic integral i/o equation, then it is locally realizable by an analytic state space system.

As in the case of differential i/o equations, to prove the result, one needs to employ some basic properties of observation spaces and observation fields associated with $F$, cf. (Wang & Sontag 1989, Wang & Sontag 1992a, Wang & Sontag 1992b). For the convenience of reference, we will put the needed materials in the Appendix.

In the following, we will use $F_c$ to denote an i/o operator defined by the generating series $c$.

Assume that an operator $F_c$ satisfies an analytic integral i/o equation (23). For each $j = 1, 2, \ldots, m$ and each $i \geq 1$, we let $d_{ij}$ be the monomial defined by $\eta_i^{j-1}\eta_j$. Observe that each $d_{ij}$ defines an operator:

$$F_{d_{ij}}[u](t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} U_j(s_i) \, ds.$$  

We use $d_i$ ($F_{d_i}$, respectively) to denote $(d_{i1}, d_{i2}, \ldots, d_{im})$ ($(F_{d_{i1}}, \ldots, F_{d_{im}})$, respectively).

Also, for each $i$, let $e_i = \eta_i^0 c$, then $e_i$ is such a series that the operator $F_{d_i}$ defined by $d_i$ is given by

$$F_{e_i}[u](t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} F_c[u](s_i) \, ds.$$  

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Hence, Eq. (23) is equivalent to

\[ A(u(t), F_{d_1}[u](t), \ldots, F_{d_k}[u](t), F_c[u](t), F_{e_1}[u](t), \ldots, F_{e_k}[u](t)) = 0. \]

We now define the *generalized observation space* \( \tilde{F}_1(c) \) by:

\[
\tilde{F}_1(c) = \text{span}_\mathbb{R}\left\{\{d_1, d_2, \ldots, d_k, e_1, e_2, \ldots, e_k\} \cup \{d : d \in F_1(c)\}\right\}.
\]

(See the Appendix for the definition of \( F_1(c) \).) The generalized observation algebra \( \tilde{A}_1(c) \) associated with \( c \) is the \( \mathbb{R} \)-algebra generated by the elements of \( F_1(c) \), under the shuffle product (see (Wang & Sontag 1992a)), and the generalized observation field \( \tilde{Q}_1(c) \) is the quotient field of \( \tilde{A}_1(c) \).

Corresponding to the second type of observation space \( F_2(c) \) studied in (Wang & Sontag 1992a), we define the second type of generalized observation space \( \tilde{F}_2(c) \) by:

\[
\tilde{F}_2(c) = \text{span}_\mathbb{R}\left\{\{d_1, d_2, \ldots, d_k, e_1, e_2, \ldots, e_k\} \cup \{d : d \in F_2(c)\}\right\}.
\]

(See again the Appendix for the definition of \( F_2(c) \).) The generalized observation algebra \( \tilde{A}_2(c) \) is the algebra generated by the elements of \( \tilde{F}_2(c) \), and the generalized observation field \( \tilde{Q}_2(c) \) is the quotient field of \( \tilde{A}_2(c) \). See also (Wang & Sontag 1992b), or the Appendix, for the meaning of meromorphically finitely generated field extension.

Following exactly the same steps of the proof of Theorem 2 in (Wang & Sontag 1992b), one proves the following:

**Lemma 4.5** If an operator \( F_c \) satisfies an analytic integral i/o equation, then the field \( \tilde{Q}_2(c) \) is meromorphically finitely generated. \( \square \)
We now return to the proof of Proposition 4.4.

Proof of Proposition 4.4: According to (Wang & Sontag 1989, Theorem 1) (see also the Appendix), one knows that $\tilde{F}_1(c) = \tilde{F}_2(c)$, hence, $\tilde{Q}_1(c) = \tilde{Q}_2(c)$. This implies that if $F_c$ satisfies an analytic integral i/o equation, then $\tilde{Q}_1(c)$ is meromorphically finitely generated. Without loss of generality, one may assume that the following are generators of $\tilde{Q}_1(c)$:

$$d_1, d_2, \ldots, d_k, e_1, e_2, \ldots, e_k, c_1, c_2, \ldots, c_n,$$

where $c_1, c_2, \ldots, c_n \in \mathcal{F}_1(c)$ and $c_1 = c$. We now let $z_i(t) = F_{d_{k-i+1}}[u](t)$, $w_i = F_{e_{k-i+1}}[u](t)$, $x_i(t) = F_{c_i}[u](t)$. It is clear that

$$z'_i(t) = z_{i+1}(t), \text{ for } 1 \leq i \leq k-1, \quad z'_k(t) = u(t), \quad z(0) = 0, \quad (24)$$

$$w'_i(t) = w_{i+1}(t), \text{ for } 1 \leq i \leq k-1, \quad w'_k(t) = x_0(t), \quad w(0) = 0. \quad (25)$$

Following the same steps as in the proof of Theorem 1 in (Wang & Sontag 1992b), one sees that there exists some analytic vector functions $g_0, g_1, \ldots, g_m$, and an analytic function $q$ satisfying

$$q(x(t), z(t), w(t)) \neq 0$$

for some analytic control $v(\cdot)$ so that the following holds for each input function $u$:

$$q(x(t), z(t), w(t))x'(t) = g_0(x(t), z(t), w(t)) + \sum_{i=1}^{m} g_i(x(t), z(t), w(t))u_i(t), \quad (26)$$
and \( x(0) = x_0 \) for some \( x_0 \in \mathbb{R}^n \). Let \( \hat{x} = (x, w, z) \). Combining (24), (25) and (26), one gets:

\[
q(\hat{x}(t)) \dot{\hat{x}}(t) = G_0(\hat{x}(t)) + \sum_{i=1}^{m} G_i(\hat{x}(t)) u_i(t)
\]

\[
\hat{x}(0) = (x_0, 0, 0)
\]

\[
y(t) = F_c[u](t) = \hat{x}_1(t),
\]

for some analytic vector fields \( G_0, G_1, \ldots, G_m \). Then the the proof of (Wang & Sontag 1992b, Theorem 1(b)) shows that the operator \( F_c \) is locally realizable by an analytic state space system.

4.2 Integral I/O Equations vs. Differential I/O Equations

According to Lemmas 3.1 and 3.3, if an operator \( F \) satisfies an algebraic integral i/o equation, then it also satisfies a differential i/o equation, and Example 4.2 shows that an operator may fail to satisfy any algebraic integral i/o equation though it admits an algebraic differential i/o equation. However, in the analytic case, an operator could satisfy an analytic integral i/o equation even if it does not satisfy any analytic differential i/o equation. The following example is based on a construction from W. Respondek.

**Example 4.6** Let \( F \) be the operator defined by the following initialized state space system:

\[
x_1' = u_1,
\]

\[
x_2' = u_2,
\]

\[
x_3' = u_3,
\]

\[
h(x) = e^{x_1} \sum_{k=0}^{\infty} a_k f_k(x_2) \frac{x_3^k}{k!},
\]
with initial state $x(0) = 0$. The functions $f_k$ and coefficients $a_k$ are defined via

$$f_k(x) = \exp\left(\exp\left(\cdots\left(\exp(x)\right)\cdots\right)\right)$$

for $k \geq 1$, and $f_0(x) = 1$, and $a_k = (f_k(1))^{-1}$, $k = 0, 1, \ldots$.

According to (Isidori 1989, Theorem III-1.5), the initialized system defines an operator $F_c$ for some $c$ (in fact $c$ is determined by the Lie derivatives of $h$ along the directions of the vector fields of the system). Apparently, $F_c[u]$ is defined for $0 \leq t \leq 1$ for all $u$ for which $\|u_2\|_{\infty} \leq 1$.

It was shown in (Wang 1990) that the operator $F$ does not satisfy any analytic i/o equation. However, it does satisfy the following analytic integral i/o equation:

$$y = e^{U_1} \sum_{k=0}^{\infty} a_k f_k(U_2) \frac{U_3^k}{k!}. $$

This again shows that, in general, the existence of integral i/o equations and the existence of differential i/o equations are not equivalent.

Acknowledgment: The author wishes to thank E. D. Sontag for many very helpful and valuable discussions. The author would also like to thank an anonymous reviewer for many valuable and detailed remarks.

References


Wang, Y. (1990), Algebraic Differential Equations and Nonlinear Control Systems, PhD thesis, Rutgers, the State University of New Jersey.


Appendix

For the convenience of reference, we provide some background materials on i/o operators defined by generating series in this appendix. For the detailed study of such operators, we

A.1 Some Continuity Properties of I/O Operators

Let $P$ be the set of $m + 1$ noncommutative variables $\{\eta_0, \eta_1, \ldots, \eta_m\}$ and let

$$P^* = \{w = \eta_{i_1}\eta_{i_2}\cdots\eta_{i_r} : 0 \leq i_s \leq m, \text{ for } s \leq r, r \geq 0\}.$$  

We use $\phi$ to denote $w$ if $w = \eta_{i_1}\cdots\eta_{i_r}$ with $r = 0$. A generating series in the variables $\eta_0, \ldots, \eta_m$ is a formal power series

$$c = \sum_{w \in P^*} \langle c, w \rangle w,$$

where $\langle c, w \rangle \in \mathbb{R}$ for all $w$.

A generating series $c$ is said to be convergent if there exist $K, M \geq 0$ such that

$$|\langle c, w \rangle| \leq KM^{|w|}k! \text{ if } |w| = k,$$  

where $|w|$ is the length of $w$, i.e., $|w| = k$ if $w = \eta_{i_1}\eta_{i_2}\cdots\eta_{i_k}$.

For each $T > 0$, consider the set $\mathcal{U}_T$ of all essentially bounded measurable functions $u : [0, T] \to \mathbb{R}^m$ with $\|u\|_\infty := \max\{\|u_i\|_\infty : 1 \leq i \leq m\} < 1$. For each $w \in P^*$, we define $V_w : \mathcal{U}_T \to \mathcal{C}[0, T]$ inductively by $V_\phi = 1$ and

$$V_w[u](t) = \int_0^t u_{i_1}(s)V_{w'}(s) \, ds,$$  

(28)
if \( w = \eta_i w' \) for some \( \eta_i \). Assume now that \( c \) is a convergent series and let \( K \) and \( M \) be as in (27). Then the series of functions

\[
F_c[u](t) := \sum_{w \in P^*} \langle c, w \rangle V_w[u](t)
\]

is uniformly and absolutely convergent on \([0, T]\) for any \( T \) such that

\[
T < \frac{1}{M m + M}, \tag{29}
\]

and for any \( u \in \mathcal{V}_T \) (recall that \( \mathcal{V}_T \) is the set of measurable functions \( u : [0, T] \to \mathbb{R}^m \) with \( \|u\|_\infty \leq 1 \)). We say that \( T \) is admissible to a convergent series \( c \) if \( T \) satisfies (29). Thus, each convergent series \( c \) defines an i/o operator \( F_c \) on \( \mathcal{V}_T \) if \( T \) is admissible to \( c \). Such an operator is (strictly) causal in the sense that \( y(t) \) only depends on \( u \) restricted to \([0, t]\). This follows from the fact that each \( V_w \) is an integral operator, and consequently, \( V_w[u](t) \) only depends on \( u \) restricted on \([0, t]\).

Lemma A.7 Let \( F_c \) be an i/o operator and \( T \) is admissible to \( c \). Then \( F_c[u] \) is analytic on \([0, T]\) if \( u \in \mathcal{V}_T \) is analytic.

To each monomial \( w_0 \in P^* \), the shift operator \( c \mapsto w_0^{-1}c \) is defined by

\[
\langle w_0^{-1}c, w \rangle = \langle c, w_0w \rangle
\]

for each \( w \in P^* \). It then follows from (28) that

\[
\frac{d}{dt} F_c[u](t) = F_{w_0^{-1}c}[u](t) + \sum_{j=1}^m u_j(t) F_{w_j^{-1}c}[u](t). \tag{30}
\]
More generally, for each \( k \geq 0 \), if \( u \) is \( C^k \), then \( F_c[u] \) is \( C^{k+1} \), and:

\[
\frac{d^k}{dt^k} F_c[u](t) = F_{c_k(u(t), \ldots, u_{k-1}(t))}(u(t))
\]

for some \( c_k(\mu_0, \ldots, \mu_{k-1}) \). See (Wang & Sontag 1989) for the precise definition of \( c_k \). Roughly speaking, for each \( k \), \( c_k(\mu_0, \ldots, \mu_{k-1}) \) is a polynomial in the \( \mu_i \)'s whose coefficients are series of the form \( w^{-1}c \). In particular,

\[
\left. \frac{d^k}{dt^k} \right|_{t=0} F_c[u](t) = \langle c_k(u(0), \ldots, u^{(k-1)}(0)), \phi \rangle
\]  

is a polynomial in \( u(0), \ldots, u^{(k-1)}(0) \) whose coefficients are real numbers.

We say that \((u, y)\) is a \( C^k \) (\( C^\omega \), respectively) i/o pair of \( F_c \) if \( y = F_c[u] \) and \( u \) is \( C^k \) (\( C^\omega \), respectively).

### A.2 Observation Spaces and Observation Fields

For any given power series \( c \), we define the observation space \( \mathcal{F}_1(c) \) as the \( \mathbb{R} \)-space spanned by all the series \( w^{-1}c \), the observation algebra \( \mathcal{A}_1(c) \) is the \( \mathbb{R} \)-algebra generated by the elements of \( \mathcal{F}_1(c) \), under the shuffle product (see (Wang & Sontag 1992a)), and the observation field \( \mathcal{Q}_1(c) \) is the quotient field of \( \mathcal{A}_1(c) \). Note that \( \mathcal{Q}_1(c) \) is always defined since \( \mathcal{A}_1(c) \) is an integral domain, see (Wang & Sontag 1992a).

For any convergent series \( c \), we say that the observation field \( \mathcal{Q}_1(c) \) is a meromorphically finitely generated field extension of \( \mathbb{R} \) if there exist an integer \( n \) and

\[
c_1, c_2, \ldots, c_n \in \mathcal{A}_1(c)
\]
such that for each element $d$ in $Q_1(c)$, there exist some analytic functions $\varphi_0$ and $\varphi_1$ defined on $\mathbb{R}^n$ such that

$$\varphi_0 (F_{c_1}[u](t), \ldots, F_{c_n}[u](t)) F_d[u](t) = \varphi_1 (F_{c_1}[u](t), \ldots, F_{c_n}[u](t))$$

for all $u \in V_T$, $t \in [0, T]$ and for any $T$ admissible for $c$, and,

$$\varphi_0 (F_{c_1}[u], \ldots, F_{c_n}[u]) \neq 0$$

for some $u \in U_T$, and some $T$ admissible for $c$. If this is the case, we call $c_1, \ldots, c_n$ the generators of the field, or, we say that the field is generated by $c_1, \ldots, c_n$.

While the finiteness properties of $F_1(c)$ and $Q_1(c)$ are related to the realizability of $F_c$ (cf (Wang & Sontag 1992b, Theorem 1)), the finiteness properties of the following type of observation spaces and fields are related to the existence of i/o equations.

The observation space $F_2(c)$ of the second type is defined to be the $\mathbb{R}$-space spanned by $c_n(\mu_0, \ldots, \mu_{n-1})$ for all $n$ and all $\mu$. The observation algebra $A_2(c)$ is defined to be the $\mathbb{R}$-algebra generated by the elements of $F_2(c)$, and the observation field $Q_2(c)$ is the quotient field of $A_2(c)$.

The following is one of the main results in (Wang & Sontag 1989):

**Lemma A.8** Let $c$ be a convergent series. Then $F_1(c) = F_2(c)$. \qed