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Input-to-state stability for discrete-time nonlinear systems☆

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The input-to-state stability property and ISS *small-gain theorems are introduced as the cornerstone of new stability criteria for discrete-time nonlinear systems.*

Abstract

In this work we study the input-to-state stability (ISS) property for discrete-time nonlinear systems. It is shown that most ISS results for continuous-time nonlinear systems in the current literature can be extended to the discrete-time case. Several equivalent characterizations of ISS are introduced and two ISS small-gain theorems are proved for nonlinear and interconnected discrete-time systems. ISS stabilizability is discussed and comparisons with the continuous-time case are made. As in the continuous time framework, where the notion ISS found wide applications, we expect that this notion will provide a useful tool in areas related to stability and stabilization for nonlinear discrete time systems as well.

Keywords: Nonlinear stability; Discrete time systems; Lyapunov functions; Input-to-state stability; Nonlinear small-gain

1. Introduction

The purpose of this paper is to study the input-to-state stability (ISS) property for discrete-time nonlinear systems of the general form

$$x(k+1) = f(x(k), u(k)), \tag{1}$$

where states $x(k)$ are in $\mathbb{R}^n$, and control values $u(k)$ in $\mathbb{R}^m$, for some n and m, and for each time instant $k \in \mathbb{Z}_+$. We assume that $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous.

We were motivated by the corresponding ISS notion which was originally proposed by Sontag (1989, 1990) for continuous-time nonlinear systems. The ISS property concerns with the continuity of state trajectories on the initial states and the inputs. Roughly speaking, a system is ISS if every state trajectory corresponding to a bounded control remains bounded, and the trajectory eventually becomes small if the input signal is small no matter what the initial state is. As shown in Sontag (1989, 1990) and many other references (Coron, Praly, & Teel, 1994; Isidori, 1999; Jiang & Mareels, 1997; Jiang, Teel, & Praly, 1994; Kazakos & Tsinias, 1994; Krstić & Li, 1998; Krstić, Kanellakopoulos, & Kokotovic, 1995; Praly & Jiang, 1993; Praly & Wang, 1994; Sontag & Wang, 1995; Sontag & Wang, 1996; Teel, 1996), ISS turns out to be a very natural stability property and, indeed, has been successfully employed in the stability analysis and control synthesis of nonlinear systems with complex structure. Our interest in discrete-time nonlinear systems (1) is due to the fact that discrete-time (or, difference) systems have their own interest and have found applications in various fields (Agarwal, 1992; LaSalle, 1986; Lakshmikantham & Trigiante, 1988). For instance, the stability theory of difference systems was recently used to design stabilizing control laws for discrete-time nonlinear systems—see, for instance, Byrnes and Lin (1994), Chen and Khalil (1995), Guo (1997), Kotsios and Kalouptsisis (1996), Nijmeijer and van der Schaft (1990), Tsinias, Kotsios, and Kalouptsidis (1990), Tsinias (1993) and references therein.

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One of the main results in this work provide an equivalence relation between ISS and ISS-Lyapunov function. The latter concept was introduced in Sontag (1989) and Sontag and Wang (1995) for continuous-time systems. In addition to this interesting result, we prove that various equivalent characterizations of the ISS condition proposed in Sontag and Wang (1996) also hold for discrete-time nonlinear systems. As in the continuous-time context, we show that a discrete-time system (1) can be rendered ISS (or, input-to-state stabilizable) if and only if it is globally stabilizable via state-feedback. This theorem is stated and illustrated in Section 5. However, new phenomena arise in the extension from continuous-time to discrete-time. For continuous-time affine systems, continuous stabilization implies ISS stabilization by means of state-feedback change $u = K(x) + v$. This is not the case any more for discrete systems as shown by an example. For a discrete-time system, whether it is an affine or non-affine system, a more complex feedback transformation of the form $u = K_1(x) + K_2(x)v$ is in general required.

We also show in this work that ISS small gain theorems in Coron et al. (1994) and Jiang et al. (1994) are extendible to the discrete-time setting. The work on small gain theorems for nonlinear systems can be traced back to Mareels and Hill (1992), where a nonlinear generalization of the classical small-gain theorem was proposed for both continuous- and discrete-time feedback systems within the input–output framework. As a consequence of not specifying the role of initial conditions, no conclusion was drawn in Mareels and Hill (1992) about the internal stability of the interconnected system. Here, with the ISS concept, the role of initial conditions can be made explicit and both external and internal stability properties are established for the whole feedback system.

The results in this paper are not surprising, considering the fact that most of them are available for continuous time systems. Most of our results can be considered as discrete analogues to the earlier results for ISS and its related characterizations and ISS nonlinear small gain theorems obtained in Coron et al. (1994), Jiang, Mareels, and Wang (1996), Jiang et al. (1994), Lin, Sontag, and Wang (1996), Sontag (1989, 1990) and Sontag and Wang (1995, 1996). Though the conceptual notions are all originated from the continuous case, and many arguments are, in some cases very straightforward or routine, generalizations of their continuous counterparts, many technical results cannot be obtained by the obvious "discretizing" generalization of the arguments used in the continuous framework. In some cases, one has to adopt completely different arguments. Considering the significant role played by ISS in the continuous case and the fact that many technical results need to be carefully examined for the discrete time case, we consider it necessary and appropriate to present the results with thorough proofs.

The remainder of the paper is comprised of five sections. Primary notations and definitions are given in Section 2. In Section 3, we introduce the ISS concept for discrete-time systems. Several popular characterizations of ISS in continuous-time are extended to the discrete-time case. Section 4 proposes two nonlinear small-gain theorems for discrete-time interconnected ISS systems. Section 5 discusses when a discrete-time system can be made ISS via state-feedback. We close this paper with some brief concluding remarks in Section 6. The technical results are discussed in Appendices A and B.

2. Notation and definitions

The notations used in this paper are quite standard. We use $\mathbb{Z}_+$ to denote the set of all nonnegative integers. For any positive real number r, $\lfloor r \rfloor$ denotes the largest integer that is less than or equal to r. For any x in $\mathbb{R}^n$, x^T is its transpose and $|x|$ its Euclidean norm. For a $n \times m$ matrix A, $|A|$ stands for its induced matrix norm. For any function $\phi : \mathbb{Z}_+ \to \mathbb{R}^n$, we denote (with slight abuse of notation) $\|\phi\| = \sup\{|\phi(k)|: k \in \mathbb{Z}_+\} \leqslant \infty$. In the case when ϕ is bounded, this is the standard l_∞ norm. For any $k \in \mathbb{Z}_+$ and any function $\phi : \mathbb{Z}_+ \to \mathbb{R}^n$, $\phi_{[k]}$ denotes the truncation of ϕ at k; i.e., $\phi_{[k]}(j) = \phi(j)$ if $j \leqslant k$, and $\phi_{[k]}(j) = 0$ if $j > k$. We denote $\phi^k := \phi - \phi_{[k]}$. We let Id denote the identity function from $\mathbb{R}$ onto $\mathbb{R}$, and we use $\gamma_1 \circ \gamma_2$ to denote the composition of two functions γ_1 and γ_2 which are from $\mathbb{R}$ to $\mathbb{R}$.

In this paper, *controls* or *inputs* are functions $u : \mathbb{Z}_+ \to \mathbb{R}^m$. For a given system, we often consider the same system but with controls restricted to take values in some subset $\Omega \subset \mathbb{R}^m$; we use $\mathscr{M}_\Omega$ to denote the set of controls taking values in Ω.

For each $\xi \in \mathbb{R}^n$ and each input u, we denote by $x(\cdot, \xi, u)$ the trajectory of system (1) with initial state $x(0) = \xi$ and the input u. Clearly such a trajectory is defined uniquely on $\mathbb{Z}_+$.

Recall that a function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathscr{K}$*-function* if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a $\mathscr{K}_\infty$*-function* if it is a $\mathscr{K}$-function and also $\gamma(s) \to \infty$ as $s \to \infty$; and it is a *positive definite* function if $\gamma(s) > 0$ for all $s > 0$, and $\gamma(0) = 0$. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathscr{KL}$*-function* if, for each fixed $t \geqslant 0$, the function $\beta(\cdot, t)$ is a $\mathscr{K}$-function, and for each fixed $s \geqslant 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.

3. Input-to-state stability properties

The main concern of this section is to understand the dependence of state trajectories on the magnitude of inputs for systems of the following type:

$$x(k+1) = f(x(k), u(k)), \tag{2}$$

where inputs $u(\cdot)$ are functions from $\mathbb{Z}_+$ to $\mathbb{R}^m$. We also assume that $f(0,0) = 0$, i.e., $\xi = 0$ is an equilibrium of the 0-input system.

3.1. ISS and ISS–Lyapunov functions

We first introduce the concepts of ISS and ISS–Lyapunov functions. Other equivalent notions of ISS will be given and demonstrated in the subsequent subsections.

Definition 3.1. System (2) is (*globally*) *input-to-state stable* (ISS) if there exist a $\mathcal{KL}$-function $\beta: \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ and a $\mathcal{K}$-function γ such that, for each input $u \in l_\infty^m$ and each $\xi \in \mathbb{R}^n$, it holds that

$$|x(k,\xi,u)| \leqslant \beta(|\xi|,k) + \gamma(\|u\|) \tag{3}$$

for each $k \in \mathbb{Z}_+$.

Note that, by causality, the same definition would result if one would replace (3) by

$$|x(k,\xi,u)| \leqslant \beta(|\xi|,k) + \gamma(\|u_{[k-1]}\|) \tag{4}$$

for every $k \in \mathbb{Z}_+ \setminus \{0\}$. Recall that $u_{[k-1]}$ denotes the truncation of u at $k-1$.

It can be seen from (4) that the ISS property implies that the 0-input system $x(k+1) = f(x(k),0)$ is globally asymptotically stable (GAS) and that (2) is "converging-input converging-state", i.e., every trajectory $x(k,\xi,u)$ goes to 0 if $u(k)$ goes to 0 as $k \to \infty$. However, the converse is not true. A simple example to consider is $x(k+1) = \frac{1}{2}(1 + \sin u(k))x(k)$. It is not hard to see that the system is 0-input GAS, and satisfies the converging-input converging-state property. But the system is not ISS, because for the constant input function $u(k) = \pi/2$, the trajectory $x(k,\xi,u)$ is identically ξ, which violates any estimation of type (3).

Definition 3.2. A continuous function $V: \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ is called an ISS–*Lyapunov function* for system (2) if the following holds:

1. There exist $\mathcal{K}_\infty$-functions α_1, α_2 such that

$$\alpha_1(|\xi|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|), \quad \forall \xi \in \mathbb{R}^n. \tag{5}$$

2. There exist a $\mathcal{K}_\infty$-function α_3 and a $\mathcal{K}$-function σ, such that

$$V(f(\xi,\mu)) - V(\xi) \leqslant -\alpha_3(|\xi|) + \sigma(|\mu|) \tag{6}$$

for all $\xi \in \mathbb{R}^n$, for all $\mu \in \mathbb{R}^m$.

A smooth ISS–Lyapunov function is one which is smooth.

Remark 3.3. As in the case of continuous time, Property 2 in the above definition is equivalent to the following property:

There exist some $\mathcal{K}_\infty$-function α_4 and some $\mathcal{K}$-function χ such that

$$\{|\xi| \geqslant \chi(|\mu|)\} \;\Rightarrow\; \{V(f(\xi,\mu)) - V(\xi) \leqslant -\alpha_4(|\xi|)\}. \tag{7}$$

It should be mentioned that it results in an equivalent property if the function α_4 in (7) is merely required to be continuous and positive definite. See Jiang and Wang (2001) for a detailed proof.

Example 3.4. As a simple illustration of these notions, we specialize (2) to linear discrete-time systems:

$$x(k+1) = Ax(k) + Bu(k) \tag{8}$$

where A is a Schur matrix, i.e., the eigenvalues of A are located strictly inside the unit disk. For such a matrix, there are constants $c > 0$ and $0 \leqslant \sigma < 1$ such that $|A^k| \leqslant c\sigma^k$ (cf. LaSalle, 1986, Chapter 5). From (8), we have

$$x(k+1) = A^{k+1}x(0) + \sum_{j=0}^{k} A^{k-j}Bu(j), \tag{9}$$

from which the ISS property (3) follows with

$$\beta(r,k) = c\sigma^k r, \quad \gamma(r) = \sum_{j=0}^{\infty} c\sigma^j |B| r = \frac{c|B|r}{1-\sigma}. \tag{10}$$

Next, we show that system (8) has a quadratic ISS–Lyapunov function. Given a symmetric and positive-definite matrix Q, let $P > 0$ be the unique solution to the matrix equation

$$A^{\mathrm{T}}PA - A = -Q.$$

Consider the positive-definite and radially unbounded function

$$V(x) = x^{\mathrm{T}}Px, \tag{11}$$

which satisfies property (5) with $\alpha_1(r) = \lambda_{\min}(P)r^2$ and $\alpha_2(r) = \lambda_{\max}(P)r^2$.

Direct computation shows

$$\begin{aligned} V(x(k+1)) - V(x(k)) &= -x^{\mathrm{T}}(k)Qx(k) + 2x^{\mathrm{T}}(k)A^{\mathrm{T}}PBu(k) \\ &\quad + u^{\mathrm{T}}(k)B^{\mathrm{T}}PBu(k). \end{aligned} \tag{12}$$

Then, by completing squares, property (6) holds with

$$\alpha_3(r) = \tfrac{1}{2}\lambda_{\min}(Q)r^2,$$

$$\sigma(r) = \left(\frac{2|A^{\mathrm{T}}PB|^2}{\lambda_{\min}(Q)} + |B^{\mathrm{T}}PB|^2\right)r^2.$$

Therefore, V defined in (11) is an ISS–Lyapunov function for (8). □

Clearly, if V is an ISS–Lyapunov function for (2), then V is a Lyapunov function for the 0-input system $x(k+1)=f(x(k),0)$. As in the classic Lyapunov stability theory, we can prove that *a system is* ISS *if and only if it admits an* ISS*–Lyapunov function*. See the main Theorem 1 below. We first demonstrate the sufficiency.

Lemma 3.5. *If system* (2) *admits a continuous ISS–Lyapunov function, then it is* ISS.

Proof. Assume that system (2) admits an ISS–Lyapunov function V. Let α_i $(i=1,2,3)$ and σ be as in (5) and (6). First observe that (6) can be rewritten as

$$V(f(\xi,\mu))-V(\xi)\leqslant -\alpha_4(V(\xi))+\sigma(|\mu|) \tag{13}$$

for all ξ and μ, where $\alpha_4=\alpha_3\circ\alpha_2^{-1}$. Without loss of generality, we assume that $\mathrm{id}-\alpha_4\in\mathscr{K}$ (cf. Lemma B.1). Fix a point $\xi\in\mathbb{R}^n$ and pick an input u. Let $x(k)$ denote the corresponding trajectory $x(k,\xi,u)$ of (2). Let ρ be any $\mathscr{K}_\infty$-function such that $\mathrm{id}-\rho\in\mathscr{K}_\infty$. Consider the set defined by

$$D=\{\xi: V(\xi)\leqslant b\}, \tag{14}$$

where $b=\alpha_4^{-1}\circ\rho^{-1}\circ\sigma(\|u\|)$.

Claim. *If there is some $k_0\in\mathbb{Z}_+$ such that $x(k_0)\in D$, then $x(k)\in D$ for all $k\geqslant k_0$.*

Proof. Assume that $x(k_0)\in D$. Then $V(x(k_0))\leqslant b$, that is, $\rho\circ\alpha_4(V(x(k_0)))\leqslant\sigma(\|u\|)$. By (13),

$$V(x(k_0+1))\leqslant(\mathrm{id}-\alpha_4)(V(x_0))+\sigma(\|u\|)$$

and since $\mathrm{id}-\alpha_4\in\mathscr{K}$, we have

$$\begin{aligned}V(x(k_0+1))&\leqslant(\mathrm{id}-\alpha_4)(b)+\sigma(\|u\|)\\&=-(\mathrm{id}-\rho)\circ\alpha_4(b)+b\\&\quad-\rho\circ\alpha_4(b)+\sigma(\|u\|)\\&\leqslant-(\mathrm{id}-\rho)\circ\alpha_4(b)+b\leqslant b.\end{aligned} \tag{15}$$

Using induction, one can show that $V(x(k_0+j))\leqslant b$ for all $j\in\mathbb{Z}_+$, that is, $V(x(k))\in D$ for all $k\geqslant k_0$. □

We now let $j_0=\min\{k\in\mathbb{Z}_+: x(k)\in D\}\leqslant\infty$. Then it follows from the above conclusion that $V(x(k))\leqslant\hat{\gamma}(\|u\|)$ for all $k\geqslant j_0$, where $\hat{\gamma}(r)=\hat{\alpha}_4^{-1}\circ\rho^{-1}\circ\sigma(r)$. For $k<j_0$, it holds that $\rho\circ\hat{\alpha}_4(V(x_k))>\sigma(\|u\|)$, and hence,

$$\begin{aligned}V(x(k+1))-V(x(k))&\leqslant-\alpha_4(V(x(k)))+\sigma(\|u\|)\\&=-(\mathrm{id}-\rho)\circ\alpha_4(V(x(k)))\\&\quad-\rho\circ\alpha_4(V(x(k)))+\sigma(\|u\|)\\&\leqslant-(\mathrm{id}-\rho)\circ\alpha_4(V(x(k))).\end{aligned} \tag{16}$$

By a standard comparison lemma (see e.g., Jiang & Wang, 2001), there exists some $\mathscr{KL}$-function $\hat{\beta}$ such that

$$V(x(k))\leqslant\hat{\beta}(V(x(0)),k)$$

for all $0\leqslant k\leqslant j_0+1$. Thus,

$$V(x(k))\leqslant\max\{\hat{\beta}(V(\xi),k),\hat{\gamma}(\|u\|)\},\quad\forall k\in\mathbb{Z}_+. \tag{17}$$

From this one gets (3) with $\beta(s,t)=\alpha_1^{-1}(\hat{\beta}(\alpha_2(|\xi|),t))$ and $\gamma(s)=\alpha_1^{-1}\circ\hat{\gamma}(s)$. □

Remark 3.6. Tracking the functions used in the proof, it can be seen that if (13) holds with some $\alpha_4\in\mathscr{K}_\infty$ such that $\mathrm{id}-\alpha_4\in\mathscr{K}$, then, for any $\rho\in\mathscr{K}_\infty$ so that $\mathrm{id}-\rho$ is of class $\mathscr{K}$, there is some $\hat{\beta}\in\mathscr{KL}$ such that (17) holds with $\hat{\gamma}=\alpha_4^{-1}\circ\rho^{-1}\circ\sigma$. This can be taken as an initial step in computing gain functions γ as in (3) from a given ISS-Lyapunov function.

Remark 3.7. Suppose (13) holds with some $\alpha_4\in\mathscr{K}_\infty$ such that $\mathrm{id}-\alpha_4\in\mathscr{K}$. If one lets $\rho=\mathrm{id}$, then, with $b=\alpha_4^{-1}\circ\sigma(\|u\|)$ and $x(k_0)\in D$, estimate (15) becomes

$$V(x(k_0+1))\leqslant b.$$

This shows that the set $D=\{\xi: V(\xi)\leqslant b\}$ is invariant. Still let $j_0=\min\{k\in\mathbb{Z}_+: x(k)\in D\}\leqslant\infty$. Then, for $k\geqslant j_0$, $V(x(k))\leqslant\alpha_4^{-1}\circ\sigma(\|u\|)$. For $k\leqslant j_0-1$, (16) yields

$$V(x(k+1))\leqslant V(x(k)).$$

Consequently, $V(x(k))\leqslant V(x(0))$ for all $0\leqslant k\leqslant j_0$. Thus, one gets the following estimate:

$$V(x(k,\xi,u))\leqslant\max\{V(\xi),\ \alpha_4^{-1}\circ\sigma(\|u\|)\}$$

for all $k\in\mathbb{Z}_+$, all $\xi\in\mathbb{R}^n$ and all u.

Also note that in the proof of Lemma 3.5, it is enough for $\mathrm{id}-\hat{\alpha}_4$ to be nondecreasing instead of being of class $\mathscr{K}$.

3.2. Asymptotic gains

Consider system (2). We say that the system has a $\mathscr{K}$-*asymptotic gain* if there exists some $\gamma_a\in\mathscr{K}$ such that

$$\overline{\lim_{k\to\infty}}\,|x(k,\xi,u)|\leqslant\gamma_a\left(\overline{\lim_{k\to\infty}}\,|u(k)|\right) \tag{18}$$

for all $\xi\in\mathbb{R}^n$.

We say that system (2) is *uniformly bounded input bounded state* (UBIBS) if bounded initial states and controls produce uniformly bounded trajectories, i.e., there exist two $\mathscr{K}$-functions σ_1 and σ_2 such that

$$\sup_k|x(k,\xi,u)|\leqslant\max\{\sigma_1(|\xi|),\sigma_2(\|u\|\},$$

$$\forall\xi\in\mathbb{R}^n,\ \forall u\in l_\infty^m. \tag{19}$$

Again, by the causality property of the system, the above is equivalent to $\sigma_1(s) \geqslant s$ and

$$|x(k,\xi,u)| \leqslant \max_{0 \leqslant j \leqslant k-1} \{\sigma_1(|\xi|), \sigma_2(|u(j)|)\},$$
$$\forall \xi \in \mathbb{R}^n,\ \forall u \in l_\infty^m,\ \forall k \geqslant 1. \tag{20}$$

Suppose that the system (2) is ISS. Without loss of generality, one may assume that the trajectories of the system satisfy the following:

$$|x(k,\xi,u)| \leqslant \max\{\beta(|\xi|,k), \gamma(\|u\|)\}, \quad \forall k \in \mathbb{Z}_+ \tag{21}$$

for some $\beta \in \mathscr{KL}$, some $\gamma \in \mathscr{K}$. Clearly, such an estimation implies UBIBS. Also note that, for any $k \geqslant l$,

$$|x(k,\xi,u)| \leqslant \max\{\beta(|x(l,\xi,u)|, k-l), \gamma(\|u^{l-1}\|)\}$$
$$\leqslant \max\{\beta(\beta(|\xi|,l),k-l), \beta(\gamma(\|u_{[l-1]}\|),k-l), \gamma(\|u^{l-1}\|)\}.$$

Letting $l = \lfloor k/2 \rfloor$, we get

$$|x(k,\xi,u)| \leqslant \max\{\tilde{\beta}(|\xi|,\lfloor k/2 \rfloor), \beta(\gamma(\|u_{[\lfloor k/2 \rfloor]}\|),$$
$$\lfloor k/2 \rfloor), \gamma(\|u^{\lfloor k/2-1 \rfloor}\|)\},$$

where $\tilde{\beta}(s,r) = \beta(\beta(s,r),r)$. From this one can see that

$$\overline{\lim_{k\to\infty}}\ |x(k,\xi,u)| \leqslant \gamma\left(\overline{\lim_{k\to\infty}}\ |u(k)|\right). \tag{22}$$

Hence we proved the following result:

Lemma 3.8. *Suppose system* (2) *is* ISS. *Then it is* UBIBS *and it admits a* $\mathscr{K}$*-asymptotic gain. Furthermore, if* (21) *holds, then the function* γ *can be taken as a* $\mathscr{K}$*-asymptotic gain of the system.*

The converse of the first statement in the above result is also true, see Theorem 1.

Remark 3.9. Suppose V is an ISS–Lyapunov function for the system satisfying an dissipation inequality (13) for some $\alpha_4 \in \mathscr{K}_\infty$ and $\sigma \in \mathscr{K}$. Suppose also that $\mathrm{Id} - \alpha_4 \in \mathscr{K}_\infty$. By Remark 3.6, for any $\rho \in \mathscr{K}_\infty$ such that $\mathrm{Id} - \rho \in \mathscr{K}_\infty$, there is some $\beta \in \mathscr{KL}$ such that (17) holds with $\hat{\gamma} = \alpha_4^{-1} \circ \rho^{-1} \circ \sigma$. Applying the same argument in deriving (22) from (21), one can show that $\hat{\gamma}$ is also a $\mathscr{K}$-asymptotic gain for $V(x(k,\xi,u))$, that is,

$$\overline{\lim_{k\to\infty}}\ V(x(\xi,k,u)) \leqslant \alpha_4^{-1} \circ \rho^{-1} \circ \sigma\left(\overline{\lim_{k\to\infty}}\ |u(k)|\right) \tag{23}$$

for all ξ and all u.

3.3. Robust stability margins

As in the case of continuous time systems, there is also an interesting connection between the ISS and the robust stability. Throughout this section, we let $\bar{B}$ denote the closed unit ball of $\mathbb{R}^m$.

Let ρ be any $\mathscr{K}_\infty$ function. Consider the system

$$x(k+1) = f(x(k), d(k)\rho(|x(k)|)) := g(x(k), d(k)), \tag{24}$$

where $d \in \mathscr{M}_{\bar{B}}$. We view the signals $d(\cdot)$ as disturbances. For such systems, there are natural definitions of global asymptotical stability (GAS) and uniform global asymptotical stability (UGAS), see Definition A.2.

We will say that system (2) is *robustly stable* if there exists a $\mathscr{K}_\infty$ function ρ (called a *stability margin*) such that system (24) is UGAS. Note that for a nonlinear GAS system, in general only small perturbations can be tolerated while preserving stability. The requirement $\rho \in \mathscr{K}_\infty$ is thus nontrivial.

Assume now system (2) is robustly stable. Let $\rho \in \mathscr{K}_\infty$ be the stability margin. This means that the corresponding system (24) is UGAS. By the Converse Lyapunov Theorem 5 presented in the appendix (also see Jiang & Wang, 2001), it follows that there exists some smooth function V such that

$$\alpha_1(\xi) \leqslant V(\xi) \leqslant \alpha_2(|\xi|) \quad \forall \xi \in \mathbb{R}^n, \tag{25}$$

holds for some $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$; and furthermore,

$$V(f(\xi, \mu\rho(|\xi|))) - V(\xi) \leqslant -\alpha_3(|\xi|),$$
$$\forall \xi \in \mathbb{R}^n,\ \forall |\mu| \leqslant 1 \tag{26}$$

for some $\mathscr{K}_\infty$-function α_3. Observe that this is equivalent to the following:

$$|v| \leqslant \rho(|\xi|) \ \Rightarrow\ V(f(\xi,v)) - V(\xi) \leqslant -\alpha_3(|\xi|). \tag{27}$$

It then follows from Remark 3.3 that V is a smooth ISS–Lyapunov function for system (2). Thus, one obtains the following result:

Lemma 3.10. *Suppose that system* (2) *is robustly stable. Then it admits a smooth* ISS*–Lyapunov function.*

The following is a key lemma in obtaining our main result given in Section 3.4. Its proof is analogous to the proof of the corresponding continuous time result in Sontag and Wang (1996, Section V), but we provide the details nonetheless, since the lemma is essential in obtaining our main results concerning the ISS property.

Lemma 3.11. *If system* (2) *is* UBIBS *and if it admits a* $\mathscr{K}$*-asymptotic gain, then it is robustly stable.*

Proof. Assume that the system is UBIBS with σ_1, σ_2 as in (19). Let γ be a $\mathscr{K}$-asymptotic gain for the system. Without loss of generality, we assume that $\sigma_2 = \gamma$ and $\sigma_1(s) \geqslant s$ for all $s \geqslant 0$. Pick any $\mathscr{K}_\infty$-function ρ such that

$$\gamma(\rho(s)) \leqslant s/2, \quad \forall s \geqslant 0.$$

Below we will show that with such a choice of ρ, the system defined by

$$x(k+1) = f(x(k), d(k)\rho(|x(k)|) \tag{28}$$

is GAS. Recall that $d \in \mathcal{M}_{\bar{B}}$.

Pick any $\xi \in \mathbb{R}^n$, $d \in \mathcal{M}_{\bar{B}}$. Let $x_\rho(k)$ denote the corresponding trajectory of (28).

Claim. *$\sigma_2(\rho(|x_\rho(k)|)) \leqslant 2\sigma_1(|\xi|)/3$ for all $k \geqslant 0$.*

Proof. First notice that the claim is true if $\xi = 0$ since both sides vanish. Assume now that $\xi \neq 0$. Again observe that the claim is true for $k = 0$, because $\sigma_2(\rho(|x_\rho(0)|)) = \gamma(\rho(|x_\rho(0)|)) \leqslant |\xi|/2 \leqslant \sigma_1(|x_\rho(0)|)/2$. Let

$$k_1 = \min\left\{k \in \mathbb{Z}_+ : \sigma_2(\rho(|x_\rho(k)|) \geqslant \frac{2\sigma_1(|\xi|)}{3}\right\}.$$

Then $k_1 > 0$. Suppose that the claim is false, and hence, $k_1 < \infty$. Then for $0 \leqslant k \leqslant k_1 - 1$, it holds that $\sigma_2(\rho(|x_\rho(k)|)) \leqslant 2\sigma_1(|\xi|)/3$, and hence, $\gamma(|d(k)\rho(|x_\rho(k)|)|) \leqslant 2\sigma_1(|\xi|)/3$ for all $0 \leqslant k \leqslant k_1 - 1$. It then follows from (20) that

$$\begin{aligned}|x_\rho(k_1)| &\leqslant \max_{0 \leqslant j \leqslant k_1 - 1} \{\sigma_1(|\xi|), \sigma_2(|d(j)\rho(|x_\rho(j)|)|)\} \\ &\leqslant \sigma_1(|\xi|),\end{aligned} \tag{29}$$

which, in turn, implies that $\sigma_2(\rho(|x_\rho(k_1)|)) \leqslant |x_\rho(k_1)|/2 \leqslant \sigma_1(|\xi|)/2 < 2\sigma_1(|\xi|)/3$. This contradicts the definition of k_1. This shows that $k_1 = \infty$, i.e., the claim is true.

An immediate consequence of the claim is that (29) holds for all $k \in \mathbb{Z}_+$, and that $\overline{\lim}_{k\to\infty} |x_\rho(k)|$ is finite for each trajectory. Taking the limits on both sides of (18), one sees that

$$\overline{\lim_{k\to\infty}} |x_\rho(k)| \leqslant \overline{\lim_{k\to\infty}} \gamma(|d(k)\rho(|x_\rho(k)|)|) \leqslant \overline{\lim_{k\to\infty}} |x_\rho(k)|/2.$$

It then follows that $\lim_{k\to\infty} |x_\rho(k)| = 0$ for each trajectory. This shows that system (28) is GAS, which, together with Theorem 5 in the appendix, implies that the system is UGAS. □

3.4. Equivalent characterizations of ISS

In this section, we present some equivalence relations among various notions.

Theorem 1. *Consider system* (2). *The following are equivalent*:

1. *It is* ISS.
2. *It is* UBIBS *and it admits a $\mathcal{K}$-asymptotic gain.*
3. *It is robustly stable.*
4. *It admits a smooth* ISS*–Lyapunov function.*

Proof. We have: [1 ⇒ 2] (clear by Definitions), [2 ⇒ 3] (see Lemma 3.11), [3 ⇒ 4] (see Lemma 3.10), [4 ⇒ 1] (see Lemma 3.5). □

Remark 3.12. A subset $\mathcal{A}$ of $\mathbb{R}^n$ is 0-input (forward) invariant for system (2) if it is (forward) invariant for the corresponding 0-input system $x(k+1) = f(x(k), 0)$, that is, if $x(0) \in A$ then $x(k) \in \mathcal{A}$ for all $k \in \mathbb{Z}_+$. As in the continuous time case, one may also define ISS and UGAS with respect to such a closed invariant set $\mathcal{A}$. It is routine to generalize Theorem 1 to the case when $\mathcal{A}$ is compact, where for statement 3, robust stability means the existence of a smooth $\rho \in \mathcal{K}_\infty$ such that the system

$$x(k+1) = f(x(k), d(k)\rho(|x(k)|_{\mathcal{A}})),$$

where $d(\cdot) \in \mathcal{M}_{\bar{B}}$, is UGAS with respect to $\mathcal{A}$.

The case when $\mathcal{A}$ is not compact is slightly trickier. Still, it can be shown that in that case statements 1, 3, and 4 are equivalent; and it is not hard to see that statement 2 is indeed strictly weaker than the other three statements. Even in the simple case when $u = 0$, the UBIBS property in combination with a $\mathcal{K}$-asymptotic gain reduces to the GAS property with respect to $\mathcal{A}$, which is in general strictly weaker than the UGAS property with respect to $\mathcal{A}$.

3.5. Remarks about gain functions

As it is well recognized in the literature, the notion of ISS is a natural nonlinear extension of classical *finite gain* stability (Desoer & Vidyasagar, 1975) in that only linear incremental gains are considered. That is why we refer γ in the ISS property (3) to as an ISS-*gain*. Naturally, we raise the question of how to compute such a gain function for a given ISS system (2).

As in the continuous-time case (Sontag, 1989), a similar ISS algorithm can be developed on the basis of an ISS–Lyapunov function V satisfying (5) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and the difference dissipation inequality

$$\begin{aligned}&V(x(k+1, \xi, u)) - V(x(k, \xi, u)) \\ &\quad \leqslant -\alpha(V(x(k, \xi, u))) + \sigma(\|u\|)\end{aligned} \tag{30}$$

for some $\alpha \in \mathcal{K}_\infty$, some $\sigma \in \mathcal{K}$. According to Lemma B.1, we assume, without loss of generality, that $\mathrm{Id} - \alpha \in \mathcal{K}$. By Remark 3.6, one sees that, for any $\mathcal{K}_\infty$-function ρ such that $\mathrm{Id} - \rho \in \mathcal{K}$, the function $\alpha_1^{-1} \circ \alpha^{-1} \circ (\mathrm{Id} + \rho) \circ \sigma(s)$ can be taken as an ISS-gain function for the system. That is, there is a $\mathcal{KL}$-function β such that

$$|x(k, \xi, u)| \leqslant \beta(|\xi|, k) + \alpha_1^{-1} \circ \alpha^{-1} \circ (\mathrm{Id} + \rho) \circ \sigma(\|u\|)$$

for all $k \in \mathbb{Z}_+$, $\xi \in \mathbb{R}^n$ and all u.

Lemma 3.13. *Suppose that system* (2) *admits an* ISS*–Lyapunov function satisfying* (5) *with some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and* (30) *for some $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$. Assume further that* id $-\ \alpha\in\mathcal{K}$. *Then it holds that*

$$\overline{\lim_{k\to\infty}}\ V(x(k,\xi,u))\leqslant\alpha^{-1}\circ\sigma\left(\overline{\lim_{k\to\infty}}\ |u(k)|\right) \tag{31}$$

for all $\xi\in\mathbb{R}^n$ *and all u. Consequently*, $\gamma_a(s):=\alpha_1^{-1}\circ\alpha^{-1}\circ\sigma(s)$ *is a* $\mathcal{K}$*-asymptotic gain such that* (18) *holds.*

Proof. Indeed, by Remark 3.9, for any $0<c<1$, it holds that

$$\overline{\lim_{k\to\infty}}\ V(x(\xi,k,u))\leqslant\alpha^{-1}\left(\frac{1}{c}\sigma\left(\overline{\lim_{k\to\infty}}\ |u(k)|\right)\right)$$

for all ξ and all u. For any given ξ and any u, letting $c\to 1$, we get (31). □

4. Nonlinear small-gain theorems

In this section, we will discuss the ISS property for interconnected and nonlinear discrete-time systems

$$x_1(k+1)=f_1(x_1(k),v_1(k),u_1(k)),$$

$$x_2(k+1)=f_2(x_2(k),v_2(k),u_2(k)), \tag{32}$$

subject to the interconnection constraints

$$v_1(k)=x_2(k),\quad v_2(k)=x_1(k), \tag{33}$$

where for $i=1,2$ and for each $k\in\mathbb{Z}_+$, $x_i(k)\in\mathbb{R}^{n_i}$, $u_i(k)\in\mathbb{R}^{m_i}$, and f_i is continuous in its arguments.

In the continuous-time case, the ISS small gain theorems are very powerful in treating stability and stabilization problems for such interconnected systems. The first such small gain theorem for nonlinear systems was obtained in Jiang et al. (1994). In Coron et al. (1994), the small gain theorem was derived by a much simpler proof in conjunction with the results in Sontag and Wang (1996). In Jiang et al. (1996), a small gain theorem was presented in terms of Lyapunov functions. In this section, we will follow the continuous-time approach used in Coron et al. (1994, Section 4) to derive an ISS small-gain theorem for discrete-time nonlinear systems. In particular, we show that Theorems 40 and 44 of Coron et al. (1994) hold in the discrete-time setting as well. In contrast to the nonlinear small-gain theorem of Mareels and Hill (1992) our ISS small-gain theorem takes into account the role of initial conditions and offers a result on the internal asymptotic stability for the interconnected discrete-time system (32).

We first introduce a useful technical result that is instrumental in the development of our discrete ISS nonlinear small-gain theorems. The continuous analogue of this result was first established in Jiang et al. (1994) under an assumption of observability. Since the completeness of solutions is not an issue in the discrete time case, we do not need the observability assumption in our context.

For notational simplicity, we use $x_i(k)$, $i=1,2$, to denote the trajectories of each individual x_i-system in (32) with inputs (v_i,u_i). If no ambiguity is possible, we also use $x(k)=(x_1(k),x_2(k))$ to mean the solutions of the interconnected system (32) subject to (33) and with inputs $u=(u_1,u_2)$.

Lemma 4.1. *Let* $h_1:\mathbb{R}^{n_1}\to\mathbb{R}^{p_1}$ *and* $h_2:\mathbb{R}^{n_2}\to\mathbb{R}^{p_2}$ *be two continuous mappings. Assume that the following* ISS*-like properties hold for the solutions of each subsystem in* (32)

$$|h_1(x_1(k))|\leqslant\max\{\beta_1(|\xi_1|,k),\gamma_1^x(\|h_2(v_1)\|),\gamma_1^u(\|u_1\|)\},$$

$$|h_2(x_2(k))|\leqslant\max\{\beta_2(|\xi_2|,k),\gamma_2^x(\|h_1(v_2)\|),\gamma_1^u(\|u_2\|)\}.$$

If $\gamma_2^x\circ\gamma_1^x(s)<s$ (*equivalently*, $\gamma_1^x\circ\gamma_2^x(s)<s$) *for all* $s>0$, *then, for the composite system* (32) *subject to the interconnection constraint* (33), *for all initial condition* $\xi=(\xi_1,\xi_2)$ *and all u, the trajectories* $(x_1(k),x_2(k))$ *satisfy the following properties*:

- *there exist some* σ, $\gamma\in\mathcal{K}$ *such that*

$$|(h_1(x_1(k,\xi,u)),h_2(x_2(k,\xi,u)))|\leqslant\max\{\sigma(|\xi|),\gamma(\|u\|)\}\quad\forall k\geqslant 0 \tag{34}$$

- *there exists some* $\lambda\in\mathcal{K}_\infty$ *such that*

$$\overline{\lim_{k\to\infty}}\ |(h_1(x_1(k)),h_2(x_2(k)))|\leqslant\overline{\lim_{k\to\infty}}\ \lambda(|u(k)|). \tag{35}$$

Proof. We first establish the boundedness property (34). Pick any initial condition $\xi=(\xi_1,\xi_2)$ and any input $u=(u_1,u_2)$, by causality, the corresponding solution $x(k)=(x_1(k),x_2(k))$ of (32) and (33) satisfies

$$|h_1(x_1(k))|\leqslant\max\{\beta_1(|\xi_1|,k),\gamma_1^x(\|(h_2(x_2))_{[k-1]}\|),\gamma_1^u(\|u_1\|)\}, \tag{36}$$

$$|h_2(x_2(k))|\leqslant\max\{\beta_2(|\xi_2|,k),\gamma_2^x(\|(h_1(x_1))_{[k-1]}\|),\gamma_2^u(\|u_2\|)\}, \tag{37}$$

which implies that

$$\|(h_1(x_1))_{[k]}\|\leqslant\max\{\beta_1(|\xi_1|,0),\gamma_1^x(\beta_2(|\xi_2|,0)),\gamma_1^x\circ\gamma_2^x(\|(h(x_1))_{[k]}\|),\gamma_1^x\circ\gamma_2^u(\|u_2\|),\gamma_1^u(\|u_1\|)\}. \tag{38}$$

Then, the small-gain condition, i.e., $\gamma_1^x\circ\gamma_2^x(s)<s$, implies

$$\|(h_1(x_1))_{[k]}\|\leqslant\max\{\beta_1(|\xi_1|,0),\gamma_1^x(\beta_2(|\xi_2|,0)),\gamma_1^x\circ\gamma_2^u(\|u_2\|),\gamma_1^u(\|u_1\|)\}\leqslant\max\{\sigma_1(|\xi|),\gamma_1(\|u\|)\} \tag{39}$$

for all $k\geqslant 0$, where $\sigma_1(s)=\max\{\beta_1(s,0),\gamma_1^x(\beta_2(s,0))\}$, and $\gamma(s)=\max\{\gamma_1^x\circ\gamma_2^u(s),\gamma_1^u(s)\}$. Similarly, one can show that, for each $k \geqslant 0$,

$$\|(h_2(x_2)_{[k]})\| \leqslant \max\{\sigma_2(|\xi|), \gamma_2(\|u\|)\} \tag{40}$$

for some $\sigma_2, \gamma_2 \in \mathcal{K}$. The combination of (39) and (40) then yields (34) with $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$ and $\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$. Observe that σ and γ are both independent of the choices of ξ and u.

Next, we establish (35). By (39) and (40), one sees that there exists some $c > 0$ (depending on $|\xi|$ and $\|u\|$) such that

$$|h_1(x_1(k))| \leqslant c \quad \text{and} \quad |h_2(x_2(k))| \leqslant c, \tag{41}$$

for all k, and hence, both $\overline{\lim}_{k\to\infty} |h_1(x_1(k))|$ and $\overline{\lim}_{k\to\infty} |h_2(x_2(k))|$ are finite. Applying (36) with $x_1(\lfloor k/2 \rfloor)$ as the initial state of x_1, we g

$$|h_1(x_1(k))| \leqslant \max\{\beta(|x_1(\lfloor k/2 \rfloor)|, \lfloor k/2 \rfloor), \gamma_1^x(\|(h_2(x_2))^{\lfloor k/2-1 \rfloor}\|), \gamma_1^u(\|u^{\lfloor k/2-1 \rfloor}\|)\}$$

This together with (41) yields

$$|h_1(x_1(k))| \leqslant \max\{\beta(c, \lfloor k/2 \rfloor), \gamma_1^x(\|(h_2(x_2))^{\lfloor k/2-1 \rfloor}\|), \gamma_1^u(\|u^{\lfloor k/2-1 \rfloor}\|)\}.$$

Taking limits in the above, we conclude that

$$\overline{\lim_{k\to\infty}}\ |h_1(x_1(k))| \leqslant \max\left\{\gamma_1^x\left(\overline{\lim_{k\to\infty}}\ |h_2(x_2(k))|\right), \gamma_1^u\left(\overline{\lim_{k\to\infty}}\ |u_1(k)|\right)\right\}.$$

Similarly, we have

$$\overline{\lim_{k\to\infty}}\ |h_2(x_2(k))| \leqslant \max\left\{\gamma_2^x\left(\overline{\lim_{k\to\infty}}\ |h_1(x_1(k))|\right), \gamma_2^u\left(\overline{\lim_{k\to\infty}}\ |u_2(k)|\right)\right\}.$$

Combining the fact that both $\overline{\lim}_{k\to\infty} |h_1(x_1(k))|$ and $\overline{\lim}_{k\to\infty} |h_2(x_2(k))|$ are well defined and the fact that $\gamma_1^x \circ \gamma_2^x(s) < s$ for each $s > 0$, one concludes that

$$\overline{\lim_{k\to\infty}}\ |h_1(x_1(k))| \leqslant \max\left\{\overline{\lim_{k\to\infty}}\ \gamma_1^x \circ \gamma_2^u(|u_2(k)|), \overline{\lim_{k\to\infty}}\ \gamma_1^u(|u_1(k)|)\right\},$$

$$\overline{\lim_{k\to\infty}}\ |h_2(x_2(k))| \leqslant \max\left\{\overline{\lim_{k\to\infty}}\ \gamma_2^x \circ \gamma_1^u(|u_1(k)|), \overline{\lim_{k\to\infty}}\ \gamma_2^u(|u_2(k)|)\right\}.$$

The asymptotic gain condition (35) then follows for some $\gamma \in \mathcal{K}$. □

Now, we are ready to state and prove our first discrete ISS small-gain theorem.

Theorem 2. *Suppose both the subsystems in* (32) *are* ISS *in the sense that*

$$|x_1(k, \xi, v_1, u_1)| \leqslant \max\{\beta_1(|\xi_1|, k), \gamma_1^x(\|v_1\|), \gamma_1^u(\|u_1\|)\},$$

$$|x_2(k, \xi, v_2, u_2)| \leqslant \max\{\beta_2(|\xi_2|, k), \gamma_2^x(\|v_2\|), \gamma_1^u(\|u_2\|)\}.$$

If $\gamma_1^x \circ \gamma_2^x(s) < s$ (equivalently, $\gamma_2^x \circ \gamma_1^x(s) < s$) for all $s > 0$, then the interconnected system (32) *and* (33) *is* ISS *with* (u_1, u_2) *as input.*

Proof. It follows immediately from Lemma 4.1 by letting h_1 and h_2 be the identity functions. Indeed, in this case, (34) yields the UBIBS condition; and (35) yields the finite $\mathcal{K}$-asymptotic gain condition. □

The above small-gain theorem can also be stated in terms of ISS–Lyapunov functions. Assume that both the subsystems in (32) are ISS. Let V_1 and V_2 be ISS–Lyapunov functions for the x_1 and x_2-subsystem of (32), respectively. That is, there exist class $\mathcal{K}_\infty$-functions α_{ij}, σ_i, ρ_i^x and ρ_i^u ($1 \leqslant i,j \leqslant 2$) such that

$$\alpha_{i1}(|\xi|) \leqslant V_i(\xi) \leqslant \alpha_{i2}(|\xi|) \tag{42}$$

and

$$V_1(f_1(\xi_1, v_1, \mu_1)) - V_1(\xi_1) \leqslant -\sigma_1(V_1(\xi_1)) + \rho_1^x(V_2(v_1)) + \rho_1^u(|\mu_1|), \tag{43}$$

$$V_2(f_2(\xi_2, v_2, \mu_2)) - V_2(\xi_2) \leqslant -\sigma_2(V_2(\xi_2)) + \rho_2^x(V_1(v_2)) + \rho_2^u(|\mu_2|). \tag{44}$$

In view of Lemma B.1, we may assume that $\mathrm{Id} - \sigma_i \in \mathcal{K}$ for $i = 1,2$.

Theorem 3. *Assume that x_1- and x_2-subsystems of* (32) *admit* ISS*–Lyapunov functions V_1 and V_2 respectively that satisfy* (43) *and* (44), *with $\mathrm{Id} - \sigma_i \in \mathcal{K}$ for $i = 1,2$. If there exists a $\mathcal{K}_\infty$-function ρ such that*

$$\sigma_1^{-1} \circ (\mathrm{Id} + \rho) \circ \rho_1^x \circ \sigma_2^{-1} \circ (\mathrm{Id} + \rho) \circ \rho_2^x < \mathrm{Id}, \tag{45}$$

then the interconnected system (32) *and* (33) *is* ISS *with* (u_1, u_2) *as the input.*

Proof. By Remark 3.6, for any $\mathcal{K}_\infty$-function η such that $\mathrm{Id} - \eta \in \mathcal{K}$, there exist $\beta_1, \beta_2 \in \mathcal{KL}$ such that, for $1 \leqslant i, j \leqslant 2$ and $j \neq i$, it holds that

$$V_i(x_i(k, \xi_i, v_i, u_i)) \leqslant \max\{\beta_i(V_i(\xi_i), k), \tilde{\gamma}_i(\rho_i^x(\|V_j(v_i)\|) + \rho_i^u(\|u_i\|))\},$$

where $\tilde{\gamma}_i(s) = \sigma_i^{-1} \circ (\mathrm{Id} + \eta)(s)$, $i = 1, 2$. Observe that, for any $\mathcal{K}$-function γ and for any $\mathcal{K}_\infty$-function χ, $\gamma(r + s) \leqslant \max\{\gamma \circ (\mathrm{Id} + \chi)(r), \gamma \circ (\mathrm{Id} + \chi^{-1})(s)\}$. Then, for $1 \leqslant i, j \leqslant 2$ and $j \neq i$, it follows that

$$\begin{aligned} &V_i(x_i(k, \xi_i, v_i, u_i)) \\ &\quad\leqslant \max\{\beta_i(V_i(\xi_i), k), \sigma_i^{-1} \circ (\mathrm{Id} + \eta) \circ (\mathrm{Id} + \chi) \\ &\qquad \circ \rho_i^x(\|V_j(v_i)\|), \sigma_i^{-1} \circ (\mathrm{Id} + \eta) \\ &\qquad \circ (\mathrm{Id} + \chi^{-1}) \circ \rho_i^u(\|u_i\|)\}. \end{aligned} \tag{46}$$

Pick $\eta, \chi \in \mathcal{K}_\infty$ so that $(\mathrm{Id} + \eta) \circ (\mathrm{Id} + \chi)(s) \leqslant (\mathrm{Id} + \rho)(s)$ for all $s \geqslant 0$. Noticing the relation (33), and applying Lemma 4.1 to (32) with $h_i(x_i) = V_i(x_i)$, and $\gamma_i^x = \sigma_i^{-1} \circ (\mathrm{Id} + \rho) \circ \rho_i^x$ for $i = 1,2$, one concludes that there exist some $\sigma, \gamma, \lambda \in \mathcal{K}$ such that along the trajectories of the interconnected system (32) and (33), it holds that, for $i = 1,2$,

$$V_i(x_i(k, \xi, u)) \leqslant \max\{\sigma(|\xi|), \gamma(\|u\|)\}, \quad \forall k,\ \forall \xi,\ \forall u$$

and

$$\overline{\lim_{k \to \infty}}\ V_i(x_i(k,\xi, u)) \leqslant \lambda(\|u\|), \quad \forall k,\ \forall \xi,\ \forall u.$$

With property (42), it follows that system (32) and (33) is UBIBS and admits a $\mathcal{K}$-asymptotic gain, and hence, it is ISS.

When $(u_1, u_2) = (0,0)$ in (32), the small-gain condition (45) can be weakened as follows.

Corollary 4.2. *Consider the interconnected discrete-time system*

$$x_1(k + 1) = f_1(x_1(k), x_2(k)), \tag{47}$$

$$x_2(k + 1) = f_2(x_2(k), x_1(k)). \tag{48}$$

Assume that both x_1-subsystem (47) *and x_2-subsystem* (48) *possess* ISS*–Lyapunov functions in the sense that*

$$V_1(f_1(\xi_1, \xi_2)) - V_1(\xi_1) \leqslant -\sigma_1(V_1(\xi_1)) + \rho_1^x(V_2(\xi_2)), \tag{49}$$

$$V_2(f_2(\xi_2, \xi_1)) - V_2(\xi_2) \leqslant -\sigma_2(V_2(\xi_2)) + \rho_2^x(V_1(\xi_1)), \tag{50}$$

with σ_i satisfying that $\mathrm{Id} - \sigma_i \in \mathcal{K}$. If the following small gain condition holds

$$\sigma_1^{-1} \circ \rho_1^x \circ \sigma_2^{-1} \circ \rho_2^x < \mathrm{Id}, \tag{51}$$

then the interconnected system (47) *and* (48) *is* GAS *at the origin.*

Proof. Let $\chi_1 = \sigma_1^{-1} \circ \rho_1^x$, $\chi_2 = \sigma_2^{-1} \circ \rho_2^x$. By Remark 3.7, one sees that

$$V_1(x_1(k)) \leqslant \max\{V_1(x_1(0)), \chi_1(\|V_2(x_2)_{[k-1]}\|)\},$$

$$V_2(x_2(k)) \leqslant \max\{V_2(x_2(0)), \chi_2(\|V_1(x_1)_{[k-1]}\|)\}.$$

By the same argument used as in deriving (39) and (40) from (38), one shows that if $(\chi_1 \circ \chi_2)(s) < s$, then

$$|(V_1(x_1(k)), V_2(x_2(k)))| \leqslant |(V_1(x_1(0)), V_2(x_2(0)))|$$

for all $k \in \mathbb{Z}_+$. By Lemma 3.13, an asymptotic gain from $V_2(x_2)$ to $V_1(x_1)$ for (47) is χ_1, and an asymptotic gain from $V_1(x_1)$ to $V_2(x_2)$ for (48) is χ_2. That is,

$$\begin{aligned} \overline{\lim_{k \to \infty}}\ V_1(x_1(k)) &\leqslant \chi_1\left(\overline{\lim_{k \to \infty}}\ V_2(x_2(k))\right) \\ &\leqslant \chi_1 \circ \chi_2\left(\overline{\lim_{k \to \infty}}\ V_1(x_1(k))\right). \end{aligned}$$

Again, the assumption that $\chi_1 \circ \chi_2 < \mathrm{Id}$ implies that

$$\overline{\lim_{k \to \infty}}\ V_1(x_1(k)) = \overline{\lim_{k \to \infty}}\ V_2(x_2(k)) = 0.$$

Thus, the system is GAS. □

Example 4.3. As an elementary application of Theorem 2, let us consider a cascade nonlinear discrete-time system of the form

$$z(k + 1) = q(z(k), x(k), u(k)), \tag{52}$$

$$x(k + 1) = g(x(k), v(k)). \tag{53}$$

Assume that z-system (52) is ISS with (x, u) as input, and let γ_1^x be an ISS-gain with respect to x. Let x-system (53) be ISS with v as input.

Clearly, $\gamma_2^z \equiv 0$ is an ISS-gain function with respect to z for the x-system since z does not affect the x-trajectories. Thus, the small-gain condition holds between γ_1^x and γ_2^z. As a direct consequence of Theorem 2, the cascade system (52) and (53) is ISS with (u, v) as input.

5. Input-to-state stabilizability

Consider system (2). We say that the system is *continuously stabilizable* if there is a continuous function $w: \mathbb{R}^n \to \mathbb{R}^m$ with $w(0) = 0$ such that under the feedback $u = w(x)$, the closed-loop system

$$x(k + 1) = f(x(k), w(x(k)))$$

is GAS.

We say that system (2) is *continuously* ISS *stabilizable* if there exist a continuous map $w: \mathbb{R}^n \to \mathbb{R}^m$ with $w(0) = 0$

and an $n\times n$ matrix Γ of continuous functions, invertible for each x, such that under the control law

$$u = w(x) + \Gamma(x)v,$$

the closed-loop system

$$x(k+1) = f(x(k), w(x(k)) + \Gamma(x(k))v(k)) \tag{54}$$

is ISS (with v as the new input).

Clearly, if a system is ISS-stabilizable, then it is stabilizable. The following result shows that the two types of stabilizability are equivalent.

Theorem 4. *System* (2) *is continuously stabilizable if and only if it is* ISS-*stabilizable*.

Proof. It is enough to show that stabilizability implies ISS-stabilizability. Assume that the system is stabilized under the continuous feedback law $u = w(x)$ with $w(0) = 0$, i.e., the system

$$x(k+1) = f(x(k), w(x(k))) \tag{55}$$

is GAS. Applying Theorem 5 to (55) one knows that there is a Lyapunov function V satisfying

$$V(f(\xi, w(\xi))) - V(\xi) < -\alpha(|\xi|), \quad \forall \xi \in \mathbb{R}^n,$$

for some positive definite function α. Define the continuous function $\delta: \mathbb{R}_{\geqslant 0}\times\mathbb{R}_{\geqslant 0} \to \mathbb{R}$ by

$$\delta(s,r) := \max_{|\xi| = s,\ |\mu| = r} \{V(f(\xi, w(\xi) + \mu)) - V(\xi) + \alpha(|\xi|)\}.$$

Note then that for every $s > 0$, $\delta(s,0) < 0$. By Lemma 3.1 in Sontag (1990), there exist a $\mathscr{K}_\infty$-function χ and a smooth function $g: \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{>0}$ with $g(s) = 1$ for all $s \in [0,1]$ such that

$$\delta(s, g(s)r) < 0 \quad \text{whenever } s \geqslant \chi(r).$$

Let now $\Gamma(\xi) = g(|\xi|)I_{m\times m}$. We would like to show that with such a choice of Γ, the corresponding closed-loop system (54) is ISS. For this purpose, we note that whenever $|\xi| \geqslant \chi(|\mu|)$,

$$V(f(\xi, w(\xi) + \Gamma(\xi)\mu)) - V(\xi) + \alpha(|\xi|) \leqslant \delta(|\xi|, |\mu|) < 0,$$

that is,

$$|\xi| \geqslant \chi(|\mu|) \Rightarrow V(f(\xi, w(\xi) + \Gamma(\xi)\mu)) - V(\xi) < -\alpha(|\xi|).$$

By Remark 3.3, V is an ISS–Lyapunov function for (54). Therefore, system (54) is ISS. □

Upon specialization of (2) to a linear system

$$x(k+1) = Ax(k) + Bu(k), \tag{56}$$

we obtain a checkable necessary and sufficient condition for ISS-stabilizability.

Corollary 5.1. *System* (56) *is* ISS-*stabilizable if and only if* (A, B) *is stabilizable, i.e. there is a matrix* K *so that the eigenvalues of* $A + BK$ *are inside the unit disk.*

For the continuous time case, Corollary 5.1 can be extended to nonlinear systems affine in controls. It was shown in Sontag (1989) that if an affine system $\dot{x} = f(x) + G(x)u$ is stabilizable, then there is some feedback $u = K(x)$ such that the closed-loop system $\dot{x} = f(x) + G(x)(K(x) + w)$ is Iss with respect to w. In contrast to the continuous time situation, this result fails in the discrete time case. As an illustration, we consider a one-dimensional discrete-time system

$$x(k+1) = x^3(k) + (|x(k)| + 1)u. \tag{57}$$

With the feedback $u = -x^3/(|x| + 1)$, the closed-loop system is GAS. In fact, the closed-loop system is even dead beat, that is, all trajectories reaches the origin in one step. On the other hand, there is no continuous feedback $u = K(x) + w$ for which the closed-loop system is ISS with respect to w. This can be seen as follows.

The closed-loop system of (57) with $u = K(x) + w$ is

$$x(k+1) = G(x(k)) + (|x(k)| + 1)w(k), \tag{58}$$

where $G(x(k)) = x^3(k) + (|x(k)| + 1)K(x(k))$. Now consider the signal $w(k)$ given by:

$$w(k) = \begin{cases} 1 & \text{if } G(x(k)) \geqslant 0, \\ -1 & \text{if } G(x(k)) < 0. \end{cases}$$

It can be seen that with such a choice of w, $|x(k+1)| \geqslant |x(k)| + 1$, and hence, $|x(k+1)| \to \infty$. This shows that the system fails to be ISS no matter what G is.

6. Conclusions

In this paper, we proved that a system is ISS if and only if it has a smooth ISS-Lyapunov function. It was shown that many recent characterizations and results related to ISS for continuous-time nonlinear systems in the literature can find their analogues in discrete-time. As in the continuous-time case, two nonlinear ISS small gain theorems have been obtained for a general interconnection of nonlinear discrete-time systems, in terms of either ISS-gains or ISS–Lyapunov functions; also see Jiang et al. (2000). Along the process of extending the corresponding

ISS theory for continuous-time systems to the discrete-time setting, new phenomena arise. For instance, every affine continuous-time finite-dimensional system that is stabilizable can be rendered ISS with respect to additive noise v as in $u = K(x) + v$ (see Sontag, 1989). This is not the case any more for discrete-time systems. In general, a far more complex feedback transformation of the type $u = K_1(x) + K_2(x)v$ is required to make a discrete-time nonlinear system ISS with respect to the new input v. While ISS has found wide applications in several control problems for continuous-time systems, we expect that the discrete ISS-related properties developed in this paper will serve as a promising toolkit for discrete-time nonlinear feedback design. Results in this direction will be explored further and reported separately.

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Appendix A. Global asymptotic stability for systems with disturbances

This work relies heavily on the work in Jiang and Wang (2001). In this appendix, we provide some key results that were used in the present work. For details, we refer the reader to Jiang and Wang (2001).

Consider the system

$$x(k+1) = f(x(k), d(k)), \quad k \in \mathbb{Z}_+, \tag{A.1}$$

where $d \in \mathcal{M}_\Omega$ for some compact subset Ω of $\mathbb{R}^m$, and f is assumed to be continuous. We will use $x(\cdot, \xi, d)$ to denote the solution of (59) with the initial state ξ and the time-varying parameter $d \in \mathcal{M}_\Omega$.

Definition A.1. System (A.1) is *uniformly globally asymptotically stable* if the following two properties hold.

1. *Uniform stability.* There exists a $\mathcal{K}_\infty$-function $\delta(\cdot)$ such that for any $\varepsilon > 0$, it holds that $|x(k, \xi, d)| \leqslant \varepsilon$ for all $k \in \mathbb{Z}_+$, all $d \in \mathcal{M}_\Omega$, and all $|\xi| \leqslant \delta(\varepsilon)$.
2. *Uniform global attraction.* For any $r, \varepsilon > 0$, there exists some $T \in \mathbb{Z}_+$ such that for every $d \in \mathcal{M}_\Omega$, $|x(k, \xi, d)| < \varepsilon$ for all $k \geqslant T$ whenever $|\xi| \leqslant r$.

It was shown in Jiang and Wang (2001) that system (A.1) is UGAS if and only if for some $\beta \in \mathcal{KL}$,

$$|x(k, \xi, d)| \leqslant \beta(|\xi|, k) \quad \forall k \in \mathbb{Z}_+, \ \forall \xi \in \mathbb{R}^n.$$

Definition A.2. System (59) is *globally asymptotically stable* (GAS) if

1. for every $\varepsilon > 0$, there exists some $\delta > 0$ such that $|x(k, \xi, d)| < \varepsilon$ for all $k \geqslant 0$, all $d \in \mathcal{M}_\Omega$, and all $|\xi| < \delta$; and
2. the attraction property $\lim_{k \to \infty} |x(k, \xi, d)| = 0$ holds for all $\xi \in \mathbb{R}^n$, all $d \in \mathcal{M}_\Omega$.

The following is one of the main results of Jiang and Wang (2001).

Theorem 5. *For system* (59), *the following are equivalent*:

1. *The system is* GAS.
2. *The system is* UGAS.
3. *The system admits a smooth Lyapunov function. That is, there exists a smooth function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *for which the following holds*:
   - *there exist two* $\mathcal{K}_\infty$*-functions* α_1 *and* α_2 *such that for any* $\xi \in \mathbb{R}^n$,

     $$\alpha_1(|\xi|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|);$$
   - *there exists some* $\mathcal{K}_\infty$*-function* α_3 *such that, for any* $\xi \in \mathbb{R}^n$ *and any* $\mu \in \Omega$,

     $$V(f(\xi, \mu)) - V(\xi) \leqslant -\alpha_3(|\xi|).$$

Appendix B. A technical lemma

In the proof of Lemma 3.5, we used the following seemingly obvious result. To make the work more self contained, we provide the result with a detailed proof.

Lemma B.1. *For any* $\mathcal{K}_\infty$*-function* α, *there is a* $\mathcal{K}_\infty$*-function* $\hat{\alpha}$ *such that the following holds*:

- $\hat{\alpha}(s) \leqslant \alpha(r)$ *for all* $r \geqslant 0$; *and*
- $\mathrm{id} - \hat{\alpha} \in \mathcal{K}$.

Observe that if a locally Lipschitz function $\hat{\alpha} \in \mathcal{K}_\infty$ satisfies the condition that $\hat{\alpha}'(r) \leqslant 1/2$ for almost all r, then $(r - \hat{\alpha}(r))' \geqslant 1/2$, and hence, $\mathrm{id} - \hat{\alpha} \in \mathcal{K}_\infty$. To prove Lemma B.1, we first prove the following result.

Lemma B.2. *Let* λ_0 *be a locally Lipschitz* $\mathcal{K}_\infty$*-function. Then, for any* $0 \leqslant a < b < \infty$, *there exists a locally Lipschitz function* $\lambda_1 \in \mathcal{K}_\infty$ *such that*

1. $\lambda_1(r) = \lambda_0(r)$ *on* $[0, a]$ *and* $\lambda_1(r) \leqslant \lambda_0(r)$ *on* $[a, b]$;
2. $\lambda_1'(r) \leqslant 1/2$ *for almost all* $r \geqslant a$.

Proof. Let λ_0 be a smooth $\mathscr{K}_\infty$-function, and let $0 \leqslant a < b$ be given. Consider the function

$$\kappa(r) = \begin{cases} \lambda_0(r) & \text{if } 0 \leqslant r \leqslant a, \\ \lambda_0(a) + \int_a^r \min\{\lambda'(s), \frac{1}{2}\}\, ds & \text{if } r \geqslant a. \end{cases}$$

Then κ is locally Lipschitz on $(0, \infty)$, $\kappa'(r) \leqslant 1/2$ for all $r \geqslant a$, $\kappa(r) \leqslant \lambda_0(r)$ for all $r \geqslant 0$, and $\kappa \in \mathscr{K}$. If $\kappa \in \mathscr{K}_\infty$, then we are done with $\lambda_1 = \kappa$.

Suppose $\kappa \notin \mathscr{K}_\infty$. Then there exists some $b_0 \geqslant b - 1$ such that $\kappa(b_0) \leqslant \lambda_0(b_0) - 1$. Let

$$\lambda_1(r) = \begin{cases} \kappa(r) & \text{if } 0 \leqslant r \leqslant b_0, \\ \kappa(b_0) + \frac{1}{2}(r - b_0) & \text{if } r \geqslant b_0. \end{cases}$$

Then λ_1 is a locally Lipschitz $\mathscr{K}_\infty$-function, $0 \leqslant \lambda_1'(r) \leqslant 1/2$ for almost every r. It is also not hard to see that $\lambda_1(r) \leqslant \lambda_0(r)$ for all $r \leqslant b$. □

Proof of Lemma B.1. Let α be a $\mathscr{K}_\infty$ function. Without loss of generality, we assume that α is smooth (otherwise, replace α by a smooth $\mathscr{K}_\infty$-function majorized by α). Applying Lemma B.2 to α with $a = 0$, $b = 10$, we know that there exists some locally Lipschitz function $\lambda_1 \in \mathscr{K}_\infty$ such that Properties 1 and 2 in Lemma B.2 hold. If $\lambda_1(r) \leqslant \alpha(r)$ for all $r \geqslant 0$, then the proof is done with $\hat{\alpha} = \lambda_1$.

Suppose for some k, we have found locally Lipschitz $\mathscr{K}_\infty$-functions $\lambda_1, \ldots, \lambda_k$ and $0 < r_1 < \cdots < r_k - 1$ such that for each $1 \leqslant j \leqslant k - 1$, the following holds:

- $r_j \geqslant r_{j-1} + 10$, where $r_0 = 0$;
- $\lambda_{j+1} = \lambda_j$ on $[0, r_j]$, and $\lambda_j(r) \leqslant \alpha(r)$ for all $r \in [0, r_j]$ with $\lambda_j(r_j) = \alpha(r_j)$, and $\lambda_k(r) \leqslant \alpha(r)$ for all $0 \leqslant r \leqslant r_{k-1} + 10$;
- $\lambda_j'(s) \leqslant 1/2$ almost everywhere.

If

$$\lambda_k \leqslant \alpha(r) \quad \forall\, r \geqslant 0, \tag{B.1}$$

then the proof is done with $\hat{\alpha} = \lambda_k$.

Suppose (B.1) fails. Then there exists some $r_k \geqslant r_{k-1} + 10$ such that $\lambda_k(r_k) = \alpha(r_k)$. Let $\tilde{\alpha}_{k+1}$ be defined by

$$\tilde{\alpha}_{k+1}(r) = \begin{cases} \lambda_k(r) & \text{if } 0 \leqslant r \leqslant r_k, \\ \alpha(r) & \text{if } r \geqslant r_k. \end{cases}$$

Applying Lemma B.2 to the locally Lipschitz $\mathscr{K}_\infty$-function $\tilde{\alpha}_{k+1}$ with $a = r_k$ and $b = r_k + 10$, we get some locally Lipschitz $\mathscr{K}_\infty$-function λ_{k+1} such that $\lambda_{k+1} = \lambda_k$ on $[0, r_k]$ and $\lambda_{k+1} \leqslant \alpha$ for all $0 \leqslant r \leqslant r_k + 10$, and $\lambda_{k+1}'(r) \leqslant 1/2$ almost everywhere.

Thus, by induction, we have shown either

1. for some k_0, there exists a locally Lipschitz $\mathscr{K}_\infty$-function λ_{k_0} such that $\lambda_{k_0}(r) \leqslant \alpha(r)$ for all $r \geqslant 0$ and $\lambda_{k_0}'(r) \leqslant 1/2$ almost everywhere; or
2. there exist $0 < r_1 < r_2 \cdots$ with $r_{k+1} \geqslant r_k + 10$ so that, for each k, there exists some locally Lipschitz $\mathscr{K}_\infty$-function λ_k such that $\lambda_k'(r) \leqslant 1/2$ almost everywhere, $\lambda_k(r) = \lambda_{k-1}(r)$ on $[0, r_{k-1}]$, and $\lambda_k(r) \leqslant \alpha(r)$ on $[0, r_k]$ with $\lambda_k(r_k) = \alpha(r_k)$.

In the first case, we complete the proof by letting $\hat{\alpha} = \lambda_{k_0}$.

Suppose the latter is the case. Define $\hat{\alpha}$ by $\hat{\alpha}(r) = \lambda_k(r)$ on $[r_{k-1}, r_k)$. Since $r_k \geqslant r_{k-1} + 10$, we see that $\hat{\alpha}$ is defined on $[0, \infty)$. Since $\lambda_k = \lambda_{k-1}$ on $[0, r_{k-1}]$, $\hat{\alpha}$ is locally Lipschitz, and $\hat{\alpha}'(r) \leqslant 1/2$ for almost all $r \geqslant 0$. Finally, since $\hat{\alpha}(r_k) = \alpha(r_k)$, we see that $\hat{\alpha}$ is of class $\mathscr{K}_\infty$. □

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