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A converse Lyapunov theorem for discrete-time systems with disturbances

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Abstract

This paper presents a converse Lyapunov theorem for discrete-time systems with disturbances taking values in compact sets. Among several new stability results, it is shown that a smooth Lyapunov function exists for a *family* of time-varying discrete systems if these systems are robustly globally asymptotically stable. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

There is a long history in the search of Lyapunov functions for dynamical systems. In this article, we present a new converse Lyapunov theorem for nonautonomous discrete-time nonlinear systems affected by disturbances $d(\cdot)$ taking values in a compact set. Our main contribution is to prove the following: a discrete-time system with disturbances, or time-varying parameters, taking values in a *compact* set, is uniformly globally asymptotically stable (UGAS) with respect to a closed, not necessarily compact, invariant set $\mathscr{A}$ if and only if there exists a smooth Lyapunov function V with respect to the set $\mathscr{A}$; see Theorem 1 in Section 2.3. To the best of the authors' knowledge, even in the special case when the systems are free of disturbances and when the invariant set $\mathscr{A}$ is compact, this result is new in the literature. For a general discussion on stability with respect to closed invariant sets in the continuous time case, the interested reader should consult [22], where several converse Lyapunov theorems were provided in the general nonautonomous case.

This work parallels the previous work [10] where a converse Lyapunov theorem was obtained for continuous-time systems with disturbances. We are motivated by the importance and application of the theory of discrete-time (or, difference) systems in various fields [1,9,8]. Recently, the stability theory of difference systems has been used to design stabilizing control laws for discrete-time nonlinear systems; see, e.g., [3,12,19–21] and references therein.

In the past literature, quite a few converse theorems have been established in the discrete-time case for systems that are free of disturbances. However, the

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results were formulated either under the assumption of (local) exponential stability, see e.g., [11,13], or under a global Lipschitz condition, see e.g., [1,8,11,17]. The Lipschitz condition was weakened in [4], but the Lyapunov functions were only constructed locally around the equilibrium. In [20], a local converse Lyapunov theorem was obtained for systems that are locally asymptotically stable with respect to compact invariant sets uniformly on time-varying parameters, assuming various conditions involving prolongations of the dynamical systems. (See also [10] for some detailed discussions of the prolongation approach in the continuous-time case.) In this paper, these overly restrictive conditions are not required and the obtained result is global in nature.

As said, the converse Lyapunov theorem established in this paper can be considered as a discrete analogue to the result obtained in [10] for continuous-time systems. The main idea in the proofs of the present paper is rooted in the proofs in [10,16], and the $\mathscr{KL}$-Lemma obtained in [15]. Nevertheless, there are quite some technical details that need to be treated differently than in the continuous case. Considering the significant role played by the converse Lyapunov theorems in the continuous-time case, we find it necessary and appropriate to present the converse Lyapunov theorem for the discrete-time case. Furthermore, the results proved in this paper provide a necessary tool for the study of input-to-state stability in [6]. Just as in the continuous case where the converse Lyapunov theorem in [10] has found wide applications, it is our belief that the converse Lyapunov theorem presented in this work will provide a useful tool for discrete-time systems analysis and synthesis. See [5,6] for preliminary applications.

In contrast to the work in continuous time where time invariance is assumed, our converse Lyapunov theorem is derived for general nonautonomous discrete systems rather than the time invariant case. Time-varying systems can be often seen in practical situations when one deals with tracking control problems. In our context, it is shown that for periodic discrete systems, the resulted Lyapunov functions are also time periodic.

In Section 2, we present the above-mentioned converse Lyapunov theorem for robust stability in discrete time. In Section 3, we discuss the special case when the invariant set $\mathscr{A}$ is compact and the system under study is periodic or time invariant. We show that when $\mathscr{A}$ is compact, the UGAS and the GAS properties are equivalent for periodic systems and time invariant systems. To preserve the smoothness of the flow, we present the proofs in Section 4 instead of scattering them with the statements of the results. Our conclusions are in Section 5.

2. Definitions and main results

Consider a system of the following general type:

$$x(k+1) = f(k, x(k), d(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where the states $x(\cdot)$ take values in $\mathbb{R}^n$, and disturbances, or time varying parameters, $d(\cdot)$ take values in Ω for some subset Ω of $\mathbb{R}^m$, and where $f : \mathbb{Z}_+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is continuous. We remark that topologically $\mathbb{Z}_+$ is treated as a subspace of $\mathbb{R}$, so a function $a(\cdot,\cdot)$ defined on $J \times S$ for some $J \subseteq \mathbb{Z}_+, S \subset \mathbb{R}^n$ is continuous if and only if $a(k,\cdot)$ is continuous on S for each $k \in J$.

We say that system (1) is periodic with period λ if $f(k,\xi,\mu)$ is periodic in k with period λ, and (1) is time invariant if $f(k,\xi,\mu)$ is independent of k.

Throughout this work, we assume that the set Ω is compact. Let $\mathscr{M}_\Omega$ be the set of all functions from $\mathbb{Z}_+$ to Ω. We will use $x(\cdot,k_0,\xi,d)$ to denote the solution of (1) with the initial state $x(k_0) = \xi$ and the external signal $d \in \mathscr{M}_\Omega$. Clearly such a trajectory is uniquely defined for all $k \geqslant k_0 \geqslant 0$. Note also that, for any $k \geqslant k_0$, $x(k,k_0,\xi,d)$ is independent of $d(j)$ for $j < k_0$.

Let $\mathscr{A}$ be a nonempty closed subset of $\mathbb{R}^n$. The set is said to be (forward) invariant if, for each $\xi \in \mathscr{A}$, $x(k,k_0,\xi,d) \in \mathscr{A}$ for all $k \geqslant k_0$. Throughout this work, we denote

$$|\xi|_{\mathscr{A}} = d(\xi,\mathscr{A}) = \inf_{\eta \in \mathscr{A}} |\xi - \eta|,$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n$.

Recall that a function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a *$\mathscr{K}$-function* if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a $\mathscr{K}_\infty$-function if it is a $\mathscr{K}$-function and also $\gamma(s) \to \infty$ as $s \to \infty$; and it is a *positive definite* function if $\gamma(s) > 0$ for all $s > 0$, and $\gamma(0) = 0$. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a $\mathscr{KL}$-function if, for each fixed $t \geqslant 0$, the function $\beta(\cdot,t)$ is a $\mathscr{K}$-function, and for each fixed $s \geqslant 0$, the function $\beta(s,\cdot)$ is decreasing and $\beta(s,t) \to 0$ as $t \to \infty$.

2.1. Uniform global asymptotic stability

We first introduce two notions of global stability for discrete systems as in (1).

Definition 2.1. Let $\mathscr{A}$ be a closed, not necessarily compact, invariant set of system (1). The system is UGAS with respect to $\mathscr{A}$ if the following two properties hold:

1. *Uniform stability*. There exists a $\mathscr{K}_\infty$-function $\delta(\cdot)$ such that for any $\varepsilon > 0$,
$$|x(k,k_0,\xi,d)|_{\mathscr{A}} \leqslant \varepsilon \qquad \forall k \geqslant k_0 \;\; \forall k_0 \in \mathbb{Z}_+ \;\; \forall d \in \mathscr{M}_\Omega, \tag{2}$$
whenever $|\xi|_{\mathscr{A}} \leqslant \delta(\varepsilon)$.
2. *Uniform global attraction*. For any $r, \varepsilon > 0$, there exists some $T \in \mathbb{Z}_+$ such that for every $d \in \mathscr{M}_\Omega$ and $k_0 \in \mathbb{Z}_+$,
$$|x(k,k_0,\xi,d)|_{\mathscr{A}} < \varepsilon \tag{3}$$
for all $k \geqslant k_0 + T$ whenever $|\xi|_{\mathscr{A}} \leqslant r$.

As in the continuous-time case, one can prove the following result. Its proof is analogous with the one in the continuous-time case and thus is omitted.

Proposition 2.2. *System* (1) is UGAS *with respect to* $\mathscr{A}$ *if and only if there exists a* $\mathscr{K}\mathscr{L}$*-function* β *such that*
$$|x(k,k_0,\xi,d)|_{\mathscr{A}} \leqslant \beta(|\xi|_{\mathscr{A}}, k-k_0) \quad \forall k \geqslant k_0 \tag{4}$$
for all $\xi \in \mathbb{R}^n$, *all* $k_0 \in \mathbb{Z}_+$, *and all* $d \in \mathscr{M}_\Omega$.

Definition 2.3. Let $\mathscr{A}$ be a closed, not necessarily compact, invariant set of system (1). The system is *globally exponentially stable* (GES) with respect to $\mathscr{A}$ if there exist constants $\kappa > 0$, $0 < \sigma < 1$ such that
$$|x(k,k_0,\xi,d)|_{\mathscr{A}} \leqslant \kappa |\xi|_{\mathscr{A}} \sigma^{k-k_0} \quad \forall \xi \in \mathbb{R}^n,$$
$$k \geqslant k_0 \;\; \forall d \in \mathscr{M}_\Omega.$$

Remark 2.4. By Proposition 7 in [15], for any $\mathscr{K}\mathscr{L}$-function β, there exist $\rho_1, \rho_2 \in \mathscr{K}_\infty$ such that
$$\beta(s,r) \leqslant \rho_1(\rho_2(s)\mathrm{e}^{-r}) \quad \forall s \geqslant 0 \quad \forall r \geqslant 0.$$
Consequently, system (1) is UGAS with respect to $\mathscr{A}$ if and only if there exist $\rho_1, \rho_2 \in \mathscr{K}_\infty$ such that
$$|x(k,k_0,\xi,d)|_{\mathscr{A}} \leqslant \rho_1(\rho_2(|\xi|_{\mathscr{A}})\mathrm{e}^{-(k-k_0)}) \quad \forall k \geqslant k_0 \tag{5}$$
for all $\xi \in \mathbb{R}^n$, all $k_0 \in \mathbb{Z}_+$, and all $d \in \mathscr{M}_\Omega$.

Remark 2.5. Suppose system (1) is periodic with period λ. By the uniqueness property of solutions, it is not hard to show that
$$x(k,k_0+m\lambda,\xi,d) = x(k-m\lambda,k_0,\xi,d_{m\lambda}) \tag{6}$$
for all $k \geqslant k_0 + m\lambda$, all ξ and all d, where $d_\tau(k) = d(k+\tau)$ for any $\tau \in \mathbb{Z}_+$. In particular, if system (1) is time invariant, then
$$x(k,k_0,\xi,d) = x(k-k_0,0,\xi,d_{k_0}) \qquad \forall \xi \; \forall k \geqslant k_0 \;\; \forall k_0 \in \mathbb{Z}_+, \text{ and } \forall d \in \mathscr{M}_\Omega.$$
Hence, if a system is periodic with period λ, then it is UGAS if and only if the estimates as in (2) and (3) hold for all $k_0 \in \{0,1,\ldots,\lambda-1\}$; and if the system is time invariant, then it is UGAS if and only if the estimates as in (2) and (3) hold for $k_0 = 0$.

2.2. Lyapunov functions

In this section, we introduce Lyapunov functions associated with the UGAS property.

Definition 2.6. A continuous function $V : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ is a *Lyapunov function for system* (1) with respect to $\mathscr{A}$ if

1. there exist two $\mathscr{K}_\infty$-functions α_1 and α_2 such that for any $\xi \in \mathbb{R}^n$,
$$\alpha_1(|\xi|_{\mathscr{A}}) \leqslant V(k,\xi) \leqslant \alpha_2(|\xi|_{\mathscr{A}}) \tag{7}$$
2. there exists a continuous, positive definite function α_3 such that
$$V(k+1, f(k,\xi,\mu)) - V(k,\xi) \leqslant -\alpha_3(|\xi|_{\mathscr{A}}) \tag{8}$$
for any $\xi \in \mathbb{R}^n$, $k \in \mathbb{Z}_+$, and any $\mu \in \Omega$.

We say that V is periodic with period λ if $V(k,\xi)$ is periodic with period λ for each ξ; and V is time invariant if $V(k,\xi)$ is independent of k.

A *smooth Lyapunov function* $V(k,\xi)$ is one which is smooth in ξ on $\mathbb{R}^n$.

It turns out that if a system admits a continuous Lyapunov function, then it also admits a smooth one. To be more precise, we have the following:

Lemma 2.7. *If there is a continuous Lyapunov function* V *with respect to* $\mathscr{A}$ *for* (1), *then there is also a smooth one* W *with respect to* $\mathscr{A}$. *Furthermore, if* V

is periodic (resp., time invariant), then W can be chosen to be periodic with the same period (resp., time invariant).

The next lemma means that one can always assume that the function α_3 as in (8) can be chosen to be of class $\mathcal{K}_\infty$. The proofs of both Lemmas 2.7 and 2.8 will be given in Section 4.1.

Lemma 2.8. *Assume that system (1) admits a Lyapunov function V. Then there exists a smooth $\mathcal{K}_\infty$-function ρ, such that, with $W = \rho \circ V$, it holds that*

$$W(k+1, f(k,\xi,\mu)) - W(k,\xi) \leqslant -\alpha(|\xi|_{\mathcal{A}}) \quad \forall \xi \in \mathbb{R}^n \ \forall \mu \in \Omega \tag{9}$$

for some $\alpha \in \mathcal{K}_\infty$.

2.3. Statement of main results

The following is our main result in this work.

Theorem 1. *Consider system (1).*

1. *The system in* UGAS *with respect to $\mathcal{A}$ if and only if it admits a smooth Lyapunov function V with respect to $\mathcal{A}$.*
2. *A periodic system with period λ is* UGAS *with respect to $\mathcal{A}$ if and only if it admits a smooth periodic Lyapunov function V with respect to $\mathcal{A}$ with the same period λ.*
3. *A time-invariant system is* UGAS *with respect to $\mathcal{A}$ if and only if it admits a smooth time invariant Lyapunov function V with respect to $\mathcal{A}$.*

The proof of the theorem will be given in Section 4.3 after we explore more properties related to UGAS and Lyapunov functions. As a by-product in proving Theorem 1, we will also establish the following.

Theorem 2. *A system of form (1) is* GES *with respect to $\mathcal{A}$ if and only if it admits a continuous Lyapunov function $V(k,\xi)$ satisfying the inequalities*

$$\begin{aligned} &|\xi|^2_{\mathcal{A}} \leqslant V(k,\xi) \leqslant c|\xi|^2_{\mathcal{A}} \\ &\qquad\qquad\qquad\qquad \forall \xi \in \mathbb{R}^n \ \forall \mu \in \Omega. \\ &V(k+1, f(k,\xi,\mu)) - V(k,\xi) \leqslant -|\xi|^2_{\mathcal{A}}, \end{aligned} \tag{10}$$

Furthermore, if a periodic, or time invariant, system is GES, *then the Lyapunov function as in (10) can be chosen to be periodic with the same period or time invariant respectively.*

3. The periodic case with compact invariant sets

In the continuous-time case, it was shown in [16] that, in the special case when $\mathcal{A}$ is compact and when the system is time invariant, the uniform property in Property 2 of Definition 2.1 can be relaxed. This result then leads to nontrivial results in the asymptotic gain characterization of input-to-state stability. In this section we show that this result still holds for periodic and time invariant systems in the discrete-time case. The proofs turn out to be far simpler, due to the fact that in the discrete case, the set $\mathcal{M}_\Omega$ is compact with the pointwise convergence topology (cf. Lemma 4.1), while in the continuous case, the corresponding set of measurable functions taking values in Ω does not possess such a compactness property.

Definition 3.1. System (1) *is globally asymptotically stable* (GAS) with respect to $\mathcal{A}$ if the following properties hold:

1. *Local uniform stability*: for every $\varepsilon > 0$, there exists some $\delta > 0$ such that
$$|x(k,k_0,\xi,d)|_{\mathcal{A}} < \varepsilon \quad \forall k \geqslant k_0 \ \forall k_0 \in \mathbb{Z}_+ \ \forall d \in \mathcal{M}_\Omega, \tag{11}$$
whenever $|\xi|_{\mathcal{A}} < \delta$.
2. *Global attraction*: for all $\xi \in \mathbb{R}^n$, all $k_0 \in \mathbb{Z}_+$, and all $d \in \mathcal{M}_\Omega$, it holds that
$$\lim_{k\to\infty} |x(k,k_0,\xi,d)|_{\mathcal{A}} = 0. \tag{12}$$

Observe that Property 1 in the above definition amounts to requiring that the map $\xi \mapsto x(k,k_0,\xi,d)$ be continuous at $\xi = 0$ uniformly on all $k, k_0 \in \mathbb{Z}_+$ and all $d \in \mathcal{M}_\Omega$. It differs from Property 1 of Definition 2.1 in that the function $\delta(\cdot)$ in (2) is required to be of class $\mathcal{K}_\infty$. Clearly UGAS implies GAS. A counterexample will be given to show that GAS with respect to closed sets is in general weaker than UGAS. On the other hand, when $\mathcal{A}$ is compact, the two notions are indeed equivalent for periodic systems.

Proposition 3.2. *Let $\mathcal{A}$ be compact. Then a periodic system, and in particular, a time-invariant system, is* UGAS *with respect to $\mathcal{A}$ if and only if it is* GAS *with respect to $\mathcal{A}$.*

The proof of Proposition 3.2 will be given in Section 4.2. Combining Proposition 3.2 with Theorem 1, we get the following:

Corollary 3.3. *Let $\mathscr{A}$ be compact. The following are equivalent for a periodic* (*resp. time-invariant*) *system*:

1. *it is* GAS *with respect to $\mathscr{A}$*;
2. *it is* UGAS *with respect to $\mathscr{A}$*;
3. *it admits a smooth periodic* (*resp. time invariant*) *Lyapunov function with respect to $\mathscr{A}$ with the same period.*

It can be seen that the compactness property of $\mathscr{A}$ is used nontrivially in the proof. Without the compactness assumption, the result is in general false even for time invariant systems. For instance, consider the system

$$x_1(k+1) = \left(1 - \frac{1}{1+|x_2(k)|}\right) x_1(k),$$

$$x_2(k+1) = x_2(k) + 1.$$

It is not hard to see that the system is GAS with respect to the set $\mathscr{A} = \{(x_1, x_2): x_1 = 0\}$. But the system is not UGAS with respect to the set $\mathscr{A}$, as the decay rate of x_1 depends on both $x_1(0)$ and $x_2(0)$.

It can also be seen that Proposition 3.2 in general fails for a time-varying system, even if the system is disturbance free. For instance, the system

$$x(k+1) = x(k)\left(1 - \frac{1}{1+k}\right)$$

is GAS, but it is not UGAS.

4. The proofs

In this section, we prove results stated in Sections 2 and 3. Along the way, we also prove some technical results of independent interest such as a comparison principle.

4.1. Proofs of Lemmas 2.7 and 2.8

Proof of Lemma 2.7. Suppose system (1) admits a continuous Lyapunov function V with respect to $\mathscr{A}$ with α_i $(i = 1, 2, 3)$ as in Definition 2.6. Fix k. It then holds that

$$V(k+1, f(k,\xi,\mu)) \leqslant V(k,\xi) \leqslant \alpha_2(|\xi|_{\mathscr{A}}) \quad \forall \mu \in \Omega \ \ \forall \xi,$$

and consequently,

$$|f(k,\xi,\mu)|_{\mathscr{A}} \leqslant \alpha_1^{-1}(V(k+1, f(k,\xi,\mu))) \leqslant \alpha_1^{-1}(\alpha_2(|\xi|_{\mathscr{A}})). \tag{13}$$

Define $\varepsilon_0 : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ by

$$\varepsilon_0(s) := \min\{\tfrac{1}{2}\alpha_1(s), \tfrac{1}{2}\alpha_2(s), \tfrac{1}{4}\alpha_3(s), \tfrac{1}{4}\alpha_3(\alpha_2^{-1}(\alpha_1(s)))\}.$$

Then $\varepsilon(\xi) := \varepsilon_0(|\xi|_{\mathscr{A}})$ is continuous, $\varepsilon(\xi) = 0$ on $\mathscr{A}$ and $\varepsilon(\xi) > 0$ for all $\xi \notin \mathscr{A}$. According to [2, Theorem 4.8, p. 197]), there is a smooth function $\tilde{V}_k$ defined on $\mathbb{R}^n \backslash \mathscr{A}$ such that

$$|\tilde{V}_k(\xi) - V(k,\xi)| \leqslant \varepsilon_0(|\xi|_{\mathscr{A}}) \tag{14}$$

for all $\xi \notin \mathscr{A}$. In particular, $\tilde{V}_k(\xi) \leqslant V(k,\xi) + \varepsilon(\xi) \to 0$ as $|\xi|_{\mathscr{A}} \to 0$. Extend $\tilde{V}_k$ continuously to $\mathbb{R}^n$ by letting $\tilde{V}_k(\xi) = 0$ for $\xi \in \mathscr{A}$.

Define $W_0 : \mathbb{Z}_+ \times \mathbb{R}^n$ by $W_0(k,\xi) = \tilde{V}_k(\xi)$. Then, for each $k \in \mathbb{Z}_+$, $W_0(k,\cdot)$ is continuous on $\mathbb{R}^n$, locally Lipschitz on $\mathbb{R}^n \backslash \mathscr{A}$, and $W(k,\xi) = 0$ for $\xi \in \mathscr{A}$. Furthermore,

$$|W_0(k,\xi) - V(k,\xi)| \leqslant \varepsilon_0(|\xi|_{\mathscr{A}}) \leqslant \alpha_3(|\xi|_{\mathscr{A}})/4 \quad \forall k \in \mathbb{Z}_+ \ \ \forall \xi \in \mathbb{R}^n, \tag{15}$$

and

$$|W_0(k+1, f(k,\xi,\mu)) - V(k+1, f(k,\xi,\mu))| \leqslant \varepsilon_0(|f(k,\xi,\mu)|_{\mathscr{A}}) \leqslant \varepsilon_0(\alpha_1^{-1}(\alpha_2(|\xi|_{\mathscr{A}})) \leqslant \alpha_3(|\xi|_{\mathscr{A}})/4 \quad \forall \xi \in \mathbb{R}^n \ \ \forall \mu \in \Omega. \tag{16}$$

It follows from (14) that

$$\hat{\alpha}_1(|\xi|_{\mathscr{A}}) \leqslant W_0(k,\xi) \leqslant \hat{\alpha}_2(|\xi|_{\mathscr{A}}) \quad \forall k \in \mathbb{Z}_+ \ \ \forall \xi \in \mathbb{R}^n,$$

where $\hat{\alpha}_1(s) = \alpha_1(s)/2$ and $\hat{\alpha}_2(s) = 2\alpha_2(s)$, and from (15), (16) and (8) that

$$W_0(k+1, f(k,\xi,\mu)) - W_0(k,\xi) \leqslant -\alpha_3(|\xi|_{\mathscr{A}})/2 \quad \forall k \in \mathbb{Z}_+ \ \ \forall \xi \in \mathbb{R}^n \ \ \forall \mu \in \Omega. \tag{17}$$

To get a Lyapunov function that is differentiable everywhere, we invoke Lemma 4.3 in [10] which shows that for the function W_0, there exists a $\mathscr{K}_\infty$-function γ such that the function $W := \gamma \circ W_0$ is smooth everywhere. To be more precise, Lemma 4.3 in [10] does not exactly apply to the function $W_0(k,\xi)$ whose first argument does not evolve in $\mathbb{R}$ but rather in $\mathbb{Z}_+$. But one can consider a function $\tilde W_0 : \mathbb{R}\times\mathbb{R}^n$ defined by $\tilde W_0(t,\xi)=0$ for $t\leqslant 0$, and

$$\tilde W_0(t,\xi)=(1-\varphi(t-k))W_0(k,\xi)+\varphi(t-k)W_0(k+1,\xi)$$

for $t\in[k,k+1)$, where $\varphi:\mathbb{R}\to[0,1]$ is a smooth function satisfying $\varphi(t)=0$ for all $t\leqslant 0$ and $\varphi(t)=1$ for all $t\geqslant 1$. Note then that $\tilde W_0$ is continuous on $\mathbb{R}^{n+1}$, locally Lipschitz on $\mathbb{R}^{n+1}\backslash\mathscr{A}_1$, $\tilde W_0(t,\xi)=0$ for all $(t,\xi)\in\mathscr{A}_1$, where $\mathscr{A}_1=\mathbb{R}\times\mathscr{A}$. Applying Lemma 4.3 in [10], one sees that there exists some $\gamma\in\mathscr{K}_\infty$ such that the function $\gamma\circ\tilde W_0$ is smooth everywhere on $\mathbb{R}^{n+1}$. Hence, $W:=\gamma\circ W_0$ is smooth everywhere, and it holds that

$$\tilde\alpha_1(|\xi|_{\mathscr{A}})\leqslant W(\xi)\leqslant\tilde\alpha_2(|\xi|_{\mathscr{A}}), \tag{18}$$

where $\tilde\alpha_i(s)=\gamma(\hat\alpha_i(s))$ $(i=1,2)$. Also note that (17) gives

$$\gamma(W_0(k+1,f(k,\xi,\mu)))-\gamma(W_0(k,\xi))\leqslant\gamma\left(W_0(k,\xi)-\tfrac12\alpha_3(|\xi|_{\mathscr{A}})\right)-\gamma(W_0(k,\xi)).$$

Let $\gamma(r)=0$ for all $r<0$ and define

$$\alpha_4(s)=\min_{\hat\alpha_1(s)\leqslant r\leqslant\hat\alpha_2(s)}\left\{\gamma(r)-\gamma\left(r-\tfrac12\alpha_3(s)\right)\right\}.$$

Then α_4 is continuous, positive definite, and satisfies

$$\alpha_4(|\xi|_{\mathscr{A}})\leqslant\gamma(W_0(k,\xi))-\gamma\left(W_0(k,\xi)-\tfrac12\alpha_3(|\xi|_{\mathscr{A}})\right)\quad\forall\xi\in\mathbb{R}^n.$$

Consequently,

$$W(k+1,f(k,\xi,\mu))-W(k,\xi)\leqslant-\alpha_4(|\xi|_{\mathscr{A}})\quad\forall\xi\in\mathbb{R}^n\quad\forall\mu\in\Omega. \tag{19}$$

This shows that W is indeed a smooth Lyapunov function for system (1).

Finally, note that if V is periodic with period λ, then, when defining W_0, one can let $W_0(k,\xi)=\tilde V_k(\xi)$ for $0\leqslant k\leqslant\lambda-1$, and let $W_0(k+m\lambda,\xi)=W_0(k,\xi)$ for all $m\in\mathbb{Z}_+$ and all $0\leqslant k\leqslant\lambda-1$. This way, W_0 is periodic with period λ, and consequently, W is periodic with period λ. Similarly, if V is time invariant, W_0 and W can be chosen time invariant. □

Proof of Lemma 2.8. Assume that system (1) admits a Lyapunov function V. By Lemma 2.7, one may always assume that V is smooth. Let α_i $(i=1,2,3)$ be as in (7) and (8). One can rewrite (8) as

$$V(k+1,f(k,\xi,\mu))-V(k,\xi)\leqslant-\hat\alpha_3(V(k,\xi))\quad\forall\xi\in\mathbb{R}^n\quad\forall\mu\in\Omega,$$

where $\hat\alpha_3(s)=\min\{\alpha_3(r):\ \alpha_2^{-1}(s)\leqslant r\leqslant\alpha_1^{-1}(s)\}$. Observe that $\hat\alpha_3$ is again continuous and positive definite. Pick any smooth $\mathscr{K}_\infty$-function ρ_0 such that $\rho_0(s/2)\hat\alpha_3(s)\geqslant s$ for all $s\geqslant 1$. Define

$$\rho(s)=s+\int_0^s\rho_0(\sigma)\,\mathrm{d}\sigma.$$

Then $\rho\in\mathscr{K}_\infty$, smooth everywhere, and $\rho'(s)=1+\rho_0(s)$. Finally, we let $W=\rho\circ V$. Then W is smooth and satisfies the following:

$$\hat\alpha_1(|\xi|_{\mathscr{A}})\leqslant W(k,\xi)\leqslant\hat\alpha_2(|\xi|_{\mathscr{A}})\quad\forall k\in\mathbb{Z}_+\quad\forall\xi\in\mathbb{R},$$

where $\hat\alpha_i=\rho\circ\alpha_i\in\mathscr{K}_\infty$ $(i=1,2)$. Below we will show that for all $\mu\in\Omega$,

$$W(k+1,f(k,\xi,\mu))-W(k,\xi)\leqslant-V(k,\xi)/2\quad\text{whenever }V(\xi)\geqslant 1. \tag{20}$$

Fix any $\xi\in\mathbb{R}^n$, $k\in\mathbb{Z}_+$, and $\mu\in\Omega$. Let $v=V(k,\xi)$, and $v_+=V(k+1,f(\xi,\mu))$. Note that $v_+\leqslant v$. Using the mean value theorem, one sees

$$\rho(v_+)-\rho(v)=\rho'(v_++\theta(v-v_+))(v_+-v)$$

for $\theta\in[0,1]$. Since $\rho'(s)\geqslant 1$ for all s, it follows that if $v_+\leqslant v/2$, then

$$\rho(v_+)-\rho(v)\leqslant-v/2. \tag{21}$$

Assume now that $v_+\geqslant v/2$. Then $\rho'(v_++\theta(v-v_+))\geqslant\rho'(v_+)\geqslant\rho'(v/2)>\rho_0(v/2)$, and hence,

$$\begin{aligned}\rho(v_+)-\rho(v)&\leqslant\rho_0(v/2)(v_+-v)\\&\leqslant-\rho_0(v/2)\hat\alpha_3(v)\\&\leqslant -v\quad\text{whenever }v\geqslant 1.\end{aligned} \tag{22}$$

Combining (21) and (22), one sees that

$$\rho(v_+)-\rho(v)\leqslant-v/2\quad\text{whenever }v\geqslant 1.$$

Also observe that when $v\leqslant 1$,

$$\begin{aligned}&\rho(v_+)-\rho(v)\\&=v_++\int_0^{v_+}\rho_0(s)\,\mathrm{d}s-\left(v+\int_0^{v}\rho_0(s)\,\mathrm{d}s\right)\\&\leqslant v_+-v\leqslant-\hat\alpha_3(v).\end{aligned}$$

We now let α_4 be any $\mathcal{K}_\infty$-function such that $\alpha_4(s) \leqslant \hat{\alpha}_3(s)$ if $s \leqslant 1$, and $\alpha_4(s) \leqslant s/2$ if $s > 1$. Let $\alpha = \alpha_4 \circ \alpha_1 \in \mathcal{K}_\infty$. It then follows that

$$W(k+1, f(k,\xi,\mu)) - W(k,\xi) \leqslant -\alpha(|\xi|_{\mathcal{A}})$$

for all $k \in \mathbb{Z}_+$, all $\xi \in \mathbb{R}^n$, and all $\mu \in \Omega$. □

4.2. Proof of Proposition 3.2

It is trivial that UGAS⇒ GAS. The converse of the implication follows from the standard fact in analysis (see, e.g., [14, Theorem 7.23]) that, for a compact set Ω, the set $\mathcal{M}_\Omega$ is compact in the pointwise convergence topology. A precise statement is as in the following:

Lemma 4.1. *For every sequence $\{d_k\}$ of functions in $\mathcal{M}_\Omega$, there exist a function $d_0 \in \mathcal{M}_\Omega$ and a subsequence $\{d_{kl}\}$ of $\{d_k\}$ such that for each $j \in \mathbb{Z}_+$, $d_{kl}(j) \to d_0(j)$ as $l \to \infty$.*

Proof of Proposition 3.2. Suppose a periodic system with period λ as in (1) in GAS. Let $k_0 = j \in \{0, 1, \ldots, \lambda - 1\}$ be given. For each $\varepsilon > 0$, and each $r > 0$, we let

$$T_j(r,\varepsilon) = \inf\{t\colon\ |x(k,j,\xi,d)|_{\mathcal{A}} \leqslant \varepsilon \quad \forall k \geqslant j+t\ \forall |\xi|_{\mathcal{A}} \leqslant r\ \forall d \in \mathcal{M}_\Omega\}$$

(where $T_j(r,\varepsilon) = \infty$ if the set is empty). Let $\delta > 0$ be as in Property 1 of Definition 3.1. Then $T_j(r,\varepsilon) \leqslant \tau := \sup\{\tau_{\xi,d}\colon\ |\xi|_{\mathcal{A}} \leqslant r,\ d \in \mathcal{M}_\Omega\}$, where

$$\tau_{\xi,d} = \inf\{k \in \mathbb{Z}_+\colon\ |x(k,j,\xi,d)|_{\mathcal{A}} \leqslant \delta/2\}.$$

By Lemma 4.1, it can be shown that $\tau < \infty$, and hence, $T_j(r,\varepsilon) < \infty$.

For any given $r > 0$ and $\varepsilon > 0$, let

$$T_{r,\varepsilon} = \max\{T_j(r,\varepsilon)\colon\ j = 0, 1, \ldots, \lambda - 1\}.$$

By definition, (3) holds for all $|\xi|_{\mathcal{A}} \leqslant r$, all $k \geqslant k_0 + T_{r,\varepsilon}$, all $k_0 \in \{0, 1, \ldots, \lambda - 1\}$ and all $d \in \mathcal{M}_\Omega$.

Next, we show that Property 1 (i.e., the global uniform stability) in Definition 2.1 holds. Again, fix $k_0 = j \in \{0, 1, \ldots, \lambda - 1\}$. For each $r > 0$, we define

$$\varphi_j(r) = \sup\{|x(k,j,\xi,d)|_{\mathcal{A}}\colon\ |\xi|_{\mathcal{A}} \leqslant r,\ k \geqslant j,\ d \in \mathcal{M}_\Omega\}.$$

By definition of $T_{r,\varepsilon}$, it holds that

$$\varphi_j(r) = \sup\{|x(k,j,\xi,d)|_{\mathcal{A}}\colon |\xi|_{\mathcal{A}} \leqslant r,\ j \leqslant k \leqslant j + T_{r,r},\ d \in \mathcal{M}_\Omega\}.$$

Using again the compactness property of $\mathcal{M}_\Omega$ (in pointwise convergence topology), one can show that $\varphi_j(r) < \infty$ for all $r > 0$. Furthermore, Property 1 in the GAS definition implies that $\varphi_j(r) \to 0$ as $r \to 0$. Let $\varphi(0) = 0$. Pick any $\mathcal{K}_\infty$-function $\tilde{\varphi}_j$ satisfying that $\tilde{\varphi}_j(r) \geqslant \varphi(r)$ for all $r \geqslant 0$. Then it holds that $|x(k,j,\xi,d)|_{\mathcal{A}} \leqslant \tilde{\varphi}_j(|\xi|_{\mathcal{A}})$ for all $k \geqslant j$, all ξ and all d. Finally, we let $\varphi(r) = \max\{\tilde{\varphi}_j(r): j = 0, 1, \ldots, \lambda\}$. It then holds that

$$|x(k,k_0,\xi,d)|_{\mathcal{A}} \leqslant \varphi(|\xi|_{\mathcal{A}})$$

for all $k \geqslant k_0$, all $0 \leqslant k_0 \leqslant \lambda$, all ξ and all d. By Remark 2.5, the system is UGAS. □

4.3. Proof of Theorem 1

To prove Theorem 1, we first discuss some preliminary results.

4.3.1. Continuity properties of trajectories

Consider system (1). It is clear that for each fixed $k \in \mathbb{Z}_+$ and each $d \in \mathcal{M}_\Omega$, the map $x(k,k_0,\xi,d)$ is continuous on ξ. Using the compactness property of $\mathcal{M}_\Omega$ with the pointwise topology (cf. Lemma 4.1), one can get the following stronger *uniform* continuity property which is an immediate consequence of the fact that compositions of uniformly continuous maps are still uniformly continuous.

Lemma 4.2. *Consider system* (1). *For any $k_0,\ T \in \mathbb{Z}_+$ and any compact subset K of $\mathbb{R}^n$, the map $\xi \mapsto (x(1+k_0,k_0,\xi,d),\ x(2+k_0,k_0,\xi,d), \ldots, x(T+k_0,k_0,\xi,d))$ is uniformly continuous on K and the continuity is uniform on $\mathcal{M}_\Omega$.*

Precisely, Lemma 4.2 means that for any compact set K, any $k_0,\ T \in \mathbb{Z}_+$, and any $\varepsilon > 0$, there exists some $\delta > 0$ such that for any $\xi_1, \xi_2 \in K$ such that $|\xi_1 - \xi_2| < \delta$,

$$|x(k,\xi_1,d) - x(k,\xi_2,d)| < \varepsilon \quad \forall k = 0, 1, \ldots, T,\ \forall d \in \mathcal{M}_\Omega. \tag{23}$$

4.3.2. A comparison principle

In the proof of the sufficiency part of Theorem 1, we need the following comparison lemma.

Lemma 4.3. *For each $\mathcal{K}$-function α, there exists a $\mathcal{KL}$-function $\beta_\alpha(s,t)$ with the following property: if $y : \mathbb{Z}_+ \to [0,\infty)$ is a function satisfying*

$$y(k+1) - y(k) \leqslant -\alpha(y(k)) \tag{24}$$

for all $0 \leqslant k < k_1$ *for some* $k_1 \leqslant \infty$, *then*

$$y(k) \leqslant \beta_\alpha(y(0), k) \quad \forall k < k_1. \tag{25}$$

Proof. Without loss of generality, one may assume that $\alpha(s) \leqslant s/2$ (otherwise one can replace $\alpha(s)$ by $\min\{\alpha(s), s/2\}$). Let, for each $r \geqslant 0$,

$$\rho(r) = \max_{0 \leqslant s \leqslant r} \{s - \alpha(s)\}.$$

Then $0 < \rho(r) < r$ for all $r > 0$, and $\rho(r)$ is non-decreasing. It follows from (24) that

$$y(k+1) \leqslant \rho(y(k)), \quad 0 \leqslant k < k_1.$$

Again, without loss of generality, one may assume that ρ is of class $\mathcal{K}$, otherwise use $(r + \rho(r))/2$ to replace $\rho(r)$. According to the standard comparison principle (see e.g. [7] or [18]), one has $y(k) \leqslant z(k)$ for all $0 \leqslant k < k_0$, where $z(t)$ is the solution for the following initial value problem:

$$z(k+1) = \rho(z(k)), \quad z(0) = y(0), \quad k \in \mathbb{Z}_+.$$

It can be seen that for each $k \in \mathbb{Z}_+$, $z(k) = \rho^k(z(0))$, and thus, $y(k) \leqslant \rho^k(y(0))$, where, for each $c \in \mathbb{R}$, $\rho^{k+1}(c) = (\rho \circ \rho^k)(c)$ for $k \geqslant 1$. Note that for each $c \geqslant 0$, $\rho^k(c) \to 0$ as $k \to \infty$.

To get an estimation as in (25), we let, for each $s \in \mathbb{R}_{\geqslant 0}$ and each $r \in [k, k+1)$, $k \in \mathbb{Z}_+$,

$$\beta_\alpha(s, r) = (k + 1 - r)\rho^k(s) + (r - k)\rho^{k+1}(s),$$

where $\rho^0(\cdot)$ denotes the identity function, i.e., $\rho^0(s) = s$ for all $s \in \mathbb{R}_{\geqslant 0}$. It is not hard to see that $\beta_\alpha \in \mathcal{KL}$. Estimation (25) follows from the fact that $\beta_\alpha(s, k) = \rho^k(s)$ for all $k \in \mathbb{Z}_+$, all $s \geqslant 0$. □

4.3.3. Proof of Theorem 1: the sufficiency

The proof of the sufficiency follows the standard argument, see e.g., [1, Chapter 5]. To make the work more self-contained, below we provide a treatment for the case with disturbances.

Assume that system (1) admits a Lyapunov function, with α_i $(i = 1, 2, 3)$ as in (7) and (8). By Lemma 2.8, we assume that $\alpha_3 \in \mathcal{K}_\infty$. Pick any $\xi \in \mathbb{R}^n$, any $k_0 \in \mathbb{Z}_+$ and any $d \in \mathcal{M}_\Omega$. Let $x(k) = x(k + k_0, k_0, \xi, d)$, and let $y(k) = V(k + k_0, x(k))$. Then, it holds that

$$\begin{aligned} &y(k+1) - y(k) \\ &\quad = V(k + k_0 + 1, x(k+1)) - V(k + k_0, x(k)) \\ &\quad \leqslant -\alpha(V(k + k_0, x(k))) = -\alpha(y(k)) \quad \forall k \in \mathbb{Z}_+, \end{aligned}$$

where $\alpha = \alpha_3 \circ \alpha_2^{-1}$. Let β_α be the $\mathcal{KL}$-function as in Lemma 4.3, then

$$y(k) \leqslant \beta_\alpha(y(0), k) = \beta_\alpha(V(k_0, \xi), k) \quad \forall k \in \mathbb{Z}_+.$$

Define $\beta(s, r) = \alpha_1^{-1} \circ \beta_\alpha(\alpha_2(s), r)$. Then $\beta \in \mathcal{KL}$ since both α_1, α_2 are $\mathcal{K}_\infty$-functions, and it holds that

$$|x(k + k_0, k_0, \xi, d)|_{\mathcal{A}} \leqslant \beta(|\xi|_{\mathcal{A}}, k) \quad \forall k \in \mathbb{Z}_+.$$

This shows that system (1) is UGAS with respect to $\mathcal{A}$. □

4.3.4. Proof of Theorem 1: the necessity

By Lemma 2.7, it is enough to prove the existence of a continuous Lyapunov function.

Assume that system (1) is UGAS. By Remark 2.4, there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that (5) holds. Let $\omega(s) = \rho_1^{-1}(s)$. Then $\omega \in \mathcal{K}_\infty$, and

$$\omega(|x(k + k_0, k_0, \xi, d)|_{\mathcal{A}}) \leqslant \rho_2(|\xi|_{\mathcal{A}})e^{-k}. \tag{26}$$

Define $V_0 : \mathbb{Z}_+ \times \mathbb{R}^n \times \mathcal{M}_\Omega \to \mathbb{R}_{\geqslant 0}$ by

$$V_0(k_0, \xi, d) = \sum_{k=0}^{\infty} \omega(|x(k + k_0, k_0, \xi, d)|_{\mathcal{A}}). \tag{27}$$

It follows from (26) that

$$\begin{aligned} \omega(|\xi|_{\mathcal{A}}) \leqslant V_0(k_0, \xi, d) &\leqslant \sum_{k=0}^{\infty} \rho_2(|\xi|_{\mathcal{A}})e^{-k} \\ &\leqslant \frac{e}{e-1}\rho_2(|\xi|_{\mathcal{A}}) \quad \forall d \in \mathcal{M}_\Omega. \end{aligned} \tag{28}$$

This shows that the series in (27) is convergent, and the convergence is uniform for $\xi \in K$ and $d \in \mathcal{M}_\Omega$ for any compact set K. Since, for each k and k_0, $\omega(|x(k + k_0, k_0, \cdot, d)|_{\mathcal{A}})$ is continuous uniformly on $d \in \mathcal{M}_\Omega$ (cf. Lemma 4.2), it follows that for each $k_0 \in \mathbb{Z}_+$, $V_0(k_0, \cdot, d)$ is continuous uniformly on $d \in \mathcal{M}_\Omega$. Define $V : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ by

$$V(k_0, \xi) = \sup_{d \in \mathcal{M}_\Omega} V_0(k_0, \xi, d). \tag{29}$$

It follows immediately from (28) that

$$\omega(|\xi|_{\mathcal{A}}) \leqslant V(k_0, \xi) \leqslant \frac{e}{e-1}\rho_2(|\xi|_{\mathcal{A}}). \tag{30}$$

Lemma 4.4. *For each* $k_0 \in \mathbb{Z}_+$, *the function* $V(k_0, \cdot)$ *is continuous on* $\mathbb{R}^n$.

Proof. Let $k_0 \in \mathbb{Z}_+$ be given. Fix $\xi \in \mathbb{R}^n$ and let $\varepsilon > 0$ be given. By the uniform continuity of V_0, there exists some $\delta > 0$ such that, whenever $|\xi - \eta| < \delta$,

$$|V_0(k_0, \xi, d) - V_0(k_0, \eta, d)| < \varepsilon/2 \quad \forall d \in \mathcal{M}_\Omega.$$

Pick such a point η. Let $d_0, d_1 \in \mathscr{M}_\Omega$ be such that

$$V(k_0,\xi) \leqslant V_0(k_0,\xi,d_0) + \varepsilon/2$$

and

$$V(k_0,\eta) \leqslant V_0(k_0,\eta,d_1) + \varepsilon/2.$$

Then, for any η such that $|\xi-\eta| < \delta$,

$$V(k_0,\xi) - V(k_0,\eta) \leqslant V_0(k_0,\xi,d_0) + \varepsilon/2 - V_0(k_0,\eta,d_0) < \varepsilon,$$

and

$$V(k_0,\eta) - V(k_0,\xi) \leqslant V_0(k_0,\eta,d_1) + \varepsilon/2 - V_0(k_0,\xi,d_1) < \varepsilon.$$

This shows that $V(k_0,\cdot)$ is continuous everywhere. □

In the following we show that V admits a desired decay estimate as in (8). Pick any k_0, any ξ and any $\mu \in \Omega$. Let ξ_+ denote $f(k_0,\xi,\mu)$, that is, $\xi_+ = x(k_0+1,k_0,\xi,d)$, where d is any function in $\mathscr{M}_\Omega$ such that $d(k_0)=\mu$. Then

$$V(k_0+1,\xi_+) = \sup_{d\in\mathscr{M}_\Omega} \sum_{k=0}^{\infty} \omega(|x(k+k_0+1,k_0+1,\xi_+,d)|_{\mathscr{A}}).$$

By the uniqueness property of solutions, one can see that, for any $d \in \mathscr{M}_\Omega$ such that $d(k_0)=\mu$, it holds that

$$x(k+k_0+1,k_0+1,\xi_+,d) = x(k+k_0+1,k_0,\xi,d) \quad \forall k \geqslant 0.$$

Hence,

$$\begin{aligned} V(k_0+1,\xi_+) &= \sup_{d\in\mathscr{M}_\Omega,\, d(k_0)=\mu} \sum_{k=0}^{\infty} \omega(|x(k+k_0+1,\xi,d)|_{\mathscr{A}}) \\ &= \sup_{d\in\mathscr{M}_\Omega,\, d(k_0)=\mu} \sum_{k=1}^{\infty} \omega(|x(k+k_0+1,k_0,\xi,d)|_{\mathscr{A}}) \\ &= \sup_{d\in\mathscr{M}_\Omega,\, d(k_0)=\mu} \sum_{k=0}^{\infty} \omega(|x(k+k_0,k_0,\xi,d)|_{\mathscr{A}}) \\ &\quad - \omega(|x(k_0,k_0,\xi,d)|_{\mathscr{A}}) \leqslant V(k_0,\xi) - \omega(|\xi|_{\mathscr{A}}). \end{aligned}$$

This shows that

$$V(k_0+1, f(k_0,\xi,\mu)) - V(k_0,\xi) \leqslant -\omega(|\xi|_{\mathscr{A}})$$

for all $\xi \in \mathbb{R}^n$ and for all $\mu \in \Omega$.

Finally, if system (1) is periodic with period λ, then (cf. (6))

$$x(k+k_0+m\lambda,k_0+m\lambda,\xi,d) = x(k+k_0,k_0,d_{m\lambda})$$

for all $k,k_0,m \in \mathbb{Z}_+$, all ξ, and all d. Thus,

$$\begin{aligned} V_0(k_0+m\lambda,\xi,d) &= \sum_{k=0}^{\infty} \omega(|x(k+k_0+m\lambda,k_0+m\lambda,\xi,d)|_{\mathscr{A}}) \\ &= \sum_{k=0}^{\infty} \omega(|x(k+k_0,k_0,\xi,d_{m\lambda})|_{\mathscr{A}}) = V_0(k_0,\xi,d_{m\lambda}). \end{aligned}$$

Consequently,

$$\begin{aligned} V(k_0+m\lambda,\xi) &= \sup_{d\in\mathscr{M}_\Omega} V_0(k_0,\xi,d_{m\lambda}) \\ &= \sup_{d\in\mathscr{M}_\Omega} V_0(k_0,\xi,d) = V(k_0,\xi) \end{aligned}$$

for all k_0, $m \in \mathbb{Z}_+$ and $\xi \in \mathbb{R}^n$. Hence, V is periodic with period λ. Similarly, if the system (1) is time invariant, then $x(k+k_0,k_0,\xi,d) = x(k,0,\xi,d_{k_0})$, from which it follows that V is also time invariant. □

4.4. *Proof of Theorem 2*

It can be seen that, in the proof of Theorem 1, the function ω can be chosen as any $\mathscr{K}_\infty$-function with the property that, for some $0<\sigma<1$,

$$\omega(|x(k+k_0,k_0,\xi,d)|_{\mathscr{A}}) \leqslant \rho_2(|\xi|_{\mathscr{A}})\sigma^k \quad \forall k \geqslant 0,\ \forall \xi \in \mathbb{R}^n,\ \forall d \in \mathscr{M}_\Omega. \qquad \square$$

In particular, in the case of GES, letting $\omega(s)=s^2$ will result in a Lyapunov function V with (7) and (8) strengthened to (10), with $c = \kappa^2/(1-\sigma^2)$.

5. Conclusions

We have presented a new converse Lyapunov theorem for general nonautonomous discrete systems subject to disturbances taking values in compact sets. Special cases such as periodic systems, time-invariant systems and GES systems are also investigated. We expect that, like the continuous-time counterpart in [10], our discrete converse Lyapunov theorem will find wide applications in the analysis and synthesis of discrete-time nonlinear systems. Preliminary results in this direction have been accomplished in [5,6].

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