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Brief paper

Nonlinear small-gain theorems for discrete-time feedback systems and applications☆

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Abstract

We derive in this work a local nonlinear small-gain theorem in the framework of input-to-state stability for discrete time systems. Our primary objective is to show that, as in the continuous-time context, these discrete-time nonlinear small-gain theorems are very effective in stability analysis and synthesis for various classes of discrete-time control systems. Two converse Lyapunov theorems for discrete exponential stability are developed to assist these applications. New results in stability and stabilization presented in this paper are significant extensions of previous work by other authors (IEEE Trans. Automat. Control 38 (1993) 1398; 39 (1994) 2340; 33 (1988) 1082).

Keywords: Discrete-time nonlinear systems; Input-to-state stability; Small-gain theorems; Stability; Stabilization

1. Introduction

One of the most important problems in control theory is concerned with system stability, especially with systems affected by external disturbances. In the late 1980s, the notion of *input to state stability* (ISS) was formulated in Sontag (1989). This property naturally incorporates an important concept "finite gain" frequently used in engineering with the classic stability notions used in ordinary differential equations, replacing the linear gain functions which represent a too strong requirement for general nonlinear systems by nonlinear gain functions. During the past decade, the ISS property has quickly become a foundational concept in nonlinear feedback analysis and design, see, for instance, Coron, Praly, and Teel (1995), Isidori (1999), Jiang and Mareels (1997), Jiang, Teel, and Praly (1994), Khalil (1996), Kokotović and Arcak (2001), Krstić, Kanellakopoulos, and Kokotović (1995), Tsinias (1993). The study of the ISS property for discrete time systems began in our previous work (Jiang, Sontag, & Wang, 1999; Jiang & Wang, 2001), where we provided a Lyapunov characterization of the ISS property as well as a discrete analogue of the nonlinear small gain theorems developed in Coron et al. (1995) and Jiang et al. (1994). More recently, Laila and Nešić (2002) provided a Lyapunov-based small-gain theorem for parameterized discrete-time interconnected ISS systems. The nonlinear small-gain theorems established in Coron et al. (1995), Jiang et al. (1994) and Jiang and Wang (2001) can be seen as state-space variants of the classical finite-gain theorems in Desoer and Vidyasagar (1975) and a more recent monotone gain theorem in Mareels and Hill (1992) that were written in terms of input–output operators. The current paper can be considered as a continuation of our previous work (Jiang et al., 1999; Jiang & Wang, 2001).

The analysis and synthesis of discrete-time control systems have received much attention in the past 15 years (see, e.g., Agarwal, 1992; Nijmeijer & van der Schaft, 1990 and

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references therein), mainly due to the increasing popularity of digital computers in control engineering applications. Our primary objective in this work is to show that, as in the continuous-time context, the discrete-time nonlinear small-gain theorems are very effective in stability analysis and synthesis for various classes of discrete-time control systems. For this purpose, we will first develop a local small-gain theorem whose proof requires a different approach from the global version (see, Jiang & Wang, 2001). This is mainly because in the local case, one has to keep tracking of the attraction domains of the subsystems. As applications of small-gain theorems obtained in this work and in Jiang and Wang (2001), we will show how some of the previous results obtained in Kazakos and Tsinias (1993), Lin and Byrnes (1994) and Magana and Zak (1988), can be extended significantly.

This paper is presented as follows. In Section 2, we derive a local version of a nonlinear small-gain theorem, and discuss some of the immediate consequences of the small-gain theorem. To assist the applications of nonlinear small-gain theorems, we also develop two converse Lyapunov theorems on uniform exponential stability for discrete-time nonlinear systems with disturbances. Section 3 contains some applications to stability and stabilization issues to illustrate the effectiveness of the ISS approach. Finally, Section 4 outlines the main contributions of this paper.

2. Local ISS for discrete time systems

In this section, we will discuss some basic stability notions as well as a local small-gain theorem.

2.1. Preliminaries

Consider a discrete-time system of the general form

$$x(k+1) = f(x(k), u(k)), \tag{1}$$

where for each $k \in \mathbb{Z}_+$ (the set of nonnegative integers), $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous. Given an initial value $x(0) = \xi$ and an input function $u : \mathbb{Z}_+ \to \mathbb{R}^m$, we use $x(k, \xi, u)$ to denote the corresponding solution of (1).

Throughout this paper, we use $|\cdot|$ to denote the usual Euclidean norm for vectors. For $u : \mathbb{Z}_+ \to \mathbb{R}^m$, $\|u\| = \max_{1\leqslant i\leqslant m} \|u_i\|$, where $\|u_i\|$ is the l_∞-norm of the ith component of u.

Definition 2.1. System (1) is *locally input-to-state stable* (LISS) if there exist some $\varepsilon > 0$, some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{K}$[1] such that, for each u and each $\xi \in \mathbb{R}^n$ satisfying $\|u\| \leqslant \varepsilon$ and $|\xi| \leqslant \varepsilon$, the solution $x(k, \xi, u)$ satisfies the property

$$|x(k, \xi, u)| \leqslant \max\{\beta(|\xi|, k), \gamma(\|u\|)\} \quad \forall k \in \mathbb{Z}_+. \tag{2}$$

The function γ is referred to as an LISS-*gain function*.

Note that if (2) holds for every bounded input u and every ξ in $\mathbb{R}^n$, then we get the standard ISS property studied in the past literature, see e.g. Jiang and Wang (2001).

For a system as in (1), we say that the system is 0-locally asymptotically stable (0-LAS) if the corresponding 0-input system $x(k+1) = f(x(k), 0)$ is LAS. We say that the system is locally asymptotically stabilizable if there is a continuous function α defined in a neighborhood of $0 \in \mathbb{R}^n$ with $\alpha(0)=0$ such that the closed-loop system $x(k+1) = f(x(k), \alpha(x(k)))$ is locally asymptotically stable. The following results are due to Lemma 6 of Gao and Lin (2000).

Lemma 2.2. *System* (1) *is* 0-*LAS if and only if it is LISS*. *In particular*, *if a feedback* $u = \alpha(x)$ *stabilizes system* (1), *then the corresponding closed-loop system with* $u = \alpha(x+v) + w$ *is* LISS *with* (v, w) *as the input*.

Lemma 2.3. *Suppose system* (1) *admits a continuous local Lyapunov function V such that for some* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, *some* $c > 0$ *and* $\sigma \in \mathcal{K}$ *it holds that*

$$\alpha_1(|\xi|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|),$$
$$V(f(\xi, \mu)) - V(\xi) \leqslant -cV(\xi) + \sigma(|\mu|)$$

for (ξ, μ) *in a neighborhood of* $(0, 0)$. *Then the system is LISS, and for any* $0 < c_0 < 1$, *the gain* γ *as in* (2) *can be chosen such that* $\gamma(s) \leqslant \alpha_1^{-1}\left(\frac{\sigma(s)}{(1-c_0)c}\right)$ *for all* $s \geqslant 0$.

2.2. A local small-gain theorem

We consider discrete-time and nonlinear interconnected systems of the form

$$\begin{aligned} x_1(k+1) &= f_1(x_1(k), v_1(k), u_1(k)),\\ x_2(k+1) &= f_2(x_2(k), v_2(k), u_2(k)) \end{aligned} \tag{3}$$

subject to the interconnection constraints

$$v_1(k) = x_2(k), \quad v_2(k) = x_1(k), \tag{4}$$

where for $i = 1, 2$, and for each $k \in \mathbb{Z}_+$, $x_i(k) \in \mathbb{R}^{n_i}$, $u_i(k) \in \mathbb{R}^{m_i}$, and f_i is continuous in its arguments.

Theorem 1. *Assume both the* x_1 *and* x_2 *subsystems in* (3) *with inputs* (v_1, u_1) *and* (v_2, u_2) *are* LISS *with gain functions* (γ_1^v, γ_1^u) *and* (γ_2^v, γ_2^u) *respectively. Suppose the small-gain condition holds locally between* γ_1^v *and* γ_2^v, *that is, there exists some* $s_0 > 0$ *such that* $\gamma_1^v \circ \gamma_2^v(s) < s$ *for all* $0 < s < s_0$. *Then the interconnected system* (3) *with* (4) *is LISS with* (u_1, u_2) *as input*.

Observe that the small-gain condition always holds in the special case when $\gamma_2^v = 0$. Also, by Lemma 2.2, the

[1] Recall that a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and vanishes at the origin. It is of class $\mathcal{K}_\infty$ if, additionally, it is unbounded. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for every fixed $t \geqslant 0$ and, for every fixed $r \geqslant 0$, $\beta(r, t)$ decreases to 0 as $t \to \infty$.

LISS condition of the x_1-system can be replaced by the 0-LAS property. Combining these observations with the global version of the ISS small-gain theorem given in Jiang and Wang (2001), we have

Corollary 2.4. *Consider a partially linear composite discrete-time system*

$$z(k+1) = q(z(k), x(k), u(k)), \quad z(k) \in \mathbb{R}^{n-r}$$
$$x(k+1) = Ax(k) + Bu(k), \quad x(k) \in \mathbb{R}^r, \; u(k) \in \mathbb{R}^m:$$

(1) *If the* 0*-input z-system, i.e.*, $z(k+1) = q(z(k), 0, 0)$, *is locally asymptotically stable, and if* (A, B) *is stabilizable, then the composite system is LAS-stabilizable.*
(2) *If the z-system* $z(k+1) = q(z(k), x(k), u(k))$ *is ISS with* $(x(k), u(k))$ *as inputs, and if* (A, B) *is stabilizable, then the composite system is GAS-stabilizable.*

The partially linear composite continuous-time systems were introduced in Saberi, Kokotović, and Sussmann (1990), where the *z*-system is only driven by the output $y = Cx$ of the linear *x*-system. A discrete-time form of such a nonlinear cascade system was considered, for instance, in Lin and Byrnes (1994). Note that Corollary 2.4 represents an extension of Corollary 2.2 of Lin and Byrnes (1994) to the global case.

2.3. Robust exponential stability

To develop our results in Section 3 on stability analysis and control synthesis, we present in this section two converse Lyapunov theorems for robust exponential stability. Consider the system

$$x(k+1) = f(x(k), u(k)), \tag{5}$$

where $f : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is continuous, and Ω is a subset of $\mathbb{R}^m$. We let $\mathscr{M}_\Omega$ denote the set of input functions taking values in Ω.

Definition 2.5. We say that system (5) is locally exponentially stable uniformly in $u \in \mathscr{M}_\Omega$ (ULES), if there exists a neighborhood D_x of 0 such that, for some $a > 0$ and $0 < b < 1$, it holds that

$$|x(k, \xi, u)| \leqslant a|\xi|b^k \qquad \forall k \in \mathbb{Z}_+ \tag{6}$$

for all $\xi \in D_x$, and all $u \in \mathscr{M}_\Omega$.

If (6) holds with $D_x = \mathbb{R}^n$, then we say that the system is uniformly globally exponentially stable (UGES).

It was shown in Jiang and Wang (2002) that the robust global asymptotic stability property for a system as in (5) is equivalent to the existence of a continuous Lyapunov function. Below we present similar results on the exponential stability property. Although the precise statements were not available in the literature, there are a fair amount of variations of converse theorems (e.g., Hahn, 1967; Jiang & Wang, 2002; Khalil, 1996) whose proofs (most of which were rooted in Massera (1956) and Nadzieja (1990)) can be easily adapted to prove the theorems presented below. For this reason, we will provide rough sketches of the proofs in the appendix instead of presenting detailed complete proofs.

Theorem 2. *Assume that system* (5) *is* ULES *with* $u \in \mathscr{M}_\Omega$ *and with* D_x *as in Definition* 2.5. *Suppose the control value set* Ω *is compact. Then, for any* $p > 0$, *there exists a continuous Lyapunov function V defined on* D_x *such that the following holds for some* $c_1, c_2, c_3 > 0$:

$$c_1|\xi|^p \leqslant V(\xi) \leqslant c_2|\xi|^p \quad \forall \xi \in D_x,$$
$$V(f(\xi, \mu)) - V(\xi) \leqslant -c_3|\xi|^p \quad \forall (\xi, \mu) \in D_x \times \Omega. \tag{7}$$

When $D_x = \mathbb{R}^n$, *that is, when the system is* UGES, *then, for any* $p > 0$, *the system admits a global Lyapunov function that fulfills* (7) *for each* $(\xi, \mu) \in \mathbb{R}^n \times \Omega$.

In many situations, the most interesting case is when $p=2$. Since the proof in the general case follows exactly the same as in the case when $p=2$, we present the result in the general case of $p > 0$.

For a system as in (5), we say that $f(\xi, \mu)$ is locally Lipschitz in ξ uniformly in $\mu \in \Omega$ if for any compact set *K*, there is some $c > 0$ such that $|f(\xi, \mu) - f(\eta, \mu)| \leqslant c|\xi - \eta|$ for all $\xi, \eta \in K$ and all $\mu \in \Omega$. If the constant *c* can be chosen independent of *K*, then we say that *f* is globally Lipschitz in ξ uniformly in $\mu \in \Omega$. In the case when *f* is either locally or globally Lipschitz, we can strengthen the result given in Theorem 2. In this case, we do not need to require the compactness property of Ω.

Theorem 3. *Assume that system* (5) *is* ULES *with* $u \in \mathscr{M}_\Omega$, *where* Ω *is not necessarily compact. Suppose f is locally Lipschitz in a neighborhood of* 0 *uniformly in* $\mu \in \Omega$. *Then for any* $p \geqslant 1$, *there exists a Lyapunov function V defined in a neighborhood* U_0 *of the origin that satisfies* (7) *and, for some* $c_4 > 0$, *the following holds*:

$$|V(\xi) - V(\eta)| \leqslant c_4(|\xi| + |\eta|)^{p-1}|\xi - \eta| \quad \forall \xi, \eta \in U_0. \tag{8}$$

In particular, if the system is UGES *and if f is globally Lipschitz uniformly in* $\mu \in \Omega$, *then for any* $p \geqslant 1$, *the system admits a global Lyapunov function that satisfies* (7) *and* (8) *for all* $\xi, \eta \in \mathbb{R}^n$ *and all* $\mu \in \Omega$.

Remark 2.6. Note that estimate (8) is stronger than the local Lipschitz continuity of *V* near 0. With $p > 1$, the Lipschitz constant of *V* can be made arbitrarily small around the origin.

Remark 2.7. In the special case when system (5) is disturbance free, i.e., when the map $f(\xi, \mu)$ in (5) is independent of μ, and if in addition *f* is differentiable, our proof in the appendix shows that *V* can be chosen differentiable. Consequently, (8) can be strengthened to $|\partial V(\xi)/\partial \xi| \leqslant c_5|\xi|^{p-1}$ (where c_5 is some constant) in a neighborhood of the origin.

2.4. Proof of Theorem 1

Although the following Lemma 2.8 only represents a special case of Theorem 1, nevertheless, in combination with Lemma 2.2, it yields Theorem 1.

Lemma 2.8. *Consider the following system*:

$$\begin{aligned} x_1(k+1) &= f_1(x_1(k), v_1(k)),\\ x_2(k+1) &= f_2(x_2(k), v_2(k)). \end{aligned} \tag{9}$$

Suppose, for $i = 1, 2$, *there exist some* $r_i > 0$, *some* $\beta_i \in \mathcal{KL}$ *and some* $\gamma_i^v \in \mathcal{K}$ *such that*

$$|x_i(k, \xi_i, v_i)| \leqslant \max\{\beta_i(|\xi_i|, k),\ \gamma_i^v(\|v_i\|)\} \tag{10}$$

holds for all $k \in \mathbb{Z}_+$, $|\xi_i| \leqslant r_i$ *and all* $\|v_i\| \leqslant r_i$. *Assume that the small-gain condition holds locally between* γ_1^v *and* γ_2^v. *Then the interconnected system* (9) *with* (4) *is LAS.*

Proof. Let $r > 0$ be such that, for $i = 1, 2$, (10) holds for all $|\xi_i| \leqslant r$ and $\|v_i\| \leqslant r$, and that both $\gamma_1^v \circ \gamma_2^v(s) < s$ and $\gamma_2^v \circ \gamma_1^v(s) < s$ hold for all $0 < s \leqslant r$. By following the same proof of Lemma 3.8 in Jiang and Wang (2001), one can show that there exists some $0 < r_0 \leqslant r$ such that

$$\lim_{k\to\infty} |x_i(k, \xi_i, v_i)| \leqslant \gamma_i^v \left(\lim_{k\to\infty} |v_i(k)| \right) \tag{11}$$

holds for all $|\xi_i| \leqslant r_0$, $\|v_i\| \leqslant r_0$, $i = 1, 2$.

Let $\beta_0(s) = \max\{\beta_1(s, 0), \beta_2(s, 0)\}$. Without loss of generality, one may assume that $\beta_0(s) \geqslant s$ and $\gamma_i^v \in \mathcal{K}_\infty$ for $i = 1, 2$. Let

$$\delta = \min\{\beta_0^{-1}(r_0),\ (\gamma_1^v \circ \beta_0)^{-1}(r_0),\ (\gamma_2^v \circ \beta_0)^{-1}(r_0)\}.$$

One can show that if $\max\{|\xi_1|, |\xi_2|\} < \delta$, then along the trajectory $x(k) = (x_1(k), x_2(k))$ of (9) with (4) with the initial state $x(0) = (\xi_1, \xi_2)$, it holds that $|x_i(k)| \leqslant r_0$ for all $k \in \mathbb{Z}_+$ ($i = 1, 2$). Applying the small-gain condition $\gamma_1^v \circ \gamma_2^v(s) < s$ on $(0, r]$, one can conclude that for $i = 1, 2$, $|x_i(k)| \leqslant \max\{\beta_0(|\xi|),\ \gamma_i^v \circ \beta_0(|\xi|)\}$ for all $k \in \mathbb{Z}_+$ and all $|\xi| \leqslant r_0$ (where $\xi = (\xi_1, \xi_2)$). Below we show the attractiveness property.

Let $|\xi| < \delta$. Still use $x(k) = (x_1(k), x_2(k))$ to denote the trajectory of (9) with (4) with the initial state ξ. Since $|x_i(k)| \leqslant r_0$ for all k, $\lim_{k\to\infty}|x_i(k)| \leqslant r_0$. Combining this with (11), one has $\lim_{k\to\infty}|x_1(k)| \leqslant \gamma_1^v(\lim_{k\to\infty}|x_2(k)|)$ and $\lim_{k\to\infty}|x_2(k)| \leqslant \gamma_2^v(\lim_{k\to\infty}|x_1(k)|)$. Consequently, we get $\lim_{k\to\infty}|x_1(k)| \leqslant \gamma_1^v \circ \gamma_2^v(\lim_{k\to\infty}|x_1(k)|)$. Again, by the small-gain condition, $\lim_{k\to\infty}|x_1(k)| = 0$.This in turn implies $\lim_{k\to\infty}|x_2(k)| = 0$. $\square$

3. Stability and stabilization

In this section, we present some applications of nonlinear small-gain theorems obtained in this work and in Jiang and Wang (2001) for interconnected systems of the following type:

$$z(k+1) = q_1(z(k), \zeta(k)), \tag{12}$$

$$\zeta(k+1) = q_2(z(k), \zeta(k)) + \omega(z(k), \zeta(k)), \tag{13}$$

where for each $k \in \mathbb{Z}_+$, $z(k) \in \mathbb{R}^{n_1}$, $\zeta(k) \in \mathbb{R}^{n_2}$. The functions q_1, q_2 and ω are continuous and vanish at $(0, 0)$. We also assume that $q_2(\cdot, \cdot)$ is C^1 in a neighborhood of $(0, 0)$ and $q_2(\cdot, 0) \equiv 0$. Note that one can view system (12)–(13) as an interconnection of two z and ζ subsystems with the uncertain term $\omega(\cdot, \cdot)$.

The continuous-time form of system (12)–(13) with $q_2(z(k), \zeta(k)) = A\zeta(k)$ was first introduced and studied in Byrnes and Isidori (1991) (see also Isidori, 1995), followed by the work of Jiang et al. (1994). Using the small-gain arguments, Jiang et al. (1994) extended (Byrnes & Isidori, 1991, Lemma 4.3) from LAS to GAS for a larger class of interconnected nonlinear systems under more relaxed conditions on the coupling term ω. The discrete-time version of Byrnes and Isidori (1991, Lemma 4.3) was established in Lin and Byrnes (1994, Lemma 2.1), which served as a basic lemma to obtain the main theorems of Lin and Byrnes (1994). In this work we will show how the small-gain approach can be used to extend the result (Lin & Byrnes, 1994, Lemma 2.1) with much relaxed conditions on q_2 and ω.

3.1. Asymptotic stability of interconnected systems

The following result covers Lemma 2.1 in Lin and Byrnes (1994) as a special case and significantly relaxes the conditions there.

Proposition 3.1. *Consider system* (12)–(13). *Assume*:

- *For some* $M > 0$, *the unperturbed* ζ*-system*

  $$\zeta(k+1) = q_2(v_1(k), \zeta(k)) \tag{14}$$

  is LES uniformly in all $v_1(\cdot)$ *such that* $\|v_1\| \leqslant M$.
- *The z-system* (12) *is LISS with* ζ *viewed as the input, i.e., for system* $z(k+1) = q_1(z(k), v_2(k))$, *there exist* $r > 0$, $\beta_z \in \mathcal{KL}$ *and* $\gamma_z \in \mathcal{K}$ *such that for all* $|z_0| \leqslant r$ *and* $\|v_2\| \leqslant r$,

  $$|z(k, z_0, v_2)| \leqslant \max\{\beta_z(|z_0|, k),\ \gamma_z(\|v_2\|)\}.$$

- *There exist some constants* $\mu_1, \mu_2 \geqslant 0$ *and some* $r > 0$ *such that for all* $|z| < r$ *and* $|\zeta| < r$

  $$|\omega(z, \zeta)| \leqslant \mu_1|\zeta| + \mu_2\gamma_z^{-1}(|z|). \tag{15}$$

Then, the interconnected system (12)–(13) is LAS if μ_1 and μ_2 are sufficiently small. Furthermore, system (12)–(13) becomes GAS if the above conditions hold globally and, additionally, $\frac{\partial q_2}{\partial \zeta}(w, \zeta)$ is bounded for all ζ and w.

Proof. Let $M>0$. By Theorem 3, there is a V defined in some neighborhood U_ζ of 0 for the ζ-system such that (7) and (8) hold for some $c_1, c_2, c_4>0$, and for some $c_3>0$,

$$V(q_2(v,\zeta)) - V(\zeta) \leqslant -c_3|\zeta| \tag{16}$$

for all $|v|\leqslant M$. Note then that, for all $|v|\leqslant M$,

$$|q_2(v,\zeta)| \leqslant \frac{V(q_2(v,\zeta))}{c_1} \leqslant \frac{V(\zeta)}{c_1} \leqslant \frac{c_2}{c_1}|\zeta|. \tag{17}$$

By hypotheses, there is a neighborhood $D_z\times D_\zeta$ of $(z,\zeta)=(0,0)$ with $D_\zeta\subset U_\zeta$ such that $q_2(z,\zeta)+\omega(z,\zeta)\in U_\zeta$ for all $(z,\zeta)\in D_z\times D_\zeta$. One may also assume that $q_2(z,\zeta)\in U_\zeta$ for all $(z,\zeta)\in D_z\times D_\zeta$ (cf. (17)). Hence,

$$\begin{aligned}&|V(q_2(z,\zeta)+\omega(z,\zeta)) - V(q_2(z,\zeta))|\\ &\quad\leqslant c_4|\omega(z,\zeta)| \leqslant c_4\,\mu_1|\zeta| + c_4\,\mu_2\,\gamma_z^{-1}(|z|).\end{aligned} \tag{18}$$

Combining this with (16), we get $V(q_2(z,\zeta)+\omega(z,\zeta)) - V(\zeta) \leqslant -(c_3-c_4\mu_1)|\zeta| + c_4\mu_2\gamma_z^{-1}(|z|)$. Thus, V is an LISS-Lyapunov function for the ζ-system if μ_1 is such that $c_4\,\mu_1<c_3$. By Lemma 2.3, there exists a function β_1 of class $\mathcal{KL}$ such that, for small initial condition ζ_0 and small input z, $|\zeta(k,\zeta_0,z)| \leqslant \max\{\beta_1(|\zeta_0|,k)\ ,\ \gamma_1(\|z\|)\}$, where γ_1 is the class $\mathcal{K}$-function given by

$$\gamma_1(r) \;=\; \frac{c_4\,\mu_2}{(1-c_0)c_1(c_3-c_4\,\mu_1)}\,\gamma_z^{-1}(r), \tag{19}$$

for any $c_0\in(0,1)$. Clearly, the small-gain condition holds between γ_z and γ_1 if μ_1 and μ_2 are sufficiently small, and as a consequence, the composed system (12)–(13) is locally asymptotically stable.

The second statement of Proposition 3.1 follows straightforwardly the global version of the small-gain theorem (see Jiang & Wang, 2001). □

3.2. A robust separation principle

As an interesting application of Proposition 3.1, we present a "robust separation principle" for discrete-time nonlinear systems with disturbances

$$x(k+1) \;=\; f(x(k),u(k)) \;+\; \Delta f(x(k),u(k)), \tag{20}$$

where Δf is an unknown disturbance.

A discrete-time separation principle was obtained independently in Kazakos and Tsinias (1993, Theorem 1) and Lin and Byrnes (1994, Theorem 4.3) in the absence of disturbances Δf. We first recover the main result, Theorem 1, in Kazakos and Tsinias (1993) by means of Theorem 1 in the current work, for undisturbed systems with outputs

$$x(k+1) = f(x(k),u(k)),\;\; y(k) = h(x(k)), \tag{21}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, and $y\in\mathbb{R}^l$. Then, using Proposition 3.1, we generalize Lin and Byrnes (1994, Theorem 4.3) to the disturbed setting (20).

We recall from Kazakos and Tsinias (1993), that a system of the form (21) is said to be *locally detectable* if there exist a neighborhood $D_e\subset\mathbb{R}^n$ of the origin, a continuous mapping $\hat f:\mathbb{R}^n\times\mathbb{R}^l\times\mathbb{R}^m\to\mathbb{R}^n$, a continuous function $W:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+$ and class $\mathcal{K}$-functions ψ_i, $1\leqslant i\leqslant 3$, such that $f(x,u)=\hat f(x,h(x),u)$ for all $(x,u)\in\mathbb{R}^n\times\mathbb{R}^m$, and the following inequalities hold:

$$\begin{aligned}&\psi_1(|x-\hat x|) \;\leqslant\; W(x,\hat x) \;\leqslant\; \psi_2(|x-\hat x|),\\ &W(f(x,u),\hat f(\hat x,h(x),u)) - W(x,\hat x) \leqslant -\psi_3(|x-\hat x|)\end{aligned}$$

for all $u\in\mathbb{R}^m$, $e=x-\hat x\in D_e$. When $D_e=\mathbb{R}^n$, system (21) is said to be *globally detectable*.

Corollary 3.2 (*Kazakos & Tsinias, 1993, Theorem 1*). *If the system* (21) *is locally detectable and LAS-stabilizable by means of* $u(k)=\alpha(x(k))$,[2] *then the detector-based output feedback law*

$$\hat x(k+1) = \hat f(\hat x(k),y(k),u(k)),\;\; u=\alpha(\hat x(k)) \tag{22}$$

locally asymptotically stabilizes the composite system (21) *and* (22).

Recall from Moraal and Grizzle (1995), Lin and Byrnes (1994), that the $\hat x$-system in (22) is said to be a *local exponential observer* for system (21) if there exists a neighborhood $D_e\subseteq\mathbb{R}^n\times\mathbb{R}^n$ of the origin such that, for some constants $a>0$ and $0<b<1$,

$$|x(k,\xi,u) - \hat x(k,\eta,u)| \;\leqslant\; a|\xi-\eta|b^k \tag{23}$$

holds for all $k\in\mathbb{Z}_+$, all $(\xi,\eta)\in D_e$, and all input functions u. When (23) holds with $D_e=\mathbb{R}^n\times\mathbb{R}^n$, the $\hat x$-system in (22) is said to be a *global exponential observer* for system (21). In case when such a local (resp. global) exponential observer exists, system (21) is said to be *exponentially detectable* (resp. globally exponentially detectable) (Lin & Byrnes, 1994). Below we assume that f and $\hat f$ are C^1 maps.

Theorem 4 (*Robust Separation Principle*). *Assume that the nominal system* (21) *is exponentially detectable and that the perturbed system* (20) *is robustly* LAS*-stabilizable by means of a* C^1 *feedback* $u=\alpha(x)$. *Let* γ_x *be an* LISS*-gain function associated with the system* (20) *in closed-loop with* $u=\alpha(x(k)-e(k))$ (*which is* LISS *with input* $e(\cdot)$ *by Lemma* 2.2). *If the following holds*

$$|\Delta f(x,\alpha(x-e))| \;\leqslant\; \mu_1|e| \;+\; \mu_2\gamma_x^{-1}(|x|) \tag{24}$$

for some sufficiently small constants $\mu_1,\mu_2\geqslant 0$ *for all* (x,e) *in a neighborhood of the origin, then, the observer-based output feedback law*

$$\hat x(k+1) = \hat f(\hat x(k),y(k),u(k)),\;\; u=\alpha(\hat x(k)) \tag{25}$$

robustly locally asymptotically stabilizes the composite system (20) *and* (25).

[2] In contrast to Kazakos and Tsinias (1993, Theorem 1), we do not require that $f(0,0)=0$ and $\alpha(0)=0$.

Moreover, if the above conditions hold globally, and if $\frac{\partial q_2}{\partial e}(x,e)$ *is bounded for all* (x,e), *where* $q_2(x,e) = f(x,\alpha(x-e)) - \hat{f}(x-e, h(x), \alpha(x-e))$, *then* (25) *is a global asymptotic stabilizer for system* (20).

Proof. In the (x,e)-coordinates, the closed-loop system (20)–(25) can be rewritten as

$$\begin{aligned} x(k+1) &= f(x(k),\alpha(x(k)-e(k))) \\ &\quad + \Delta f(x(k),\alpha(x(k)-e(k))), \\ e(k+1) &= q_2(x(k),e(k)) + \Delta f(x(k),\alpha(x(k)-e(k))). \end{aligned} \tag{26}$$

Applying Proposition 3.1 with $z = x$, $\zeta = e$, $\omega = \Delta f$ and $q_1(x,e) = f(x,\alpha(x-e)) + \Delta f(x,\alpha(x-e))$, one sees that the composite system (26) is locally asymptotically stable. The second statement of Theorem 4 follows again from Proposition 3.1. □

3.3. Robust global stabilization via static output feedback

Consider a discrete-time system with outputs

$$\begin{aligned} z(k+1) &= q(z(k),x(k)), \\ x(k+1) &= Ax(k) + B(u(k) + \omega(z(k),x(k))), \\ y(k) &= Cx(k), \end{aligned} \tag{27}$$

where $(z(k),x(k)) \in \mathbb{R}^{n_0} \times \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^l$ represent the state, the input and the output, respectively. The map $\omega : \mathbb{R}^{n_0} \times \mathbb{R}^n \to \mathbb{R}^m$ is an unknown coupling function, that is, the x-system can be viewed as a linear system $x(k+1) = Ax(k) + Bu(k)$ subject to an actuator disturbance $\omega(z,x)$. The z-system is referred to as *dynamic uncertainties* with unmeasured state z and uncertain function q. Note that the system in the absence of z-dynamics was studied in Magana and Zak (1988). System (27) can also be seen as an input-output linearizable system with matched uncertainty (Isidori, 1995; Nijmeijer & van der Schaft, 1990).

The following assumptions are made on (27):

(H1) The z-system is ISS with respect to x and with an ISS-gain $\gamma_z \in \mathscr{K}_\infty$.

(H2) The linear system $x(k+1) = Ax(k) + Bu(k)$ with output $y(k) = Cx(k)$ is output-feedback stabilizable (Magana & Zak, 1988), i.e., there is a matrix G such that the spectral radius of $A_0 = A - BGC$ is strictly less than one.

(H3) There are two constant matrices $Q = Q^{\mathrm{T}} > 0$ and F, of appropriate dimensions, such that $B^{\mathrm{T}}PA_0 = FC$, with $P = P^{\mathrm{T}} > 0$ the solution of the discrete Lyapunov equation $A_0^{\mathrm{T}}PA_0 - P = -Q$.

(H4) There exist a class $\mathscr{K}_\infty$ function ρ and a constant $\varepsilon \geqslant 0$ such that

$$|\omega(z,x)| \leqslant \rho(|z|) + \varepsilon|x| \quad \forall (z,x) \in \mathbb{R}^{n_0} \times \mathbb{R}^n. \tag{28}$$

Our objective is to design a linear, memoryless, output feedback law of the form $u(k) = Ky(k)$ with K a constant $m \times l$ matrix, in such a way that the closed-loop system is GAS. Our solution is presented below that extends the main theorems of Magana and Zak (1988) to a larger class of nonlinear discrete systems.

For a positive semi-definite symmetric matrix F, we use $\lambda^F_{\min}$ ($\lambda^F_{\max}$, resp) to denote the minimal (maximal, resp) eigenvalue of F.

Proposition 3.3. *Assume* (H1)–(H4) *holds. Suppose the matrices P and Q as in* (H3), *the function ρ and the constant ε as in* (H4) *can be chosen such that the following holds for some* $\varepsilon_1, \varepsilon_2 > 0$:

$$\varepsilon < \sqrt{\frac{\lambda^Q_{\min}}{(1+\varepsilon_1)\lambda^{B^{\mathrm{T}}PB}_{\max}}}, \tag{29}$$

$$\rho(s) < \sqrt{\frac{(1-\varepsilon_2)(\lambda^Q_{\min} - \lambda^{B^{\mathrm{T}}PB}_{\max}(1+\varepsilon_1)\varepsilon^2)\lambda^P_{\min}}{(1+\varepsilon_1^{-1})\lambda^P_{\max}\lambda^{B^{\mathrm{T}}PB}_{\max}}}\,\gamma_z^{-1}(s)$$

for all $s > 0$. *Then the static output feedback controller given by*

$$u(k) = -Gy(k) - (B^{\mathrm{T}}PB)^{-1}Fy(k) \tag{30}$$

globally asymptotically stabilizes the system (27).

Proof. Consider (27) with (30). As it can be directly checked, by Assumptions (H2)–(H4), the function $V(x) = x^{\mathrm{T}}Px$ satisfies $V(x(k+1)) - V(x(k)) \leqslant -x^{\mathrm{T}}(k)Qx(k) + \lambda^R_{\max}\varepsilon^2(1+\varepsilon_1)|x(k)|^2 + \lambda^R_{\max}(1+\varepsilon_1^{-1})\rho(|z(k)|)^2$, where $R := B^{\mathrm{T}}PB$. As a result, with (29), V is an ISS-Lyapunov function for the closed-loop x-subsystem with z as the input. Furthermore, by Lemma 2.3, an ISS gain function can be chosen as

$$\gamma_x(r) = \sqrt{\frac{(1+\varepsilon_1^{-1})\lambda^R_{\max}\lambda^P_{\max}}{(1-\varepsilon_2)[\lambda^Q_{\min} - \lambda^R_{\max}\varepsilon^2(1+\varepsilon_1)]\lambda^P_{\min}}}\,\rho(r).$$

By the assumption on ρ, the small-gain condition $\gamma_x \circ \gamma_z(r) < r$ holds for all $r > 0$. Hence, the closed-loop system is GAS by the ISS small-gain Theorem 2 of Jiang and Wang (2001). □

Note that a local version of Proposition 3.3 can be established as a direct application of Theorem 1.

4. Conclusions

In this paper, we have developed a nonlinear, local ISS small-gain theorem for discrete-time interconnected systems and two converse Lyapunov theorems for uniform exponential stability. It is shown that these theorems together with a global ISS small-gain theorem, recently introduced in Jiang and Wang (2001), are very effective in stability analysis and control synthesis for discrete-time nonlinear systems with complex structures. As appealing applications, these tools yield significant extensions of earlier work in discrete-time nonlinear systems and control, cf Kazakos and Tsinias (1993), Lin and Byrnes (1994) and Magana and Zak (1988).

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Appendix A. Proofs of Theorems 2 and 3

Proof of Theorem 2 (*Sketch*). Let $p > 0$ be given. Define $V : D_x \to \mathbb{R}_{\geqslant 0}$ by

$$V(\xi) = \sup_{u \in \mathscr{M}_\Omega} V_0(\xi, u), \tag{A.1}$$

where V_0 is defined by $V_0(\xi, u) = \sum_{k=0}^{K} |x(k, \xi, u)|^p$ on $D_x \times \mathscr{M}_\Omega$ with K chosen large enough so that $a^p b^{p(K+1)} \leqslant 1/2$, and where a, b are as in (6). It follows from (6) that $|\xi|^p \leqslant V(\xi) \leqslant \frac{a^p}{1-b^p}|\xi|^p$. By the compactness of Ω and the continuity of f, the function V_0 is continuous in ξ uniformly in $u \in \mathscr{M}_\Omega$, and consequently, $V(\xi)$ is continuous.

With the property that $x(k, f(\xi, \mu), u) = x(k+1, \xi, v)$ for some v, one sees that for all $\xi \in D_x$ and all $\mu \in \Omega$,

$$V(f(\xi, \mu)) \leqslant \sup_{w \in \mathscr{M}_\Omega} \sum_{k=1}^{K+1} |x(k, \xi, w)|^p \leqslant V(\xi) - |\xi|^p + (a|\xi| b^{K+1})^p \leqslant V(\xi) - \frac{1}{2}|\xi|^p.$$

The last statement of Theorem 2 about the global case follows readily from the above reasoning. □

Proof of Theorem 3 (*Sketch*). Let $p \geqslant 1$ be given. We define the functions V and V_0 as in (A.1). The only point different from the proof of Theorem 2 is to show the stronger Lipschitz continuity property (8).

The proof of (8) follows the same idea in Hahn (1967, Theorem 56.1). Let U be a neighborhood of 0 on which $f(\xi, \mu)$ is Lipschitz uniformly in μ with a Lipschitz constant c. Choose U_0 such that $x(k, \xi, u) \in U$ for all k, all $\xi \in U_0$ and all u. By induction, one can show that, for each $u \in \mathscr{M}_\Omega$ and each $k \in \mathbb{Z}_+$, $|x(k, \xi, u) - x(k, \eta, u)| \leqslant c^k |\xi - \eta|$ for all $\xi, \eta \in U_0$. By the mean value theorem, one can show that

$$\begin{aligned} |V_0(\xi, u) - V_0(\eta, u)| &\leqslant \sum_{k=0}^{K} p(|x(k, \xi, u)| \\ &\quad + |x(k, \eta, u)|)^{p-1} \big| |x(k, \xi, u)| \\ &\quad - |x(k, \eta, u)| \big| \\ &\leqslant c_4(|\xi| + |\eta|)^{p-1} |\xi - \eta| \end{aligned} \tag{A.2}$$

for all $\xi, \eta \in U_0$, all $u \in \mathscr{M}_\Omega$, where $c_4 = p a^{p-1} \sum_{k=0}^{K} c^k$. Passing through the supremum, one can show that (A.2) also holds for V.

The last statement of Theorem 3 about the global case again follows from the above proof in an obvious way. □

References

Agarwal, R. P. (1992). *Difference equations and inequalities: theory, methods and applications*. New York: Marcel Dekker.

Byrnes, C. I., & Isidori, A. (1991). Asymptotic stabilization of minimum-phase nonlinear systems. *IEEE Transactions on Automatic Control*, *36*, 1122–1137.

Coron, J.-M., Praly, L., & Teel, A. (1995). Feedback stabilization of nonlinear systems: sufficient conditions and Lyapunov and input–output techniques. in: Isidori (Ed.), *Trends in control*. Berlin: Springer.

Desoer, C., & Vidyasagar, M. (1975). *Feedback systems: input–output properties*. New York: Academic Press.

Gao, K., Lin, Y. (2000). On equivalent notions of input-to-state stability for nonlinear discrete time systems. In *Proceedings of IASTED international conference on control and applications*, Cancun (pp. 81–87).

Hahn, W. (1967). *Stability of motion*. Berlin: Springer.

Isidori, A. (1995). *Nonlinear control systems: an introduction*. (3rd ed.), New York: Springer.

Isidori, A. (1999). Nonlinear control systems II. London: Springer.

Jiang, Z. P., & Mareels, I. (1997). A small-gain control method for nonlinear cascaded systems with dynamic uncertainties. *IEEE Transactions on Automatic Control*, *42*, 292–308.

Jiang, Z. P., Sontag, E. D., Wang, Y. (1999). Input-to-state stability for discrete-time nonlinear systems. In *Proceedings of 14th IFAC World Congress*, Beijing (Vol. E. pp. 277–282).

Jiang, Z. P., Teel, A., & Praly, L. (1994). Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals, and Systems*, *7*, 95–120.

Jiang, Z. P., & Wang, Y. (2001). Input-to-state stability for discrete-time nonlinear systems. *Automatica*, *37*, 857–869.

Jiang, Z. P., & Wang, Y. (2002). A converse Lyapunov theorem for discrete time systems with disturbances. *Systems and Control Letters*, *45*(1), 49–58.

Kazakos, D., & Tsinias, J. (1993). Stabilization of nonlinear discrete-time systems using state detection. *IEEE Transactions on Automatic Control*, *38*, 1398–1400.

Khalil, H. K. (1996). *Nonlinear systems*. 2nd ed., Upper Saddle River, NJ: Prentice-Hall.

Kokotović, P. V., & Arcak, M. (2001). Constructive nonlinear control: a historical perspective. *Automatica*, *37*, 637–662.

Krstić, M., Kanellakopoulos, I., & Kokotović, P. V. (1995). *Nonlinear and adaptive control design*. New York: Wiley.

Laila, D.S., Nešić, D., 2002. Lyapunov based small-gain theorem for parameterized discrete-time interconnected ISS systems. *Proceedings of* 41st *IEEE conference on decision control*, Las Vegas (pp. 2292–2297).

Lin, W., & Byrnes, C. I. (1994). Design of discrete-time nonlinear control systems via smooth feedback. *IEEE Transactions on Automatic Control*, *39*, 2340–2346.

Magana, M. E., & Zak, S. H. (1988). Robust output feedback stabilization of discrete-time uncertain dynamical systems. *IEEE Transactions on Automatic Control*, *33*, 1082–1085.

Mareels, I. M. Y., & Hill, D. J. (1992). Monotone stability of nonlinear feedback systems. *Journal of Mathematical Systems Estimation Control*, *2*, 275–291.

Massera, J. (1956). Contributions to stability theory. *Annals of Mathematics*, *64*, 182–206.

Moraal, P. E., & Grizzle, J. W. (1995). Observer design for nonlinear systems with discrete-time measurements. *IEEE Transactions on Automatic Control*, *40*, 395–404.

Nadzieja, T. (1990). Construction of a smooth Lyapunov function for an asymptotically stable set. *Czechoslovak Mathematical Journal*, *115*(40), 195–199.

Nijmeijer, H., & van der Schaft, A. (1990). *Nonlinear dynamical control systems*. New York: Springer.

Saberi, A., Kokotović, P. V., & Sussmann, H. J. (1990). Global stabilization of partially linear composite systems. *SIAM Journal on Control and Optimization*, *28*, 1491–1503.

Sontag, E. D. (1989). Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, *34*, 435–443.

Tsinias, J. (1993). Versions of Sontag's "input to state stability conditions" and the global stabilizability problem. *SIAM Journal on Control and Optimization*, *31*, 928–941.

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