Analytic Constraints and Realizability for Analytic Input/Output Operators*

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Abstract

For input/ouput (i/o) operators, an equivalence is shown between realizability by state space systems and the existence of analytic constraints on higher order derivatives of i/o signals. This provides a precise characterization of realizability, extending to the general analytic case previous work that dealt with the equivalence between algebraic realizability and algebraic i/o equations.

1 Introduction

In the previous work [20], it was shown that a Fliess input/output operator $u(\cdot) \mapsto y(\cdot)$ is representable by a type of *polynomial* state space (finite-dimensional differential equation) system if and only if the operator admits an equation of the type

$$A(u(t), u'(t), \ldots, u^{(k)}(t), y(t), y'(t), \ldots, y^{(k)}(t)) = 0\,,$$

where A is likewise a *polynomial*. Thus there is an elegant one to one correspondence between realizability and the existence of equations, just as there is in the classical linear case. This

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motivates the question: is there a similar correspondence in the general analytic case? A partial answer was obtained in [21]; there, relying both upon the above algebraic results and the rich theory of nonlinear realizability (for which see for instance [8, 9, 10, 14]), it was shown that the existence of an analytic i/o equation implies local realizability. But in contrast to the algebraic case, the *converse* of the result does not hold in general. Based on a construction provided by W. Respondek, an example was given in [21] showing that an operator may be realizable by an analytic state space system but yet may fail to satisfy any possible analytic i/o equation.

The topic of relating i/o equations to realizability is one that has been extensively studied by many authors; see for instance [3, 6, 16]. In the algebraic case treated in [20], the existence of an i/o equation is trivial, as it can be deduced by an elementary argument involving finiteness of transcendence degree. However, the analogous argument fails in the analytic case. One would need an elimination theorem for what we call "meromorphically finitely generated field extensions," but no such theorem exists in general. (Under nonsingularity assumptions, and assuming suitable observability conditions, local equations do exist, as one may apply the implicit mapping theorem; see for instance [10, 17, 2].) One also knows (cf. [22]) that there cannot exist i/o equations of order less than the dimension of any observable analytic realization.

The solution we propose in this paper is to generalize the notion of i/o equation so as to obtain a one to one correspondence that holds in the general case. We introduce an abstract notion of "analytic constraints," based on subanalytic set theory. The main result of this work is that local analytic realizability is indeed equivalent to the existence of such analytic constraints. As a consequence of our techniques, we also show that an operator realizable by an analytic system can always be approximated by a sequence of operators that do admit local i/o equations.

Outline of this article: The formalism is based on the *generating series* suggested by Fliess.

In Section 2, after briefly recalling the definition of i/o operators, we introduce the definitions of analytic constraints and local i/o equations. The main result and its proof are given in Section 3. The techniques used to deal with analytic constraints are based on the analytic stratification theory developed in [15] and [7]. The converse implication (realizability under the assumptions of the theorem) is obtained by a perturbation approach, using with the Lie rank condition for realizability given in [4, 13].

2 Basic Definitions

In this section we first recall the definition of analytic input/output operators, and briefly discuss their basic properties. (For a detailed discussion of such operators, we refer the readers to [8, 20, 21].) Then we introduce the definitions of analytic constraints and local input/output equations.

2.1 Analytic Input/Output Operators

Let P^* be the set of monomials in the noncommutative variables $\eta_0, \eta_1, \ldots, \eta_m$, i.e.,

$$P^* = \{\eta_{i_1}\eta_{i_2}\cdots\eta_{i_r} : 0 \le i_s \le m, \text{ for } s \le r, r \ge 0\} .$$

We use ϕ to denote $\eta_{i_1}\eta_{i_2}\cdots\eta_{i_r}$ if $r = 0$. A *generating series* is a formal power series:

$$c = \sum_{w \in P^*} \langle c, w\rangle w ,$$

where $\langle c, w\rangle \in \mathbb{R}$ for all w. We let $\mathfrak{S}$ denote the set of all generating series in the variables $\eta_0, \eta_1, \ldots, \eta_m$. This set can be identified with $\mathbb{R}^{\mathbb{N}}$, the set of all the mappings from $\mathbb{N}$ to $\mathbb{R}$.

We adopt the weak topology on this set, for which a basis of open sets consists of all sets of the form $\prod_{i=1}^{\infty} U_i$, where each U_i is an open subset of $\mathbb{R}$ and only finitely many of them are proper subsets of $\mathbb{R}$. With this topology, that a sequence $\{c_j\}$ converges to c means

$$\lim_{j\to\infty} \langle c_j,\ w\rangle \ = \ \langle c,\ w\rangle$$

for each $w \in P^*$.

A series c is *convergent* if there exist $K,\, M \geq 0$ such that

$$|\langle c,\ w\rangle| \leq KM^k k! \quad \text{if } |w| = k\,, \tag{1}$$

where $|w|$ is the length of w, i.e., $|w| = k$ if $w = \eta_{i_1}\eta_{i_2}\cdots\eta_{i_k}$.

For each $T > 0$, consider the set $\mathcal{U}_T$ of all essentially bounded measurable functions $u : [0, T) \to \mathbb{R}^m$ with $\|u\|_\infty := \max\{\|u_i\|_\infty : 1 \leq i \leq m\} < 1$. For each $w \in P^*$, we define $V_w : \mathcal{U}_T \to \mathcal{C}[0,\, T)$ inductively by $V_\phi = 1$ and

$$V_w[u](t) \ = \ \int_0^t u_i(s) V_{w'}(s)\, ds\,,$$

if $w = \eta_i w'$ for some η_i, where u_0 is the constant function given by $u(t) \equiv 1$. Assume now that c is a convergent series and let K and M be as in (1). Then the series of functions

$$F_c[u](t) \ := \ \sum_{w\in P^*} \langle c,\ w\rangle V_w[u](t)$$

is uniformly and absolutely convergent on $[0,\,T)$ for any T such that

$$T \,<\, \frac{1}{Mm+M}\,, \tag{2}$$

and for any $u \in \mathcal{U}_T$ (cf. [8, 21]). We say that T is admissible for c if T satisfies (2). Thus, each convergent series c defines an i/o operator F_c on $\mathcal{U}_T$ if T is admissible to c.

Let $I = [a,\ b)$ with $a < b$. A function $\sigma : I \to \mathbb{R}$ is $\mathcal{C}^k$ on I if σ can be extended to a $\mathcal{C}^k$ function defined on $(a',\ b)$ for some $a' < a$. A function σ is said to be piecewise $\mathcal{C}^k$ on I if there exist some integer l and

$$a \,=\, t_0 \,<\, t_1 \,<\, \cdots \,<\, t_{l-1} \,<\, t_l \,=\, b$$

such that σ is $\mathcal{C}^k$ on $[t_{i-1},\ t_i)$ for each $i = 1, 2, \ldots, l$. A control function u is piecewise $\mathcal{C}^k$ if each of its components is.

In [5, 21], it was shown that for any $\mathcal{C}^{k-1}$ input function $u \in \mathcal{U}_T$ with T admissible to c, $F_c[u]$ is $\mathcal{C}^k$. The same proof also shows the following:

Lemma 2.1 Let c be a convergent series for which T is admissible. Assume that $k \geq 1$. If $u \in \mathcal{U}_T$ is piecewise $\mathcal{C}^{k-1}$, then $F_c[u]$ is piecewise $\mathcal{C}^k$, and $F_c[u]$ is $\mathcal{C}^k$ at any point where u is $\mathcal{C}^{k-1}$. □

We call (u, y) a (piecewise, respectively) $\mathcal{C}^k$ i/o pair of F_c if $y = F_c[u]$ and u is (piecewise, respectively) $\mathcal{C}^k$.

An i/o operator F_c is *realizable* by an (initialized) analytic system of dimension n if there exist an analytic manifold $\mathcal{M}$ of dimension n, some $p_0 \in \mathcal{M}$, $(m+1)$ analytic vector fields $g_0, g_1, \ldots, g_m$, an analytic function $h : \mathcal{M} \to \mathbb{R}$, and some $\tau > 0$, such that for each $u \in \mathcal{U}_\tau$, there exists a solution $x(\cdot)$ of the equation

$$x' = g_0(x) + \sum_{i=1}^{m} g_i(x)u_i,$$

with $x(0) = p_0$ such that

$$F_c[u](t) = h(x(t)),$$

for all $t \in [0, \tau)$.

Note that the realizability definition given here appears different from the one given in [8]. Using Theorem III-1.5 of [8], however, one can show that the definitions are equivalent.

In [21], it was shown that an operator F_c is realizable if it satisfies an analytic input/output equation, that is, there exist some integer $k > 0$ and some nontrivial analytic function A on $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1}$ such that for all $\mathcal{C}^k$ i/o pair (u, y) of F_c, it holds that

$$A(u(t),\, u'(t),\, \ldots,\, u^{(k)}(t),\, y(t), \ldots, y^{(k)}(t)) \;=\; 0 \tag{3}$$

for all $0 \leq t < T$ for some $T > 0$. But in contrast to the result obtained in [20] for the algebraic case, the converse of the statement is in general not true for the analytic case. The following example, obtained by slightly modifying a construction by Respondek, was studied in detail in [18] to illustrate this fact. The example, with $\mathcal{M} = \mathbb{R}^3$ and $m = 3$, is as follows:

$$\begin{aligned} x_1' &= u_1\,, \\ x_2' &= u_2\,, \\ x_3' &= u_3\,, \\ h(x) &= e^{x_1} \sum_{k=0}^{\infty} a_k f_k(x_2)\, \frac{x_3^k}{k!}\ , \end{aligned}$$

with initial state $x(0) = 0$. The functions f_k and coefficients a_k are defined via

$$f_k(x) = \underbrace{\exp(\exp(\cdots(\exp(x))\cdots))}_{k}$$

for $k \geq 1$, and $f_0(x) = 1$, and $a_k = (f_k(1))^{-1}$, $k = 0, 1, \ldots$.

According to [8, Theorem III-1.5], the initialized system defines an operator F_c for some c (in fact c is determined by the Lie derivatives of h along the directions of the vector fields of the system). It is not hard to see that $F_c[u]$ is defined for $0 \leq t \leq 1$ for all u for which $\|u_2\|_\infty \leq 1$. Furthermore, the state space system defining the operator has the minimal dimension. Still, this operator does not satisfy any analytic i/o equations (cf. [18]). This shows that the existence of i/o equations is not an equivalent characterization for realizability. To provide a more precise characterization for realizability, we introduce a notion of analytic constraints in the next section.

2.2 Analytic Constraints

Throughout this work, by an analytic submanifold we mean an analytic *embedded* submanifold. Let $\mathcal{N}$ be an analytic manifold. A subset S of $\mathcal{N}$ is said to be an analytic submanifold of $\mathcal{N}$ of codimension k if for every point $p \in S$, there exist a neighborhood $\mathcal{V}$ of p, and k analytic functions $\sigma_1, \sigma_2, \ldots, \sigma_k$ defined on $\mathcal{V}$ such that $\mathrm{d}\sigma_1(q), \ldots, \mathrm{d}\sigma_k(q)$ are linearly independent for $q \in \mathcal{V}$ and it holds that $S \cap \mathcal{V} = \{q : \sigma_i(q) = 0,\ 1 \leq i \leq k\}$.

Definition 2.2 Assume that $\mathcal{M}$ is an analytic manifold. A subset S of $\mathcal{M}$ is said to be an *analytically stratified union* of a family $\mathcal{T}$ (whose members are called strata) of submanifolds of $\mathcal{M}$ if the following properties hold:

1. each stratum is a connected analytic submanifold of $\mathcal{M}$,

2. if $T_1,\ T_2 \in \mathcal{T}$ are strata and $\bar{T}_1 \cap T_2 \neq \emptyset$, then T_2 is a subset of $\bar{T}_1$, where $\bar{T}_1$ denotes the closure of T_1.

3. if a stratum $T_2 \neq T_1$ is a subset of $\bar{T}_1$, then $\operatorname{codim} T_2 > \operatorname{codim} T_1$. ∎

A subset S of an analytic manifold $\mathcal{M}$ is said to be an *analytically thin* subset if S is a locally finite analytically stratified union of a family $\mathcal{T}$ of strata of codimensions at least 1. Observe that if S is analytically thin, then S is nowhere dense, that is, the closure of S does not have any interior point.

Definition 2.3 An operator F_c is said to satisfy an *analytic constraint* if there exist some integer k, an analytically thin subset S of $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1}$ and some $\tau > 0$ such that for each piecewise $\mathcal{C}^k$ input function $u \in \mathcal{U}_\tau$, it holds that

$$\left(u(t),\, u'(t),\, \ldots,\, u^{(k)}(t),\, y(t),\, y'(t),\, \ldots, y^{(k)}(t)\right) \in S$$

for all $t \in [0, \tau)$. □

Remark 2.4 Assume that an i/o operator F_c satisfies an analytic i/o equation (3). Let S be the subset of $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1}$ defined by $S = \{p :\ A(p) = 0\}$. Then S is an analytic set as defined in [15], and therefore, a stratified union of analytic submanifolds (cf. [15]). Since S is nowhere dense, none of the submanifolds that compose S can have codimension 0. Thus, S is an analytically thin subset of $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1}$. One sees from here that if F_c satisfies an analytic i/o equation, then it satisfies an analytic constraint. □

2.3 Input/Output Equations

By a *local analytic input/output equation of order k* we mean an equation of the type

$$A(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)) = 0, \tag{4}$$

where A is an analytic function defined on some open subset Ω of $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1}$, and nontrivial in the last variable, i.e., there exists some $(\bar{\mu}_0, \ldots, \bar{\mu}_k, \bar{\nu}_0, \ldots, \bar{\nu}_{k-1})$ such that $(\bar{\mu}_0, \ldots, \bar{\mu}_k, \bar{\nu}_0, \ldots, \bar{\nu}_{k-1}, \bar{\nu}_k) \in \Omega$ for some $\bar{\nu}_k$ and

$$A(\bar{\mu}_0, \ldots, \bar{\mu}_k, \bar{\nu}_0, \ldots, \bar{\nu}_{k-1}, \nu_k)$$

is not a constant function.

For $\mu = (\mu_0, \ldots, \mu_k) \in \mathbb{R}^{m(k+1)}$ with $|\mu_0| < 1$ and $r,\ T > 0$, we let $B_k(\mu, r, T)$ be the set of piecewise $\mathcal{C}^k$ input functions defined by:

$$B_k(\mu, r, T) = \{u \in \mathcal{U}_T : \ |u^{(i)}(t) - \mu_i| < r,\ 0 \le i \le k,\ 0 \le t < T\}.$$

Let F_c be an i/o operator. We say that F_c *admits a local analytic i/o equation of order k* if there exist some $\mu \in \mathbb{R}^{m(k+1)}$, some $r > 0$ and some $T > 0$ such that (4) holds for all i/o pairs (u, y) of F_c for which $u \in B_k(\mu, r, T)$. In that case, (4) is called a local i/o equation of F_c.

Observe that if an operator (globally) admits an i/o equation (4) as defined in [21], i.e., the function A in (4) is analytic everywhere on $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1}$ and the equation holds for each $\mathcal{C}^k$ i/o pair (u, y) (and hence for each *piecewise* $\mathcal{C}^k$ i/o pairs (u, y), cf. [21, Remark 4.2]), then the operator satisfies the equation also in the local sense as defined above.

3 Main Results

The following theorem is the main result of this work.

Theorem 1 *Let F_c be an i/o operator defined by a convergent series c. Then the following are equivalent:*

1. *F_c is locally realizable by an analytic state space system;*
2. *F_c satisfies an analytic constraint;*
3. *There exist a sequence of convergent series $\{c_j\}$ and an integer $k > 0$ such that each F_{c_j} admits a local analytic i/o equation of order lower than or equal to k, and that*

$$\lim_{j\to\infty} c_j = c\,.$$

We will prove the theorem in the following sections.

3.1 From Realizability to Analytic Constraints $(1 \Longrightarrow 2)$

Lemma 3.1 If F_c is locally realizable by an analytic state space system, then it satisfies an analytic constraint.

Proof. The proof of the lemma involves the concepts of subanalytic sets and analytic stratification of analytic manifolds. For the detailed definitions of these concepts, we refer the readers to [15, 7].

Assume F_c is realizable by $(\mathcal{M},\ (g_0,\ \ldots,\ g_m),\ p_0,\ h)$ of dimension n. Let $\mathcal{N}$ be a compact neighborhood of p_0. For instance, one may assume that $p_0 \in \mathbb{R}^n$ and choose $\mathcal{N}$ to be the set

$\{p:\ \|p-p_0\| \le r\}$ for some small $r>0$. Define a map $\varphi:\ \mathbb{R}^{m(n+1)} \times \mathcal{N} \to \mathbb{R}^{m(n+1)} \times \mathbb{R}^{n+1}$ by

$$\varphi : (\mu_0,\, \mu_1,\, \ldots,\, \mu_n,\, p) \mapsto (\mu_0,\, \mu_1,\, \ldots,\, \mu_n,\, h(p),\, y_1(p),\, \ldots,\, y_n(p))$$

where $\mu_i \in \mathbb{R}^m$,

$$y_i(p) = \left.\frac{d^i}{dt^i}\right|_{t=0} h(x(t)),$$

and $x(t)$ is the solution of the equations

$$x'(t) = g_0(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t), \quad x(0) = p\,,$$

and u is any control with the initial values $u^{(i)}(0) = \mu_i$ for $0 \le i \le n$.

Clearly φ is an analytic map defined on a subanalytic set. It is also not hard to see that φ is proper on $\mathbb{R}^{m(n+1)} \times \mathcal{N}$. It then follows from Corollary 5 of [15] that the image $\mathcal{W}$ of φ is a subanalytic set. Again by Theorem 3 in [15], one knows that there is an analytic stratification $\mathcal{S}$ of $\mathcal{M}$ so that $\mathcal{W}$ is a union of some strata of the stratification. Since the preimage of $\mathcal{W}$ is $m(n+1)+n$ dimensional, by Sard's Theorem (cf. [1]), one knows that none of the strata contained in $\mathcal{W}$ could have codemision 0. Thus we reach the conclusion that $\mathcal{W}$ is an analytically thin set.

Now we turn back to the initialized system $(\mathcal{M},\, (g_0,\, \ldots,\, g_m), p_0,\, h)$. First note that there exists some $T>0$ such that $x(t) \in \mathcal{N}$ for all $0 \le t < T$, and for all piecewise $\mathcal{C}^k$ function $u \in \mathcal{U}_T$. For each $0 \le t < T$, one has:

$$(u(t),\, u'(t),\, \ldots,\, u^{(n)}(t),\ y(t),\, \ldots, y^{(n)}(t)) = \varphi(u(t),\, u'(t),\, \ldots,\, u^{(n)}(t),\, x(t)).$$

It then follows that

$$(u(t),\, u'(t),\, \ldots,\, u^{(n)}(t),\; y(t),\, \ldots, y^{(n)}(t)) \in \mathcal{W},$$

for any $0 \le t < T$. We conclude that F_c satisfies an analytic constraint. ∎

3.2 From Analytic Constraints to I/O Equations ($2 \Longrightarrow 3$)

Lemma 3.2 For any operator F_c satisfying an analytic constraint in $\mathbb{R}^{m(k+1)+(k+1)}$, there exists a sequence of convergent series $\{c_j\}$ such that each F_{c_j} admits a local analytic i/o equation of order less than or equal to k, and that $\lim_{j\to\infty} c_j = c$.

To prove the lemma, we need the following results.

Consider an analytically thin set S in $\mathbb{R}^r$ for some $r > 0$. Let $\mathcal{T}$ denote the family of the strata that compose S. By definition of analytically thin sets, one knows that $\mathcal{T}$ is locally finite. We also have the following conclusion:

Lemma 3.3 If T_0 is a stratum of $\mathcal{T}$ such that $\operatorname{codim} T_0 = \min\{\operatorname{codim} T :\; T \in \mathcal{T}\}$, then T_0 is open relative to S.

Proof. Let T_0 be such that

$$\operatorname{codim} T_0 = \min\{\operatorname{codim} T :\; T \in \mathcal{T}\}\,.$$

Assume that T_0 is not open relative to S. Then there exists a point $p \in T_0$ such that in any neighborhood of p, there is at least one point $q \in S \setminus T_0$. It follows that there exists a sequence $\{q_k\}$ in $S\backslash T_0$ such that $q_k \to p$. By local finiteness of $\mathcal{T}$, one may assume that there exists some stratum T_1 such that $q_k \in T_1$ for all k. From here one sees that $p \in \bar{T}_1 \cap T_0$. By the definition of stratified unions, one knows that T_0 is a proper subset of $\bar{T}_1$ and $\operatorname{codim} T_0 > \operatorname{codim} T_1$, which contradicts the minimality of $\operatorname{codim} T_0$. ∎

Let F_c be an i/o operator that satisfies an analytic constraint, that is, there exist some integer $k > 0$ and an analytically thin set $S \in \mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1}$, and some $\tau > 0$ such that for every piecewise $\mathcal{C}^k$ function $u \in \mathcal{U}_\tau$, it holds that

$$\xi(t,u) := \Big(u(t),\, u'(t),\, \ldots,\, u^{(k)}(t),\, y(t),\, y'(t),\, \ldots,\, y^{(k)}(t)\Big) \in S$$

for every $t \in [0,\, \tau)$.

We still let $\mathcal{T}$ denote the family of strata that compose S, and $\mathcal{T}_0$ denote the set of strata defined by

$$\mathcal{T}_0 = \{T \in \mathcal{T} :\ \exists\, u \text{ such that } \xi(0,\, u) \in T\}\,,$$

and let

$$r \;=\; \min\{\operatorname{codim} T :\ T \in \mathcal{T}\}\,, \quad r_0 \;=\; \min\{\operatorname{codim} T :\ T \in \mathcal{T}_0\}\,.$$

Let $\lambda = r_0 - r$. In the following we shall prove Lemma 3.2 by applying induction on λ. We first show the following result for the case when $\lambda = 0$.

Lemma 3.4 If F_c satisfies an analytic constraint in $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1}$ with $\lambda = 0$, then F_c admits a local analytic i/o equation of order less than or equal to k.

Proof. By assumption, there exists some input u so that $\xi(0,\, u) \in T_0$ for some T_0 such that

$$\operatorname{codim} T_0 = \min\{\operatorname{codim} T :\ T \in \mathcal{T}\}\,.$$

By Lemma 3.3, one sees that there exists some neighborhood $\mathcal{V}_0$ of $\xi(0, u)$ in $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1}$ so that $\mathcal{V}_0 \cap S \subseteq T_0$.

Since $\operatorname{codim} T_0 \geq 1$, there is at least one nontrivial analytic function α defined in a neighborhood $\mathcal{V}_1$ of $\xi(0, u)$ such that

$$\mathcal{V}_1 \cap T_0 = \{q \in \mathcal{V}_1 : \ \alpha(q) = 0\}.$$

Let $\mathcal{V} = \mathcal{V}_0 \cap \mathcal{V}_1$. Then $S \cap \mathcal{V} \subseteq \{q \in \mathcal{V}_1 : \ \alpha(q) = 0\}$. Let

$$\bar{\mu} = (\bar{\mu}_0, \bar{\mu}_1, \ldots, \bar{\mu}_k) = (u(0), u'(0), \ldots, u^{(k-1)}(0)) \tag{5}$$

and

$$\bar{\nu} = (\bar{\nu}_0, \bar{\nu}_1, \ldots, \bar{\nu}_k) = (y(0), y'(0), \ldots, y^{(k)}(0)). \tag{6}$$

Without loss of generality, one may assume that $\mathcal{V} = \mathcal{W} \times \mathcal{N}$ where

$$\mathcal{W} = \{\mu : \ |\mu_i - \bar{\mu}_i| < r, \ 0 \leq i \leq k\}$$

for some $r > 0$, and $\mathcal{N}$ is some neighborhood of $\bar{\nu}$ in $\mathbb{R}^{k+1}$. By Lemma A.11 in the Appendix, there exists some $0 < \tau_1 \leq \tau$ so that for all $u \in B_k(\bar{\mu}, r, \tau_1)$ and for all $0 \leq t < \tau_1$, it holds that

$$(y(t), y'(t), \ldots, y^{(k)}(t)) \in \mathcal{N}.$$

Thus, it follows that, for every $u \in B_k(\bar{\mu}, r, \tau_1)$,

$$\alpha(u(t), u'(t), \ldots, u^{(k)}(t), y(t), y'(t), \ldots, y^{(k)}(t)) \ = \ 0 \tag{7}$$

for all $t \in [0,\, \tau_1)$.

To conclude that F_c admits a local analytic equation of order less than or equal to k, we note that for some $j = 0, 1, \ldots, k$, the function

$$\frac{\partial}{\partial \nu_j} \alpha(\mu_0,\, \mu_1, \ldots, \mu_k,\, \nu_0,\, \ldots, \nu_k)$$

cannot be identically zero on $\mathcal{W} \times \mathcal{N}$. Let r be such largest number. Since for any $j \leq r$, $y^{(j)}$ does not depend on $u^{(i)}$ for $i \geq r$, Eq. (7) yields a local equation of order less than or equal to k for F_c. ∎

We now return to prove Lemma 3.2 for the more general cases. Assume that the conclusion of Lemma 3.2 is true when $\lambda = s$, that is, for every operator that satisfies an analytic constraint in $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1}$ with $\lambda = s \geq 0$, there exists a sequence of convergent series $\{c_j\}$ such that $c_j \to c$, and each c_j admits a local analytic i/o equation of order less than or equal to k. In the following we show that the same conclusion holds for the case when $\lambda = s + 1$.

By assumption, one knows that there exist some stratum $T_1 \in \mathcal{T}_0$ with $\operatorname{codim} T_1 = r_0$ and some u such that $\xi(0,\, u) \in T_1$. We still let $\bar{\mu}$ and $\bar{\nu}$ be as defined in (5) and (6). We now consider the following two cases.

Case 1: Assume that there exist some $r > 0$ and some $0 \leq \tau_2 \leq \tau$ such that for any input $u \in B_k(\bar{\mu}, r, \tau)$ it holds that $\xi(t,\, u) \in T_1$ for all $0 \leq t < \tau_2$. Using the same argument used in the proof of Lemma 3.4, one shows that F_c admits a local analytic i/o equation of order less than or equal to k.

Case 2: Assume now that the assumption in case 1 does not hold. Then there exist a sequence of piecewise $\mathcal{C}^k$ inputs $\{u_j\}$ with $u_j \in B_k(\bar{\mu},\, 1/j,\, \tau)$, and a sequence $t_j \to 0$ such that, for each j, $\xi(t_j,\, u_j) \notin T_1$. Again, by Lemma A.11, one knows that $\xi(t_j,\, u_j) \to (\bar{\mu},\, \bar{\nu})$ as $j \to \infty$.

By the local finiteness of $\mathcal{T}$, one may assume that $\xi(t_j,\, u_j) \in T_2$ for some stratum T_2. Thus $(\bar{\mu},\, \bar{\nu}) \in T_1 \cap \bar{T}_2 \neq \emptyset$. From here it follows that $\operatorname{codim} T_2 < \operatorname{codim} T_1$.

Now, for any $u \in \mathcal{U}_{t_1}$ and $v \in \mathcal{U}_{t_2}$, we use $u\#_{t_1}\nu$ to denote the concatenated function:

$$(u\#_{t_1}v)(t) \;=\; \begin{cases} u(t), & \text{if } 0 \le t < t_1, \\ v(t-t_1), & \text{if } t_1 \le t < t_1 + t_2. \end{cases}$$

Note then if $u \in \mathcal{U}_{t_1}$ and $v \in \mathcal{U}_{t_2}$ are both piecewise $\mathcal{C}^k$, then $u\#_{t_1}v \in \mathcal{U}_{t_1+t_2}$ is again piecewise $\mathcal{C}^k$.

Corresponding to the sequence $\{u_j\}$ and $\{t_j\}$, we define a sequence of operators $\{G_j\}$ in the following way. For each integer j, we let G_j be defined by

$$G_j[v](t) \;=\; F[u_j\#_{t_j}v](t_j + t)\,.$$

According to Lemma 2.4 in [21], for each j, G_j is an operator defined by the series d_j, i.e., $G_j = F_{d_j}$, where d_j is the series given by:

$$\langle d_j,\; w\rangle \;=\; F_{w^{-1}c}[u_j](t_j)\,,$$

(see the Appendix for the notion $w^{-1}c$) and if T is admissible to c, then $T - t_j$ is admissible to d_j. Furthermore, we have the following:

Lemma 3.5 For each monomial $w \in P^*$,

$$\langle d_j,\; w\rangle \,\to \langle c,\; w\rangle\,, \qquad \text{as } j \to \infty\,.$$

The proof of Lemma 3.5 will be given in the Appendix.

Observe that for each j, G_j also satisfies the analytic constraint S, and moreover, with

$$\xi_j(t,u) := \Big(u(t),\, u'(t),\, \ldots,\, u^{(k)}(t),\, G_j[u](t),\, \frac{d}{dt}G_j[u](t),\, \ldots,\, \frac{d^k}{dt^k}G_j[u](t)\Big)\,,$$

it holds that $\xi_j(0,v_j) \in T_2$, where $v_j(t) = u_j(t+t_j)$. Since $\operatorname{codim} T_2 < \operatorname{codim} T_1$, it follows that the index λ for G_j is less than or equal to s. By the induction assumption, one knows that for each j, there exists a sequence of convergent series $\{d_{ji}\}_{i=1}^{\infty}$ such that each d_{ji} admits a local analytic i/o equation of order less than or equal to k, and

$$\lim_{i\to\infty} d_{ji} \;\to\; d_j\,, \qquad \text{as } i \to \infty\,.$$

For each j, we let $c_j = d_{jj}$. Then $\{c_j\}$ is a sequence converging to c, and every F_{c_j} admits a local analytic i/o equation of order less than or equal to k.

By induction, Lemma 3.2 holds for all $\lambda \geq 0$.

3.3 From I/O Equations to Realizability ($3 \Longrightarrow 1$)

For each integer k, we define

$$\beta(k) = \sum_{i=0}^{k} (m+1)^i\,.$$

Note that $\beta(k)$ is in fact the number of the elements in the set

$$P^k := \{w \in P^* :\ |w| \leq k\}\,.$$

Lemma 3.6 If F_c admits a local analytic i/o equation of order lower than or equal to k, then F_c is locally realizable by an analytic state space system of dimension less than or equal to

$\beta(k)$.

The proof of Lemma 3.6 basically follows the same steps as in the proof of [21, Theorem 3(b)]. The only difference between Lemma 3.6 and [21, Theorem 3(b)] is that in Lemma 3.6, F_c is only assumed to admit a local analytic i/o equation, while in the context of [21], F_c is assumed to admit an analytic equation in a global sense. Still, the proof of [21, Theorem 3(b)] can be used to prove Lemma 3.6 with very minor modifications. In the following, we provide some details of the proof.

First of all, by the definition of local i/o equations, one immediately gets the following:

Lemma 3.7 Let $\bar{\mu} \in \mathbb{R}^{mk}$ and let $r, T > 0$. An operator F_c satisfies i/o equation (4) for all $u \in B_k(\bar{\mu}, r, T)$ if and only if

$$A\left(\mu_0,\, \mu_1,\, \ldots,\, \mu_k,\, F_c,\, F_{c_1(\mu_0)},\, \ldots,\, F_{c_k(\mu_0,\ldots,\mu_{k-1})}\right) = 0 \tag{8}$$

for all μ with $|\mu_i - \bar{\mu}_i| < r$. □

For the definition of the series $c_l(\mu_0, \ldots, \mu_{l-1})$, we refer the readers to [19]. Roughly speaking, $c_k(\mu_0, \ldots, \mu_{l-1})$ is such a series that

$$F_{c_k(\mu_0,\ldots,\mu_{l-1})}[u](t) \;=\; \left.\frac{d^l}{d\tau^l}\right|_{\tau=0} F_c[u\#_t v](t+\tau)$$

for any v such that $v^{(i)}(0) = \mu_i, 0 \le i \le l-1$, and in particular,

$$\frac{d^l}{dt^l} F_c[u](t) \;=\; F_{c_k(u(t),\ldots,u^{(l-1)}(t))}[u](t)$$

if $u \in \mathcal{C}^{l-1}$.

Eq. (8) means that

$$A\left(\mu_0,\,\mu_1,\,\ldots,\,\mu_k,\,F_c[u](t),\,F_{c_1(\mu_0)}[u](t),\,\ldots,\,F_{c_k(\mu_0,\ldots,\mu_{k-1})}[u](t)\right)=0$$

for all $u\in B_k(\bar{\mu},\,r,\,T)$ and for all $0\le t<T$.

To prove Lemma 3.6, we need to study the structures of observation spaces and observation fields associated with i/o operators as we did in [20, 21]. For a detailed study of such objects, see [19, 20, 21]. To make this work more self contained, we provide the definitions and some basic properties of observation spaces $\mathcal{F}_1(c)$, $\mathcal{F}_2(c)$ and observation fields $\mathcal{Q}_1(c)$ and $\mathcal{Q}_2(c)$ in the Appendix. By Theorem 1 in [19] (see also Lemma A.12 in the Appendix), one knows $w^{-1}c\in\mathcal{F}_2(c)$ for any $w\in P^*$.

In analogue to [21, Theorem 2], we have the following:

Lemma 3.8 Assume that F_c admits a local analytic i/o equation of order k. Then $\mathcal{Q}_2(c)$ is meromorphically generated by the elements in

$$\mathfrak{S}_k:=\{w^{-1}c:\ w\in P^k\}\,. \tag{9}$$

Proof. The proof of Lemma 3.8 basically follows the same steps as in the proof of [21, Theorem 2(b)]. The only modification is that the sets Ω and Ω_1 are changed to be open dense subsets of some open ball

$$\{\mu\in\mathbb{R}^{m(k+2)}:\ |\mu_i-\bar{\mu}_i|<r,\ 0\le i\le k+1\}\,\subseteq\,\mathbb{R}^{m(k+2)}\,,$$

(while in the context of [21], Ω and Ω_1 are open dense in $\mathbb{R}^{m(k+2)}$). Still, Lemma 12.11 in [11] can be used to show that for any $r \geq 1$,

$$c_{k+r}(\mu_0, \mu_1, \ldots, \mu_{k+r-1})$$

is meromorphically generated by the elements in $\mathfrak{S}_k$ (note that the set Φ in the context of [21] is contained in the set $\mathfrak{S}_k$). This then implies that $\mathcal{Q}_2(c)$ is meromorphically generated by the elements in $\mathfrak{S}_k$. ∎

Proof of Lemma 3.6. First note that one of the main results in [19] (see also the Appendix) implies that $\mathcal{Q}_1(c) = \mathcal{Q}_2(c)$. From this we conclude that if F_c admits a local i/o equation of order k, then $\mathcal{Q}_1(c)$ is meromorphically generated by $\mathfrak{S}_k$. By Theorem 1(b) in [21], one sees that F_c is locally realizable by an analytic system. Furthermore, by tracking the proof of the theorem, one can see that the dimension of the system is lower than or equal to $\beta(k)$, the number of elements in $\mathfrak{S}_k$. ∎

In [4, 8, 13], the authors used the Lie rank associated with series to study the realizability for operators. Let $\mathfrak{P}$ denote the set of polynomials in $\eta_0, \ldots, \eta_m$. One defines the Lie bracket $[\cdot,\ \cdot]$ on $\mathfrak{P}$ by:

$$[P_1,\ P_2] = P_1 \cdot P_2 - P_2 \cdot P_1\,,$$

where "$\cdot$" stands for the standard product defined for polynomials. With $[\cdot,\ \cdot]$ defined as above, $\mathfrak{P}$ forms a Lie algebra. Let L be the subalgebra of $\mathfrak{P}$ generated by $\eta_0, \ldots, \eta_m$. For each series c, we define $\psi_c : \mathfrak{P} \to \mathfrak{S}$ by:

$$\psi_c(P) \;=\; \sum \langle P,\ w\rangle c w^{-1}$$

if $P = \sum \langle P,\ w \rangle w$. (See the Appendix for the definition of cw^{-1}.) The *Lie rank* $\rho(c)$ of c is then defined as the dimension of the space

$$\mathcal{L}_c := \operatorname{span}_{\mathbb{R}} \{ \psi_c(w) :\ w \in L \} .$$

Lemma 3.9 If $c_j \to c$ as $j \to \infty$, then

$$\rho(c) \ \leq\ \liminf_{j \to \infty} \rho(c_j) \,. \tag{10}$$

Proof. Let

$$\liminf_{j \to \infty} \rho(c_j) \ =\ n \,. \tag{11}$$

Clearly the conclusion of the lemma is true if $n = \infty$. We now assume that $n < \infty$. Assume that $\rho(c) > n$. Then there exist $w_1, \ldots, w_{n+1} \in L$ such that the series

$$\psi_c(w_1),\ \psi_c(w_2),\ \ldots,\ \psi_c(w_{n+1})$$

are linearly independent.

We now enumerate the elements of P^*, the set of monomials in $\eta_0, \ldots, \eta_m$, by $z_1, z_2, \ldots$, and let A_r be the matrix of $(n+1)$ columns and infinitely many rows whose (i, j)-th entry is

$\langle \psi_{c_r}(w_j),\ z_i \rangle$, that is,

$$A_r = \begin{pmatrix} \langle \psi_{c_r}(w_1),\ z_1 \rangle & \langle \psi_{c_r}(w_2),\ z_1 \rangle & \dots & \langle \psi_{c_r}(w_{n+1}),\ z_1 \rangle \\ \langle \psi_{c_r}(w_1),\ z_2 \rangle & \langle \psi_{c_r}(w_2),\ z_2 \rangle & \dots & \langle \psi_{c_r}(w_{n+1}),\ z_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle \psi_{c_r}(w_1),\ z_l \rangle & \langle \psi_{c_r}(w_2),\ z_l \rangle & \dots & \langle \psi_{c_r}(w_{n+1}),\ z_l \rangle \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

and let A_0 be the matrix whose (i,j)-th entry is $\langle \psi_c(w_j),\ z_i \rangle$. Then A_0 is full column rank in the sense that if there is $v \in \mathbb{R}^{n+1}$ such that $A_0 v = 0$, then $v = 0$. Let B_l be the matrix formed by the first l rows of A_0. Then it can be shown that there exists some $l_0 > 0$ such that $\operatorname{rank} B_{l_0} = n+1$ (see also the proof of Theorem 1 in [19]). Let B_{r,l_0} be the matrix formed by the first l_0 rows of A_r.

For any matrix A, we let $A(i,j)$ denote the (i,j)-th entry of A. Note that for $1 \le i \le l_0$ and $1 \le j \le n+1$,

$$B_{r,l_0}(i,j) = \langle \psi_{c_r}(w_j),\ z_i \rangle = \langle c_r,\ z_i w_j \rangle \to \langle c,\ z_i w_j \rangle\,, \qquad \text{as } r \to \infty\,.$$

This then implies that there exists some $K > 0$ such that for $k \ge K$,

$$\operatorname{rank} B_{k,l_0} \ge \operatorname{rank} B_{l_0} = n+1\,,$$

from which it immediately follows that

$$\rho(c_k) \ge \operatorname{rank} A_k \ge n+1$$

for all $k \ge K$. This is impossible because of (11). Hence, $\rho(c) \le n$. ∎

It is well-known that the i/o operator F_c defined by a convergent series is locally realizable by an analytic state space system if and only if $\rho(c) < \infty$, and if F_c is realizable by a system of dimension n, then $\rho(c) \leq n$ (cf. [4, 8, 13]).

Now assume that $\{c_j\}$ is a sequence of convergent series such that each c_j admits a local analytic equation of order k, and assume that $c_j \to c$. Then by Lemma 3.2, one knows that c_j is locally realizable by an analytic system of dimension lower than or equal to $\beta(k)$, and by the second part of the above statement, the Lie rank $\rho(c_j)$ of c_j is bounded by $\beta(k)$. By Lemma 3.9, one knows that $\rho(c) \leq \beta(k)$, which, in turn, implies that F_c is realizable. Thus we proved the following:

Lemma 3.10 Let c be a convergent series. If there exists a squence of convergent series $\{c_j\}$ such that $c_j \to c$ as $j \to \infty$ and that each c_j admits a local analytic i/o equation of order less than or equal to k for some $k > 0$, then F_c is locally realizable. □

Combining Lemmas 3.1, 3.2 and 3.10, one obtains Theorem 1.

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Appendix

Analytic i/o operators defined by generating series have been extensively studied in the realizability literature, see for instance [4, 5, 8, 20, 21]. For convenience of reference, we provide some background material on such operators.

A.1 Some Continuity Properties of I/O Operators

Consider the set $\mathfrak{S}$ of all generating series in the variables $\eta_0, \eta_1, \ldots, \eta_m$. With the usual "+" defined coefficientwise for series, the set $\mathfrak{S}$ forms a vector space. To each monomial $w_0 \in P^*$, we associate a "shift" operator $c \mapsto w_0^{-1}c$ defined by

$$\langle w_0^{-1}c,\ w\rangle \ = \ \langle c,\ w_0 w\rangle$$

for all $w \in P^*$. The map $c \mapsto cw_0^{-1}$ is defined similarly by

$$\langle cw_0^{-1},\ w\rangle \ = \ \langle c,\ ww_0\rangle$$

for all $w \in P^*$. It was shown in [12] that if c is a convergent series and T is admissible for c, then T is admissible for $w_0^{-1}c$ for any $w_0 \in P^*$. The same result also holds for cw_0^{-1}.

Proof of Lemma 3.5: Let K and M be the constants for which (1) holds for c, and fix any monomial w with $|w| = l$. Then for any $u \in \mathcal{U}_\tau$ and $t \in [0, \tau)$, it holds that

$$\begin{aligned} |F_{w^{-1}c}[u](t) - \langle w^{-1}c,\ \phi\rangle| &= \left|\sum_{|z|\geq 1}\langle c,\ wz\rangle V_w[u](t)\right| \\ &\leq \sum_{k=1}^{\infty} KM^{l+k}(m+1)^{l+k}\,\frac{t^k}{k!}\,(k+l)! \\ &= KM^l(m+1)^l\sum_{k=1}^{\infty}\frac{s^k}{k!}\,(l+k)!\,, \end{aligned} \qquad (12)$$

where $s = M(m+1)t$. Note that in the above argument we used the fact that for $z \in P^*$ with $|z| = k$, it holds that $V_z[u](t) \leq t^k/k!$ for any $u \in \mathcal{U}_\tau$, for any $0 \leq t < \tau$. For the series (12), we have, for $0 \leq s < 1$,

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{s^k}{k!}(l+k)! &= s \sum_{k=0}^{\infty} \frac{s^k}{(k+1)!}(l+k+1)! \\
&\leq s \sum_{k=0}^{\infty} s^k (k+l+1)(k+l)\cdots(k+2)(k+1) = s \frac{d^{l+1}}{ds^{l+1}} \sum_{k=0}^{\infty} s^k \\
&= \frac{s}{(1-s)^{l+2}}\,(l+1)!\,.
\end{aligned}
$$

Thus, one gets

$$
|F_{w^{-1}c}[u](t) - \langle w^{-1}c,\ \phi\rangle| \;\leq\; KM^l(m+1)^l(l+1)!\,\frac{s}{(1-s)^{l+1}}\,. \tag{13}
$$

Applying (13) to c_j, one sees that

$$
|\langle c_j,\ w\rangle - \langle c,\ w\rangle| \;=\; \left|F_{w^{-1}c}[u_j](t_j) - \langle w^{-1}c,\ \phi\rangle\right| \;\leq\; KM^l(m+1)^l(l+1)!\,\frac{M(m+1)t_j}{(1-M(m+1)t_j)^{l+2}}\,.
$$

Since $t_j \to 0$ as $j \to \infty$, it follows that $\langle c_j,\ w\rangle \to \langle c,\ w\rangle$. ∎

In [5] it was shown that for an operator F_c, it holds that

$$
\frac{d}{dt}F_c[u](t) \;=\; F_{\eta_0^{-1}c}[u](t) \;+\; \sum_{j=1}^{m} u_j(t) F_{\eta_j^{-1}c}[u_j](t)\,.
$$

For higher order derivatives, it was shown in [19] that

$$
\frac{d^k}{dt^k}F_c[u](t) \;=\; F_{c_k(u(t),u(t),\ldots,\,u^{(k-1)}(t))}[u](t)\,.
$$

See [19] for the detailed definition on $c_k(\mu_0, \ldots, \mu_{k-1})$. Basically, $c_k(\mu_0, \ldots, \mu_{k-1})$ is a polynomial in $(\mu_0, \ldots, \mu_{k-1})$ whose coefficients are the elements in the set $\{w^{-1}c : \ w \in P^k\}$. An immediate implication is that $F_{c_k(\mu_0,\ldots,\mu_{k-1})}$ is a polynomial in $(\mu_0, \ldots, \mu_{k-1})$ whose coefficients are $F_{w^{-1}c}$ for some $w \in P^k$. Note that the proof of Lemma 3.5 also shows that for $w \in P^*$,

$$\lim_{t \to 0^+} F_{w^{-1}c}[u](t) \, = \, \langle w^{-1}c, \, \phi \rangle$$

uniformly on the set $\mathcal{U}_T$ (where T is admissible to c). From this one gets the following:

Lemma A.11 Let c be a convergent series with T admissible to c. Then for any $k \geq 0$, and any $r > 0$,

$$\lim_{t \to 0^+} \frac{d^k}{dt^k} F_c[u](t) \, = \, \langle c_k(\mu_0, \, \ldots, \mu_{k-1}), \, \phi \rangle$$

uniformly on the set $B_k(0, \, r, \, T)$. □

A.2 Observation Spaces and Observation Fields

For any given power series c, we define the *observation space* $\mathcal{F}_1(c)$ as the $\mathbb{R}$-space spanned by all the series $w^{-1}c$, the *observation algebra* $\mathcal{A}_1(c)$ is the $\mathbb{R}$-algebra generated by the elements of $\mathcal{F}_1(c)$, under the shuffle product (see [20]), and the *observation field* $\mathcal{Q}_1(c)$ is the quotient field of $\mathcal{A}_1(c)$. Note that $\mathcal{Q}_1(c)$ is always defined since $\mathcal{A}_1(c)$ is an integral domain (cf. [20]).

For any convergent series c, we say that the observation field $\mathcal{Q}_1(c)$ is a *meromorphically finitely generated field extension of* $\mathbb{R}$ if there exists an integer n and

$$c_1, \, c_2, \, \ldots, \, c_n \, \in \mathcal{A}_1(c)$$

such that for each element d in $\mathcal{Q}_1(c)$, there exist some analytic functions φ_0 and φ_1 defined on

$\mathbb{R}^n$ such that

$$\varphi_0\left(F_{c_1}[u](t),\, \ldots,\, F_{c_n}[u](t)\right)\, F_d[u](t) \,=\, \varphi_1\left(F_{c_1}[u](t),\, \ldots,\, F_{c_n}[u](t)\right)$$

for all $u \in \mathcal{V}_T,\ t \in [0,\, T]$ and for any T admissible for c, and,

$$\varphi_0\left(F_{c_1}[u],\, \ldots,\, F_{c_n}[u]\right) \,\neq\, 0$$

for some $u \in \mathcal{U}_T$, and some T admissible for c. If this is the case, we call $c_1, \ldots, c_n$ the generators of the field, or, we say that the field is generated by $c_1, \ldots, c_n$.

While the finiteness properties of $\mathcal{F}_1(c)$ and $\mathcal{Q}_1(c)$ are related to the realizability of F_c (cf. [21, Theorem 1]), the finitenss properties of the following type of observation spaces and fields are related to existence of i/o equations.

For a convergent series c, the observation space $\mathcal{F}_2(c)$ of the second type is defined to be the $\mathbb{R}$-space spanned by $c_n(\mu_0, \ldots, \mu_{n-1})$ for all n and all μ. The observation algebra $\mathcal{A}_2(c)$ is defined to be $\mathbb{R}$-algebra generated by the elements of $\mathcal{F}_2(c)$, and the observation field $\mathcal{Q}_2(c)$ is the quotient field of $\mathcal{A}_2(c)$.

One of the main results in [19] is the following:

Lemma A.12 Let c be a convergent series. Then $\mathcal{F}_1(c) = \mathcal{F}_2(c)$. □

An immediate consequence of Lemma A.12 is that $\mathcal{A}_1(c) = \mathcal{A}_2(c)$ and $\mathcal{Q}_1(c) = \mathcal{Q}_2(c)$.