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Fliess operators on L_p spaces: convergence and continuity

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Abstract

Fliess operators as input–output mappings are particularly useful in a number of fundamental problems concerning nonlinear realization theory. In the classical analysis of these operators, certain growth conditions on the coefficients in their series representations insure uniform and absolute convergence, provided every input is uniformly bounded by some fixed upperbound. In some emerging applications, however, it is more natural to consider other classes of inputs. In this paper, L_p function spaces are considered. In particular, it is shown that the classic growth conditions also provide sufficient conditions for convergence and continuity when the admissible inputs are from a ball in $L_p[t_0, t_0 + T]$, where T is bounded and $p \geqslant 1$. In addition, stronger global growth conditions are given that apply even for the case where T is unbounded. When the coefficients of a Fliess operator have a state space representation, it is shown that the state space model will always locally realize the corresponding input–output map on $L_p[t_0, t_0 + T]$ for sufficiently small $T > 0$. If certain well-posedness conditions are satisfied then the state space model will globally realized the input–output mapping for unbounded T when the coefficients satisfy the global growth condition. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Consider the set of input–output operators having m inputs and ℓ outputs. Let $I = \{0, 1, \ldots, m\}$, and let I^k be the set of all sequences $(i_k i_{k-1} \cdots i_1)$, where $i_r \in I$ for $1 \leqslant r \leqslant k$. For $k=0$, I^0 denotes the set whose only element is the empty sequence ϕ and $I^* := \bigcup_{k \geqslant 0} I^k$. A formal power series in the $m+1$ noncommutative variables $\{\theta_0, \theta_1, \ldots, \theta_m\}$has the form

$$c = \sum_{\eta \in I^*} c(\eta) w_\eta,$$

where $w_\eta = \theta_{i_k} \theta_{i_{k-1}} \cdots \theta_{i_1}$ if $\eta = (i_k i_{k-1} \cdots i_1)$, and $c(\eta) \in \mathbb{R}^\ell$. Such a series is nothing but a mapping from I^* into $\mathbb{R}^\ell$. For each formal power series c, one can formally associate a corresponding input–output operator F_c in the following manner. Let $p \geqslant 1$ and $a < b$ be given. For a measurable function $u : [a, b] \to \mathbb{R}^m$, define $\|u\|_p = \max\{\|u_i\|_p : 1 \leqslant i \leqslant m\}$, where $\|u_i\|_p$ is the usual L_p-norm for a measurable real-valued function, u_i, defined on $[a, b]$. Let $L_p^m[a, b]$ denote the set of all measurable functions defined on $[a, b]$ having a finite $\|\cdot\|_p$ norm. Clearly, a function $u = (u_1, \ldots, u_m) \in L_p^m[a, b]$ if and only if $u_i \in L_p[a, b]$ for each i, and $\|u\|_p \geqslant \|u_i\|_p$ for all i. Now with $t_0, T \in \mathbb{R}$ fixed and $T > 0$, define inductively for each $\eta \in I^*$, the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to \mathscr{C}[t_0, t_0 + T]$

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by $E_\phi = 1$, and

$$E_{i_k i_{k-1}\cdots i_1}[u](t) = \int_{t_0}^{t} u_{i_k}(\tau) E_{i_{k-1}\cdots i_1}[u](\tau)\,d\tau,$$

where $u_0=1$. The input–output operator corresponding to c is then

$$F_c[u](t) = \sum_{\eta\in I^*} c(\eta) E_\eta[u](t),$$

which is referred to as a *Fliess operator*. In the classical literature where these operators first appeared [2–4,8,9,12], it is normally assumed that there exists real numbers $K > 0$ and $M > 0$ such that

$$|c(\eta)| \leqslant K M^{|\eta|} |\eta|! \quad \forall \eta \in I^*, \tag{1}$$

where $|x| = \max\{|x_1|, |x_2|, \ldots, |x_\ell|\}$ when $x \in \mathbb{R}^\ell$, and $|\eta|$ denotes the number of symbols in η. These growth conditions on the coefficients of c insure that there exist positive real numbers R and T_0 such that for all measurable functions u with $\|u\|_\infty \leqslant R$ and $T \leqslant T_0$, the series defining F_c converges uniformly and absolutely on $[t_0, t_0 + T]$. In some emerging applications of this theory, however, it is more natural to consider other classes of inputs, for example, finite energy inputs [5,6,10,11]. Hence, the question arises as to whether there exists suitable growth conditions which guarantee the necessary convergence properties for these alternative input sets. In this paper, the function spaces $L_p[t_0, t_0 + T]$ are considered when T is both bounded and unbounded and $p \geqslant 1$. In Section 2 it is shown that the classic growth condition is a sufficient condition for the convergence of the series defining F_c and for the continuity of the corresponding output function $y = F_c[u]$ when u is restricted to a ball in $L_p[t_0, t_0 + T]$ of sufficiently small radius and $T > 0$. In Section 3 a stronger global growth condition is developed which applies even for the case where T is unbounded.

Fliess operators are particularly useful in a number of fundamental problems in realization theory for nonlinear input–output maps [2,3,7,13,14]. Let $\mathscr{M}$ be an n-dimensional analytic manifold, and let

$$\dot{x} = f(x) + g(x)u, \tag{2}$$

$$y = h(x)$$

be a state space system defined in terms of local coordinates on $\mathscr{M}$. Unless stated otherwise, f, g, and h are assumed to be analytic on $\mathscr{M}$. The triple (f, g, h) defined locally about $x_0 \in \mathscr{M}$ is said to *represent* a formal power series c if for every $\eta = (i_k \cdots i_0) \in I^*$

$$c(\eta) = L_{g_\eta} h(x_0) := L_{g_{i_0}} L_{g_{i_1}} \cdots L_{g_{i_k}} h(x_0), \tag{3}$$

where $g_0 := f$, g_i is the ith column of g when $i > 0$, and $L_{g_i} h$ denotes the Lie derivative of h with respect to g_i. On the other hand, (f, g, h, x_0) is said to *realize* the operator F_c on an input space $\mathscr{U}[t_0, t_0 + T]$ when $F_c[u](t) = h(x(t))$ for every $t \in [t_0, t_0 + T]$, where $x(t)$ is the solution of (2) with $x(t_0) = x_0$ and $u \in \mathscr{U}[t_0, t_0 + T]$. It is well known that if the coefficients of c satisfy (1) then a representation of c exists if and only if the Lie rank of the Hankel matrix of c is finite [3,8,9]. It is equally well established in this literature that a representation of c produces a realization of F_c on the set of uniformly bounded measurable functions defined over $[t_0, t_0 + T]$ provided that T is sufficiently small. In Section 4 it is shown that a state space representation of c is also a local realization of F_c on $L_p[t_0, t_0 + T]$ for any given $p \geqslant 1$ when T is chosen sufficiently small. It is then shown that a global realization is also possible if the coefficients of c satisfy the global growth condition and the state space model satisfies a certain well-posedness condition.

2. Convergence and continuity theorems: the local case

Two numbers p and q in $[1, \infty]$ are called conjugate exponents if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We adopt the convention that $1/\infty = 0$. Thus, 1 and ∞ are conjugate exponents. For $R > 0$, define

$$B_p^m(R)[t_0, t_0 + T] = \{u \in L_p^m[t_0, t_0 + T], \|u\|_p \leqslant R\}.$$

The main result of this section is that if a power series c has coefficients that satisfy the growth condition (1) then for any $p \geqslant 1$, F_c is well defined on $B_p^m(R)[t_0, t_0 + T]$ for some $R > 0$ and some $T > 0$. In which case, the result is local in two senses, over a finite interval T and within a ball of finite radius R. First consider the case when $p = 1$.

Theorem 2.1. *Suppose c is a series with coefficients that satisfy the growth condition* (1). *Then there exist $R>0$ and $T>0$ such that for each $u \in B_1^m(R)[t_0, t_0+T]$, the series*

$$y(t) = F_c[u](t) = \sum_{\eta \in I^*} c(\eta) E_\eta[u](t) \tag{4}$$

converges absolutely and uniformly on $[t_0, t_0+T]$. Furthermore, the function y is absolutely continuous on $[t_0, t_0+T]$.

Since for any $p>1$ and on any interval $[t_0, t_0+T]$ it can be shown that

$$\|u\|_1 \leqslant \|u\|_p T^{1/q}$$

when p and q are conjugate exponents, the following corollary of Theorem 2.1 is immediate.

Corollary 2.1. *Suppose c is a series with coefficients that satisfy the growth condition* (1). *Then for any $1 \leqslant p \leqslant \infty$, there exist $R>0$ and $T>0$ such that for each $u \in B_p^m(R)[t_0, t_0+T]$, the series*

$$y(t) = F_c[u](t) = \sum_{\eta \in I^*} c(\eta)\, E_\eta[u](t)$$

converges absolutely and uniformly on $[t_0, t_0+T]$, and the function y is absolutely continuous on $[t_0, t_0+T]$.

To prove Theorem 2.1, a certain set of upperbounds for the iterated integrals $E_\eta[u]$ is needed. For any $\eta = (i_k \cdots i_1) \in I^k$ and any $0 \leqslant j \leqslant m$, let ω_η^j denote the number of symbols in $(i_k \cdots i_1)$ that take on the value j. (For instance, when $\eta = (102011)$ then $\omega_\eta^0 = 2$, $\omega_\eta^1 = 3$ and $\omega_\eta^2 = 1$.) Clearly, $\sum_{j=0}^m \omega_\eta^j = k$ for any $\eta \in I^k$. Also, without loss of generality, it is assumed below that $t_0 = 0$.

Lemma 2.1. *For any $\eta = (i_k \cdots i_1) \in I^k$ and any $u \in L_1[0,T]$, it follows that*

$$|E_\eta[u](t)| \leqslant U_0(t)^{r_0} U_1(t)^{r_1} \cdots U_m(t)^{r_m} \frac{1}{r_0! r_1! \cdots r_m!} \tag{5}$$

for all $t \in [0,T]$, where for each j, $U_j(t) := \int_0^t |u_j(s)|\,\mathrm{d}s$, and $r_j = \omega_\eta^j$. In particular, if on $[0,T]$, $\max\{\|u\|_1, \|u_0\|_1\} \leqslant R_1$ then

$$|E_\eta[u](t)| \leqslant \frac{R_1^k}{r_0! r_1! \cdots r_m!} \quad \forall t \in [0,T]. \tag{6}$$

Proof. The proof is by induction on k. It is clear that the result is true when $k=1$. Suppose it is known to be true up to some fixed $k \geqslant 1$. Pick any $(i_{k+1} i_k \cdots i_1) \in I^{k+1}$. Suppose $i_{k+1} = j$ and $\omega_{i_k \cdots i_1}^l = r_l$. Then $\omega_{i_{k+1} i_k \cdots i_1}^l = r_l$ if $l \neq j$, and $\omega_{i_{k+1} i_k \cdots i_1}^j = r_j + 1$. Hence,

$$\begin{aligned}
&|E_{i_{k+1} i_k \cdots i_1}[u](t)| \\
&\leqslant \int_0^t |u_j(\tau)| \cdot |E_{i_k \cdots i_1}[u](\tau)|\,\mathrm{d}\tau \\
&\leqslant \int_0^t |u_j(\tau)| U_0(\tau)^{r_0} \cdots U_m(\tau)^{r_m} \\
&\quad \times \frac{1}{r_0! \cdots r_m!}\,\mathrm{d}\tau \\
&\leqslant U_0(t)^{r_0} \cdots U_j(t)^{r_j+1} \cdots U_m(t)^{r_m} \\
&\quad \times \frac{1}{r_0! \cdots (r_j+1)! \cdots r_m!}.
\end{aligned}$$

The result then holds for all $k \geqslant 1$. □

The proof of Theorem 2.1 is now possible.

Proof of Theorem 2.1. Suppose the coefficients of c satisfy the growth condition (1) for some $K>0$ and $M>0$. Without loss of generality it is assumed that $\ell = 1$. Fix some $T>0$. Pick any $u \in L_1^m[0,T]$ and let $R = \max\{\|u\|_1, T\}$ $(=\max\{\|u\|_1, \|u_0\|_1\})$. For any fixed $k>0$

$$|c(i_k \cdots i_1) E_{i_k \cdots i_1}[u](t)| \leqslant K M^k k! \frac{R^k}{r_0! r_1! \cdots r_m!}$$

for all $t \in [0,T]$, where $r_j = \omega_{i_k \cdots i_1}^j$. Observe that for any $r_0, \ldots, r_m$ such that $r_0 + \cdots + r_m = k$, the set

$$\{(i_k \cdots i_1) \colon \omega_{i_k \cdots i_1}^j = r_j, 0 \leqslant j \leqslant m\}$$

consists of $k!/(r_0! r_1! \cdots r_m!)$ elements. Thus for

$$a_k(t) = \sum_{i_k, \ldots, i_1 = 0}^m c(i_k \cdots i_1) E_{i_k \cdots i_1}[u](t),$$

$$\begin{aligned}
|a_k(t)| &\leqslant \sum_{i_k, \ldots, i_1 = 0}^m |c(i_k \cdots i_1) E_{i_k \cdots i_1}[u](t)| \\
&\leqslant K M^k R^k \sum_{i_k, \ldots, i_1 = 0}^m \frac{k!}{r_0! r_1! \cdots r_m!}
\end{aligned}$$

$$= KM^kR^k \sum_{r_0+\cdots+r_k=k} \frac{k!}{r_0!r_1!\cdots r_m!} \cdot \frac{k!}{r_0!r_1!\cdots r_m!}$$

$$\leqslant KM^kR^k \left[\sum_{r_0+\cdots+r_k=k} \frac{k!}{r_0!r_1!\cdots r_m!} \right]^2$$

$$= KM^kR^k(m+1)^{2k}, \tag{7}$$

where the following fact has been used:

$$\sum_{r_0+\cdots+r_k=k} \frac{k!}{r_0!r_1!\cdots r_m!} = (z_0+z_1+\cdots+z_m)^k\big|_{z_0=\cdots=z_m=1} = (m+1)^k.$$

It then follows that (with $a_0(t)=0$):

$$\sum_{k=0}^{\infty} |a_k(t)| \leqslant \sum_{k=0}^{\infty} KM^kR^k[(m+1)^2]^k.$$

This shows that if $R < 1/M(m+1)^2$, i.e., if

$$\max\{\|u\|_1,\ T\} \leqslant \frac{1}{M(m+1)^2},$$

then series (4) converges absolutely and uniformly on $[0,T]$.

The absolute continuity of y follows from the fact that each $E_{i_k\cdots i_1}[u]$ is absolutely continuous on $[0,T]$ and that series (4) converges uniformly on the same interval. □

The above proof shows that for $1 \leqslant p \leqslant \infty$, if the coefficients of c satisfy the growth condition (1), then the series defines a Fliess operator from $B_p^m(R)[t_0,t_0+T]$ to a bounded subset of $\mathscr{C}[t_0,t_0+T]$ for T sufficiently small. Of course since T is finite, this implies that in general such operators map $B_p^m(R)[t_0,t_0+T]$ into $B_q^{\ell}(S)[t_0,t_0+T]$ for some $S>0$.

3. Convergence and continuity theorems: the global case

For any fixed $t_0 \in \mathbb{R}$ and any $p \geqslant 1$, define the function space $L_{p,e}^m(t_0)$ by

$$L_{p,e}^m(t_0) = \{u\colon u \in L_p^m[t_0,t_1] \quad \forall t_0 < t_1 < \infty\}.$$

The main result of this section is given in the theorem below. Here u is no longer restricted to a finite interval, but a stricter growth condition on the coefficients is needed to insure convergence.

Theorem 3.1. *Suppose for a given power series c there exists real numbers $K>0$ and $M>0$ such that*

$$|c(\eta)| \leqslant KM^{|\eta|}, \quad \forall \eta \in I^*. \tag{8}$$

Then for any $u \in L_{1,e}^m(t_0)$, the series

$$y(t) = F_c[u](t) = \sum_{\eta \in I^*} c(\eta)E_\eta[u](t) \tag{9}$$

converges absolutely and uniformly on $[t_0,t_0+T]$ for any $T>0$. In particular, the series in (9) converges absolutely on $[t_0,\infty)$, and y is continuous on $[t_0,\infty)$.

Proof. Again assume that $t_0=0$. Suppose the coefficients of c satisfy the growth condition (8) for some $K>0$ and $M>0$. Choose any $T>0$, and pick any $u \in L_{1,e}^m(0)$. Let

$$R = \max\{\|u_{[0,T]}\|_1, T\},$$

where $u_{[0,T]}$ denotes the restriction of u to the interval $[0,T]$. Using the same notation as in the proof of Theorem 2.1, estimate (7) can be strengthened to

$$|a_k(t)| \leqslant \frac{KM^kR^k(m+1)^{2k}}{k!}.$$

Consequently,

$$\sum_{k=0}^{\infty}|a_k(t)| \leqslant \sum_{k=0}^{\infty} \frac{KM^kR^k(m+1)^{2k}}{k!} = K\mathrm{e}^{MR(m+1)^2}.$$

Thus, the series in (9) converges absolutely and uniformly on $[0,T]$ for arbitrary $T>0$, and the continuity of y follows immediately. □

Observe that for any $p \geqslant 1$, $L_{p,e}^m(t_0) \subseteq L_{1,e}^m(t_0)$. As a consequence of Theorem 3.1, the following corollary is true.

Corollary 3.1. *Suppose for a given power series c, the coefficients satisfies the growth condition (8) for some $K>0$ and $M>0$. Then for any $u \in L_{p,e}^m(t_0)$, $p \geqslant 1$, the series*

$$y(t) = F_c[u](t) = \sum_{\eta \in I^*} c(\eta)E_\eta[u](t)$$

converges absolutely on $[t_0,\infty)$. Furthermore, y is continuous on $[t_0,\infty)$.

The above corollary says that if c satisfies the growth condition (8), then F_c defines an operator from $L^m_{p,e}(t_0)$ to $\mathscr{C}[t_0,\infty)$ for any $p \geqslant 1$.

Remark 3.1. The following observation will be useful in the next section where state space realizations are considered. Let c be a series that satisfies the global growth condition (8). Suppose u is analytic on $[0,T]$. Let $\tilde{u}$ be a complex extension of u that is analytic in a neighborhood $W_{\mathbb{C}}$ of $[0,T]$ in the complex plane. Without loss of generality, one may assume that $W_{\mathbb{C}}$ is simply connected, that the closure $\overline{W}_{\mathbb{C}}$ of $W_{\mathbb{C}}$ is compact, and that $\tilde{u}$ is analytic on $\overline{W}_{\mathbb{C}}$. Thus, there is some $\tilde{R} > 0$ such that $|\tilde{u}(z)| \leqslant \tilde{R}$ for all $z \in W_{\mathbb{C}}$. Now for any complex vector-valued function $\tilde{u}(z) = (\tilde{u}_1(z),\ldots,\tilde{u}_m(z))$ which is analytic on $W_{\mathbb{C}}$, define $E_\eta[\tilde{u}](z) : W_{\mathbb{C}} \to \mathbb{C}$ by

$$E_{i_{k+1}i_k\cdots i_1}[\tilde{u}](z) = \int_0^z \tilde{u}_{i_{k+1}}(\zeta)E_{i_k\cdots i_1}[\tilde{u}](\zeta)\,\mathrm{d}\zeta,$$

where $E_\phi = 1$ and $\tilde{u}_0 = 1$. By induction, the integrand is analytic, so the value of the integral is independent of the path (chosen inside $W_{\mathbb{C}}$), and the resulting function is analytic in $W_{\mathbb{C}}$ as well. As in the proof of Theorem 3.1, it can be shown that the series

$$\tilde{y}(z) = \sum_{\eta \in I^*} c(\eta)E_\eta[\tilde{u}](z)$$

converges uniformly on $W_{\mathbb{C}}$. Since each $E_\eta[\tilde{u}](z)$ is analytic, it follows that $\tilde{y}(z)$ is analytic on $W_{\mathbb{C}}$ (cf., [1, Theorem 5.1]). But clearly $\tilde{y}(z)$ is an analytic complex extension of the function

$$y(t) = \sum_{\eta \in I^*} c(\eta)E_\eta[u](t).$$

Therefore y is analytic on $[0,T]$. This shows that if u is analytic on $[0,T]$, then so is $F_c[u]$. Furthermore, if u is analytic on $[0,\infty)$, then so is $F_c[u]$.

4. State space systems

In this section Fliess operators, F_c, are considered where c has at least one analytic state space representation (f,g,h,x_0). It is first shown that such a representation always realizes F_c on the input sets $B^m_p(R_p)[0,T_p]$ for some constants $R_p, T_p > 0$ and $p \geqslant 1$. In the case where the global growth condition is satisfied, it is next shown that an analytic state space representation of c will realize F_c globally on $L_{p,e}(0)$ for any $p \geqslant 1$ if an additional well-posedness condition is met. It is demonstrated by example in the subsequent section that this result can fail when this latter condition is not present.

Recall that an analytic state space system (f,g,h,x_0) realizes F_c on $B^m_p(R)[0,T]$ for some $p \geqslant 1$ and some $R,T > 0$ if the output function $y(t) = h(x(t))$ with

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = x_0 \qquad (10)$$

is defined for all $t \in [0,T]$ and all $u \in B^m_p(R)[0,T]$, and $y(t) = F_c[u](t)$ for all $t \in [0,T]$ and all $u \in B^m_p(R)[0,T]$.

Theorem 4.1. *Suppose c is a series with coefficients that satisfy the growth condition* (1). *If there exists an analytic state space representation* (f,g,h,x_0) *of c then this state space system also realizes* F_c *on* $B^m_p(R)[0,T]$ *if* $R > 0$ *and* $T > 0$ *are sufficiently small.*

Proof. The proof amounts to verifying *Fliess's fundamental formula* when the admissible inputs are taken from $L_p[0,T]$. By carefully examining the proofs of Lemma 3.1.4 and Theorem 3.1.5 in [7], one sees that Theorem 3.1.5 in [7] applies to any set $\mathscr{U}$ of input functions (which is not necessarily a subset of $L^m_\infty[0,T]$), provided that all series of the form $F_c[u]$ involved in the proofs converge absolutely and uniformly on the interval of interest $[0,T]$ for any $u \in \mathscr{U}$, and that $F_c[u]$ stays in a neighborhood of $F_c[0]$ for all $t \in [0,T]$ and all $u \in \mathscr{U}$. Combining this observation with our result Theorem 2.1, one sees that (f,g,h,x_0) above realizes F_c on $B^m_1(R)[0,T]$ if $R > 0$ and $T > 0$ are sufficiently small. The more general result then follows from the fact that $L^m_p[0,T] \subset L^m_1[0,T]$ for $p \geqslant 1$. □

The next lemma constitutes our well-posedness condition for the global case. It is an application of the results in the previous section, which allows one to determine when a state space system has a well-defined trajectory for all $t \geqslant 0$ and every $u \in L^m_{1,e}(0)$.

Lemma 4.1. *Consider an analytic system as in* (10) *with initial state* x_0. *Let* $H(x) = x$. *If for some* $K \geqslant 0$ *and* $M \geqslant 0$,

$$|L_{g_\eta}H(x_0)| \leqslant KM^{|\eta|}, \quad \forall \eta \in I^*, \qquad (11)$$

then for any $u \in L^m_{1,e}(0)$, the trajectory $x(t)$ of the system with initial state $x(0) = x_0$ and input u is defined for all $t \geqslant 0$.

Proof. Let $c(\eta) = L_{g_\eta} H(x_0)$. Then it follows from Theorem 3.1 that for any $u \in L^m_{1,e}(0)$, the function $\lambda(t) = \sum c(\eta) E_\eta[u](t)$ is defined for all $t \geqslant 0$. It is thus sufficient to show that λ is the solution of (10) with initial state x_0 and input u.

Fix an analytic input u. From the observation in Remark 3.1, $\lambda(t)$ is analytic on $[0, \infty)$. Consequently, $g_i(\lambda(t))$ is analytic on $[0, \infty)$ for each $i = 0, 1, \ldots, m$. By Isidori [7, Theorem 3.1.5], one sees that

$$\dot{\lambda}(t) = f(\lambda(t)) + g(\lambda(t))u(t) \tag{12}$$

on $[0, T)$ for some $T > 0$. Since all functions involved in (12) are analytic, it follows that (12) holds for all $t \geqslant 0$. This shows that λ is the trajectory of (10) defined on $[0, \infty)$.

Suppose now that $u \in L^m_{1,e}(0)$. Fix any $T > 0$. Then there exists a sequence $\{u_j\}$ of analytic functions such that $u_j \to u$ in the L_1-norm on $[0, T]$. Let $\lambda_j(t) = F_c[u_j](t)$ and let $\lambda(t) = F_c[u](t)$. Then $\lambda_j(t) \to \lambda(t)$ uniformly on $[0, T]$ (cf. [14, Lemma 2.2]). By what was just shown for analytic input functions, λ_j is the solution of (10) with the initial state x_0 and input u_j, that is

$$\lambda_j(t) = x_0 + \int_0^t f(\lambda_j(\tau)) + g(\lambda_j(\tau))u_j(\tau)\,d\tau \tag{13}$$

for all $t \in [0, T]$. Letting $j \to \infty$ on both sides of (13) and using the fact that $u_j \to u$ in the L_1-norm, it is immediate that

$$\lambda(t) = x_0 + \int_0^t f(\lambda(\tau)) + g(\lambda(\tau))u(\tau)\,d\tau$$

for all $t \in [0, T]$. Hence, $\lambda(t)$ is the solution of (10) with the initial state x_0 and the input u which is defined on $[0, T]$. Since T can be chosen arbitrarily, it follows that λ is the solution with initial state x_0 and input u on $[0, \infty)$. □

As a consequence of Lemma 4.1, we get the following global result.

Corollary 4.1. *Suppose c is a series with coefficients that satisfy the global growth condition* (8). *If there exists an analytic state space representation (f, g, h, x_0) of c that satisfies the growth condition* (11), *then this state space system realizes F_c on $L^m_{p,e}(0)$ for any $p \geqslant 1$.*

Proof. Let c be a series with coefficients that satisfy the growth condition (8). Suppose (f, g, h, x_0) is a representation of c that satisfies the growth condition (11). As in the previous proofs, it is sufficient to prove the case when $p = 1$. By Lemma 4.1, it follows that $y = h(x(t))$ is defined for all $t \geqslant 0$ and all $u \in L_{1,e}(0)$. Furthermore, the proof of Lemma 4.1 also shows that if u is analytic on $[0, T)$ for some $T \leqslant \infty$, then both y and $F_c[u]$ are also analytic on $[0, T)$. Again, by Isidori [7, Theorem 3.1.5], for any analytic u, $y(t) = F_c[u](t)$ on $[0, \tau)$ for some $\tau > 0$. Hence, by analyticity, $y(t) = F_c[u](t)$ for all $t \in [0, T)$.

For a general input $u \in L^m_{1,e}(0)$ which is not analytic, we again apply the approximation argument used in the proof of Lemma 4.1. Fix any $T > 0$. Pick a sequence of analytic input functions $\{u_j\}$ that converge to u in the L_1-norm on $[0, T]$. With the growth condition on c, it is not hard to show that $F_c[u_j](t) \to F_c[u](t)$ uniformly on $[0, T]$. On the other hand, $y_j(t) \to y(t)$ uniformly on $[0, T]$, where y_j is the output function of the initialized state space system corresponding to the input function u_j. With the conclusion that $y_j(t) = F_c[u_j](t)$ for all $t \in [0, T]$, one must conclude that $y(t) = F_c[u](t)$ for all $t \in [0, T]$. Since T can be chosen arbitrarily, $y(t) = F_c[u](t)$ for all $t \geqslant 0$. □

5. Examples

In this section, a variety of examples are presented to illustrate the concepts from the previous sections. The first example shows that all bilinear systems satisfy the most restrictive growth condition (8), which insures global convergence and continuity properties for the corresponding Fliess operator. A similar analysis holds for linear systems. The second example is one that fails the global growth condition (8) but satisfies the local growth condition (1). The next example involves a series with finite Lie rank but fails both the global and local growth conditions. The final example shows how Corollary 4.1 can fail if the well-posedness condition (11) is not satisfied.

Example 5.1. Consider a bilinear system

$$\dot{x} = Ax + \sum_{i=1}^{m} N_i x u_i,$$
$$y = Cx,$$

where $C \in \mathbb{R}^{\ell \times n}$. Let x_0 be any nonzero vector in $\mathbb{R}^n$. Using the induced matrix norm, define $K = \|C\||x_0|$, and $M = \max_{0 \leqslant i \leqslant m} \|N_i\|$, where $N_0 = A$. Then observe that for any $k \geqslant 0$ it follows directly that

$$|c(i_k \cdots i_1)| = |CN_{i_k} \cdots N_{i_1} x_0| \leqslant \|C\| M^k |x_0| = KM^k.$$

Hence, the global growth condition (8) is satisfied.

Example 5.2. Consider the scalar system

$$\dot{z} = z^2 u,$$
$$y = z$$

in which case the effective index set is $I^* = \{(\underbrace{1 \cdots 1}_{k}) : k \geqslant 0\}$. The coefficients for $z_0 = 1$ are easily shown to be

$$c(\underbrace{1 \cdots 1}_{k}) = L_g^k h(1) = k!, \quad k \geqslant 0.$$

Hence, $|c(\underbrace{1 \cdots 1}_{k})| = k!$ satisfies only the local growth condition (1). If one applies the local coordinate transformation $x = \phi(z) = (z-1)/z$ in a neighborhood of $z_0 = 1$, then the resulting Wiener system

$$\dot{x} = u,$$
$$y = \frac{1}{1-x}$$

has identical coefficients for $x_0 = 0$.

Example 5.3. If on the index set $I^* = \{(\underbrace{1 \cdots 1}_{k}) : k \geqslant 0\}$, c is defined as $c : \eta \mapsto (|\eta|!)^2$, then clearly the growth condition (1) is violated. In which case, it is not even possible for c to have an analytic representation. This may also be taken as a situation where c has a finite Lie rank (equal to 1), but F_c fails to have a local state space realization.

Finally, the following example shows that Corollary 4.1 may fail if condition (11) is not satisfied.

Example 5.4. Define c on the index set I^*, where $I = \{0, 1\}$, as

$$c(\eta) = \begin{cases} 1, & \eta = (1 \cdots 1), \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, c satisfies the global growth condition (8), and for any $u \in L_{1,e}(0)$,

$$F_c[u](t) = \sum_{k=0}^{\infty} \frac{U^k(t)}{k!} = \mathrm{e}^{U(t)}, \quad t \geqslant 0,$$

where $U(t) = \int_0^t u(s)\,\mathrm{d}s$. The one-dimensional system

$$\dot{x} = xu, \quad x_0 = 1,$$
$$y = x$$

is a state space representation of c and also realizes F_c on $L_{1,e}(0)$. However, not every state space representation of c realizes F_c on $L_{1,e}(0)$. For instance,

$$\dot{x}_1 = x_1^2 + x_2, \quad x_0 = (0, 1)^T,$$
$$\dot{x}_2 = x_2 u,$$
$$y = x_2$$

is also a state space representation of c, but it fails the well-posedness condition (11). Observe that the system is not even complete, i.e., there are finite escape times for some obvious choices of $u \in L_{1,e}(0)$. So the system cannot be a global realization for F_c on $L_{1,e}(0)$.

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