PROBLEMS

11516. Proposed by Elton Bojaxhiu, Albania, and Enkel Hysnelaj, Australia. Let $T$ be the set of all nonequilateral triangles. For $T \in T$, let $O$ be the circumcenter, $Q$ the incenter, and $G$ the centroid. Show that $\inf_{T \in T} \angle O G Q = \pi/2$.

11517. Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanta, Romania. Let $f$ be a three-times differentiable real-valued function on $[a,b]$ with $f(a) = f(b)$. Prove that

$$\left| \int_a^{(a+b)/2} f(x) \, dx - \int_{(a+b)/2}^b f(x) \, dx \right| \leq \frac{(b-a)^4}{192} \sup_{x \in [a,b]} |f'''(x)|.$$  

11518. Proposed by Mihaly Bence, Brasov, Romania. Suppose $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n$ be positive numbers such that $\sum_{k=1}^{n} 1/\lambda_k = 1$. Prove that

$$\frac{\zeta(\lambda_1)}{\lambda_1} + \sum_{k=2}^{n} \frac{1}{\lambda_k} \left( \zeta(\lambda_k) - \sum_{j=1}^{k-1} j^{-\lambda_k} \right) \geq \frac{1}{(n-1)(n-1)!}.$$  

11519. Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania. Find

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n+m} \frac{H_{n+m}}{n + m},$$

where $H_n$ denotes the $n$th harmonic number.

11520. Proposed by Peter Ash, Cambridge Math Learning, Bedford, MA. Let $n$ and $k$ be integers with $1 \leq k \leq n$, and let $A$ be a set of $n$ real numbers. For $i$ with $1 \leq i \leq n$, let $S_i$ be the set of all subsets of $A$ with $i$ elements, and let $\sigma_i = \sum_{s \in S_i} \max(s)$. Express the $k$th smallest element of $A$ as a linear combination of $\sigma_0, \ldots, \sigma_n$.

doi:10.4169/000298910X496796
Let \( n \) be a positive integer and let \( A_1, \ldots, A_n, B_1, \ldots, B_n, C_1, \ldots, C_n \) be points on the unit two-dimensional sphere \( S_2 \). Let \( d(X, Y) \) denote the geodesic distance on the sphere from \( X \) to \( Y \), and let \( e(X, Y) \) be the Euclidean distance across the chord from \( X \) to \( Y \). Show that
\[
\begin{align*}
(a) & \text{ There exists } P \in S_2 \text{ such that } \sum_{i=1}^n d(P, A_i) = \sum_{i=1}^n d(P, B_i) = \sum_{i=1}^n d(P, C_i). \\
(b) & \text{ There exists } Q \in S_2 \text{ such that } \sum_{i=1}^n e(Q, A_i) = \sum_{i=1}^n e(Q, B_i). \\
(c) & \text{ There exist a positive integer } n, \text{ and points } A_1, \ldots, A_n, B_1, \ldots, B_n, C_1, \ldots, C_n \text{ on } S_2, \text{ such that for all } R \in S_2, \sum_{i=1}^n e(R, A_i), \sum_{i=1}^n e(R, B_i), \text{ and } \sum_{i=1}^n e(R, C_i) \text{ are not all equal. (That is, part (b) cannot be strengthened to read like part (a).)}
\end{align*}
\]

**11522. Proposed by Moubinoole Omarjee, Lycée Jean Lurçat, Paris, France.** Let \( E \) be the set of all real 4-tuples \((a, b, c, d)\) such that if \( x, y \in \mathbb{R} \), then \((ax + by)^2 + (cx + dy)^2 \leq x^2 + y^2\). Find the volume of \( E \) in \( \mathbb{R}^4 \).

**SOLUTIONS**

### Cevian Subtriangles

**11404 [2009, 83]. Proposed by Raimond Struble, North Carolina State at Raleigh, Raleigh, NC.** Any three non-concurrent cevians of a triangle create a subtriangle. Identify the sets of non-concurrent cevians which create a subtriangle whose incenter coincides with the incenter of the primary triangle. (A cevian of a triangle is a line segment joining a vertex to an interior point of the opposite edge.)

*Solution by M. J. Englefield, Monash University, Victoria, Australia.* Label the vertices of the primary triangle \( ABC \) in counterclockwise order, and let \( I \) be the incenter. The following construction identifies the required triples of cevians. Take an arbitrary cevian \( AA' \) not passing through \( I \) and consider the circle \( \kappa \) centered at \( I \) tangent to \( AA' \), say at \( P_A \). There are two points on \( \kappa \) for which the line joining them to \( B \) is tangent to \( \kappa \). Choose for \( P_B \) the one that is counterclockwise from \( P_A \) on \( \kappa \), and take \( B' \) to be the intersection of the line through \( B \) and \( P_B \) with \( AC \). Similarly choose \( P_C \) to lie counterclockwise from \( P_B \) on \( \kappa \), and let \( C' \) be the intersection of \( AB \) with the tangent from \( C \) to \( \kappa \) at \( P_C \). By construction, \( \kappa \) is the incircle of the subtriangle.

*Editorial comment.* Little attention has been given to the subtriangle that is the topic of this problem. If the non-concurrent cevians divide the sides of \( \triangle ABC \) in ratios \( \lambda, \mu, \nu \), Routh’s theorem gives the area of the subtriangle as \( (\lambda \mu \nu - 1)^2/((\lambda \mu + \lambda + 1)(\mu \nu + \mu + 1)(\nu \lambda + \nu + 1)) \) times the area of \( \triangle ABC \). It is also known (H. Bailey, Areas and centroids for triangles within triangles, *Math. Mag.* 75 (2002) 371) that the centroids of the two triangles coincide if and only if \( \lambda = \mu = \nu \).

Also solved by R. Chapman (U. K.), C. Curtis, J. H. Lindsey II, M. D. Meyerson, J. Scher (Canada), R. A. Simon (Chile), R. Stong, Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), and the proposer.

### A Limit of an Alternating Series

**11412 [2009, 179]. Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.** Let \( f \) be a monotone decreasing function on \([0, \infty)\) such that \( \lim_{x \to \infty} f(x) = 0 \). Define \( F \) on \((0, \infty)\) by \( F(x) = \sum_{n=0}^{\infty} (-1)^n f(nx) \).

(a) Show that if \( f \) is continuous at 0 and convex on \([0, \infty)\), then \( \lim_{x \to 0^+} F(x) = f(0)/2 \).
(b) Show that the same conclusion holds if we drop the second condition on \( f \) from (a) and instead require that \( f \) have a continuous second derivative on \([0, \infty)\) such that \( \int_0^\infty |f''(x)| \, dx < \infty \).

(c) Dropping the conditions of (a) and (b), find a monotone decreasing function \( f \) on \([0, \infty)\) with \( f(0) > 0 \) such that

\[
\limsup_{x \to 0^+} \sup_{0 < y < x} F(y) = f(0), \quad \limsup_{x \to 0^+} \inf_{0 < y < x} F(y) = 0.
\]

Solution by Richard Bagby, New Mexico State University, Las Cruces, NM. For \( f \) a monotone decreasing function on \([0, \infty)\) with \( \lim_{x \to \infty} f(x) = 0 \), define

\[
F(x) = \sum_{n=0}^\infty (-1)^n f(nx) = \sum_{n=0}^\infty [f(2nx) - f((2n+1)x)], \quad x > 0.
\]

By the alternating series test, the series defines \( F(x) \) with \( 0 \leq F(x) \leq f(0) \).

(a) If \( f \) is convex, then for each \( x > 0 \), the difference \( f(kx) - f((k+1)x) \) is a nonincreasing function of the positive integer \( k \). Therefore, we have

\[
F(x) \geq \sum_{n=0}^\infty [f((2n+1)x) - f((2n+2)x)] = f(0) - F(x),
\]

as well as

\[
F(x) \leq f(0) - f(x) + \sum_{n=1}^\infty [f((2n-1)x) - f(2nx)] = 2f(0) - f(x) - F(x).
\]

Thus we see that \( f(0) \leq 2F(x) \leq 2f(0) - f(x) \) for all \( x > 0 \) when \( f \) is convex. In particular, \( \lim_{x \to 0^+} F(x) = \frac{1}{2}f(0) \) if \( f(x) \) is also continuous at the origin.

(b) Suppose that instead of assuming that \( f(x) \) is convex, we assume that \( f \in C^2[0, \infty) \) with \( \int_0^\infty |f''(x)| \, dx < \infty \). Observe that since \( f(x) \to 0 \) as \( x \to \infty \), we may write

\[
F(x) = \frac{1}{2}f(0) + \frac{1}{2} \sum_{n=0}^\infty (-1)^n [f(nx) - f((n+1)x)]
\]

\[
= \frac{1}{2}f(0) + \frac{1}{2} \sum_{n=0}^\infty \left[ \int_{(n+1)x}^{(2n+2)x} f'(t) \, dt - \int_{2nx}^{(2n+1)x} f'(t) \, dt \right]
\]

\[
= \frac{1}{2}f(0) + \frac{1}{2} \sum_{n=0}^\infty \int_{2nx}^{(2n+1)x} f''(s + t) \, ds \, dt
\]

\[
= \frac{1}{2}f(0) + \frac{1}{2} \int_0^x \left( \sum_{n=0}^\infty \int_{2nx}^{(2n+1)x} f''(s + t) \, dt \right) ds.
\]

This implies that

\[
\left| F(x) - \frac{1}{2}f(0) \right| \leq \frac{x}{2} \int_0^\infty |f''(t)| \, dt,
\]

so once again \( F(x) \to \frac{1}{2}f(0) \) as \( x \to 0 \) from the right.

(c) A simple choice of a monotone decreasing function \( f \) with \( f(0) > 0 \) for which

\[
\limsup_{x \to 0^+} F(x) = f(0), \quad \liminf_{x \to 0^+} F(x) = 0.
\]
is given by \( f(x) = 1 \) for \( 0 \leq x < 1 \) and \( f(x) = 0 \) for \( 1 \leq x < \infty \). For each positive integer \( k \), we then have \( F(1/(2k)) = 1 \) and \( F(1/(2k + 1)) = 0 \).


### A Definite Hyperbolic

11418 [2009, 276]. Proposed by George Lamb, Tucson, AZ. Find

\[
\int_{-\infty}^{\infty} \frac{t^2 \operatorname{sech}^2 t}{a - \tanh t} \, dt
\]

for complex \( a \) with \( |a| > 1 \).

**Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.** The answer is \( \frac{1}{2} \left( \log^3 \left( \frac{a+1}{a-1} \right) + \pi^2 \log \left( \frac{a+1}{a-1} \right) \right) \), where \( \log \) is the principal branch of the logarithm defined on the complex plane cut along the negative real numbers. The formula is valid for every complex number \( a \) with \( a \notin [-1, 1] \).

For \( a \notin [-1, 1] \) the integral is convergent. Denote its value by \( I(a) \). Compute

\[
I(a) = \int_{-\infty}^{\infty} \frac{t^2 \, dt}{(a \cosh t - \sinh t) \cosh t} = \int_{-\infty}^{\infty} \frac{4t^2 e^{2t} \, dt}{(a - 1)e^{2t} + a + 1}(e^{2t} + 1)
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2 e^x \, dx}{((a-1)e^x + a + 1)(e^x + 1)}
\]

\[
= \frac{1}{2(a-1)} J \left( \frac{a+1}{a-1} \right),
\]

with

\[
J(b) = \int_{-\infty}^{\infty} \frac{x^2 e^x \, dx}{(e^x + b)(e^x + 1)}.
\]

In order to evaluate \( J(b) \) for \( b \in \mathbb{C} \setminus (-\infty, 0] \), let

\[
F(z) = \frac{(z^2 + \pi^2 z)e^z}{(1 - e^z)(b - e^z)}.
\]

For large positive \( R \), consider the contour \( \gamma_R \) consisting of a positively oriented rectangle \( ABCD \) with vertices \( A, B, C, D \) at \( -R - i\pi, R - i\pi, R + i\pi, \) and \( -R + i\pi \), respectively. The only points inside the rectangle \( \gamma_R \) where the denominator of \( F \) vanishes are 0 and \( \log b \), but 0 is a removable singularity for \( F \) and \( \log b \) is a simple pole with residue

\[
\text{Res}(F, \log b) = \frac{\log^3 b + \pi^2 \log b}{b - 1}.
\]

The residue formula says that

\[
\int_{\gamma_R} F(z) \, dz = \frac{2\pi i}{b - 1}(\log^3 b + \pi^2 \log b).
\]

© THE MATHEMATICAL ASSOCIATION OF AMERICA  [Monthly 117]
However,
\[
\int_{AB} F(z) \, dz + \int_{CD} F(z) \, dz = \int_{-R}^{R} F(x - i\pi) \, dx - \int_{-R}^{R} F(x + i\pi) \, dx
\]
\[
= \int_{-R}^{R} \frac{(x + i\pi)^3 + \pi^2(x + i\pi) - (x - i\pi)^3 - \pi^2(x - i\pi)}{(1 + e^x)(b + e^x)} \, dx
\]
\[
= 6\pi i \int_{-R}^{R} \frac{x^2 e^x \, dx}{(1 + e^x)(b + e^x)},
\]
so
\[
\lim_{R \to \infty} \left( \int_{AB} F(z) \, dz + \int_{CD} F(z) \, dz \right) = 6\pi i \int_{-\infty}^{\infty} \frac{x^2 e^x \, dx}{(1 + e^x)(b + e^x)} = 6\pi i J(b).
\]

Next, \( \int_{BC} F(z) \, dz = i \int_{-\pi}^{\pi} F(R + it) \, dt \), so if \( R > 1 + |b| \), then
\[
\left| \int_{BC} F(z) \, dz \right| \leq 2\pi \sup_{t \in [-\pi, \pi]} |F(R + it)| \leq 2\pi \frac{\sqrt{R^2 + \pi^2(R^2 + 2\pi^2)e^R}}{(e^R - 1)(e^R - |b|)}.
\]
Therefore, \( \lim_{R \to \infty} \int_{BC} F(z) \, dz = 0 \). Similarly, \( \lim_{R \to \infty} \int_{DA} F(z) \, dz = 0 \). Combining our results, we conclude that
\[
6\pi i J(b) = \frac{2\pi i}{b - 1} (\log^3 b + \pi^2 \log b),
\]
or, equivalently,
\[
J(b) = \frac{1}{3(b - 1)} (\log^3 b + \pi^2 \log b).
\]
Therefore, as claimed, we get
\[
I(a) = \frac{a - 1}{2} J \left( \frac{a + 1}{a - 1} \right) = \frac{1}{12} \left( \log^3 \left( \frac{a + 1}{a - 1} \right) + \pi^2 \log \left( \frac{a + 1}{a - 1} \right) \right).
\]


**A Triangle Construction**

11419 [2009, 276]. Proposed by Vasile Mihai, Belleville, Ontario, Canada. Let \( G \) be the centroid, \( H \) the orthocenter, \( O \) the circumcenter, and \( P \) the circumcircle of a triangle \( ABC \) that is neither isosceles nor right.
Let $A', B',$ and $C'$ be the orthic points of $ABC$, that is, the respective feet of the altitudes from $A$, $B$, and $C$. Let $A_1$ be the point on $P$ such that $AA_1$ is parallel to $BC$, and define $B_1$, $C_1$ similarly. Let $A'_1$ be the point on $P$ such that $A_1A'_1$ is parallel to $AA'$, and define $B'_1$, $C'_1$ similarly (see sketch).

Show that
(a) $A_1A'_1$, $B_1B'_1$, and $C_1C'_1$ are concurrent at the point $I$ opposite $H$ from $O$ on the Euler line $HO$.
(b) $A_1A'$, $B_1B'$, and $C_1C'$ are concurrent at the centroid $G$.
(c) the circumcircles of $OA_1A'_1$, $OB_1B'_1$, and $OC_1C'_1$ (which are clearly concurrent at $O$) are concurrent at a second point $K$ lying on $HO$, and $|OH| \cdot |OK| = abc/p$, where $a$, $b$, and $c$ are the edge lengths of $ABC$, and $p$ is the perimeter of $A_1B_1C_1$.

Solution by Paul Yiu, Florida Atlantic University, Boca Raton, FL.
(a) Each of the lines $A_1A'_1$, $B_1B'_1$, and $C_1C'_1$ is the reflection of an altitude in the perpendicular bisector of the corresponding side, and these bisectors each contain the circumcenter $O$. Since the altitudes intersect at the orthocenter $H$, these reflected lines intersect at the reflection of $H$ in $O$.
(b) Let $D$ be the midpoint of $BC$. Since $AA_1$ and $BC$ are parallel and $AA_1 = 2 \cdot DA'$, the lines $A_1A'$ and $AD$ intersect at a point that divides each of $A_1A'$ and $AD$ in the ratio $2 : 1$. This point is the centroid $G$ of triangle $ABC$. The same holds for $B_1B'$ and $C_1C'$.
(c) The inverse of the line $A_1A'_1$ in the circumcircle $P$ is the circle $OA_1A'_1$. This circle contains the inverse $K$ of $I$ in $P$. The same holds for the lines $B_1B'_1$ and $C_1C'_1$. Note that $|OH| \cdot |OK| = |OI| \cdot |OK| = R^2$, where $R$ is the circumradius.

If $ABC$ is acute, then the angles of $A_1B_1C_1$ are $\pi - 2A$, $\pi - 2B$, and $\pi - 2C$. The perimeter $p$ of triangle $A_1B_1C_1$ is given by

\[
p = 2R(\sin 2A + \sin 2B + \sin 2C) = 2a \cos A + 2b \cos B + 2c \cos C
\]

\[
= \frac{a^2(b^2 + c^2 - a^2) + b^2(c^2 + a^2 - b^2) + c^2(a^2 + b^2 - c^2)}{abc}
\]

\[
= \frac{16\Delta^2}{abc} = \left(\frac{abc}{R}\right)^2 \cdot \frac{1}{abc} = \frac{abc}{R^2}.
\]

Therefore, $R^2 = abc/p$.

This formula is correct only for acute triangles. If angle $A$ is obtuse, the angles of triangle $A_1B_1C_1$ are $2A - \pi$, $2B$, and $2C$.

Matrix Normality

11422 [2009, 277]. Proposed by Christopher Hillar, The Mathematical Sciences Research Institute, Berkeley, CA. Let $H$ be a real $n \times n$ symmetric matrix with distinct eigenvalues, and let $A$ be a real matrix of the same size. Let $H_0 = H$, $H_1 = AH_0 - H_0A$, and $H_2 = AH_1 - H_1A$. Show that if $H_1$ and $H_2$ are symmetric, then $AA' = A'A$; that is, $A$ is normal.

Solution by Patrick Corn, St. Mary’s College of Maryland, St. Mary’s City, MD. If we conjugate $H_0, H_1, H_2$, and $A$ by the same orthogonal matrix, then the hypotheses, definitions, and conclusion remain unchanged. There exists an orthogonal matrix that diagonalizes $H_0$, since $H_0$ is a real, symmetric matrix. Without loss of generality, then, we may assume that $H_0$ is diagonal with distinct entries.

Since $H_1$ is symmetric, it follows that $AH_0 - H_0A = (AH_0 - H_0A)' = H_0A' - A'H_0$, and thus $(A + A'H_0 = H_0(A + A')$. Since the matrix $A + A'$ commutes with $H_0$, it must be diagonal. Now write $A = D + S$, where $D = (1/2)(A + A')$ is diagonal and $S = (1/2)(A - A')$ is skew-symmetric.

Since $H_2$ is symmetric, we have $H_1(A + A') = (A + A')H_1$, and $H_1D = DH_1$. That is, $(AH_0 - H_0A)D = D(AH_0 - H_0A)$. Since $D$ and $H_0$ commute, $AH_0 - H_0A = S H_0 - H_0S$, and then $(SH_0 - H_0S)D = D(SH_0 - H_0S)$, so $H_0(DS - SD) = (DS - SD)H_0$. Thus $DS - SD$ commutes with $H_0$, so it must be diagonal. However, $DS$ and $SD$ both have zero diagonals, since $S$ does, and therefore $DS = SD$.

Expanding and using $DS = SD$, we conclude that

$$AA' - A'A = (D + S)(D' + S') - (D' + S')(D + S) = 2(SD - DS) = 0.$$

This gives the desired result.


A Lobachevsky Integral

11423 [2009, 277]. Proposed by Gregory Minton, D. E. Shaw Research, LLC, New York, NY. Show that if $n$ and $m$ are positive integers with $n \geq m$ and $n - m$ even, then $\int_{x=0}^{\infty} x^{-m} \sin^nx \, dx$ is a rational multiple of $\pi$.

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA. We use induction on $m$. Let $I(n, m) = \int_{x=0}^{\infty} x^{-m} \sin^n x \, dx$. First, for any odd positive integer $n = 2k + 1$, we recall that

$$\sin^{2k+1} x = \frac{1}{2^{2k}} \sum_{i=0}^{k} (-1)^{k-i} \binom{2k+1}{i} \sin ((2k - 2i + 1)x)$$

and

$$\int_{0}^{\infty} \frac{\sin(ax)}{x} \, dx = \frac{\pi}{2}$$

for $a > 0$. Hence

$$I(2k + 1, 1) = \frac{1}{2^{2k+1}} \sum_{i=0}^{k} (-1)^{k-i} \binom{2k+1}{i} \pi$$

August–September 2010] PROBLEMS AND SOLUTIONS 655
is a rational multiple of $\pi$. For $m = 2$, note that integration by parts gives

$$I(n, 2) = n \int_0^\infty \frac{\sin^n x \cos x}{x} \, dx.$$ 

Using the product to sum formula for sine and cosine, for $n = 2k$ we can expand $\sin^{2k-1} x \cos x$ as

$$\frac{1}{2^{2k-1}} \sum_{i=0}^{k-1} (-1)^{k-i+1} \binom{2k-1}{i} \left( \sin \left( (2k-2i)x \right) + \sin \left( (2k-2i-2)x \right) \right),$$

so

$$I(2k, 2) = \frac{k}{2^{2k-2}} \left( \frac{1}{2} \binom{2k-1}{k-1} + \sum_{i=0}^{k-2} (-1)^{k-i+1} \binom{2k-1}{i} \right) \pi$$

is also a rational multiple of $\pi$. For $m \geq 2$, integrating by parts twice leads to

$$I(n, m + 1) = -\frac{n^2}{m(m-1)} I(n, m - 1) + \frac{n(n-1)}{m(m-1)} I(n-2, m-1).$$

When $n - (m + 1)$ is even and nonnegative, the right side is a rational multiple of $\pi$ by the induction hypothesis. Therefore, the left side is also such a multiple, which completes the proof.

**Editorial comment.** The integrals $I(n, m)$ were apparently first considered by N. I. Lobachevskii, *Probabilité des résultats moyens tirés d’observations répétées*, *J. Reine Angew. Math.* 24 (1842) 164–170.

T. Hayashi, in “On the integral $\int_0^\infty \frac{\sin^n x}{x^m} \, dx”$, *Nieuw Arch. Wiskd.* (2) 14 (1923) 13–18, gave the following explicit evaluation:

$$I(n, m) = \frac{\pi (-1)^{(n-m)/2}}{2^{n-m+1}(m-1)!} \sum_{0 \leq j \leq (n-1)/2} (-1)^j \binom{n}{j} \left( \frac{n}{2} - j \right)^{m-1}$$

which for $m = 1$ or 2 simplifies to

$$I(2k + 1, 1) = \frac{\pi \binom{2k}{k}}{2^{k+1} k!} = \frac{\pi}{2^{2k+1}} \binom{2k}{k},$$

$$I(2k, 2) = \frac{\pi \binom{2k-3}{k-1}}{2^k (k-1)!} = \frac{\pi}{2^{2k-1}} \binom{2k-2}{k-1},$$

and these more than suffice for the current problem.