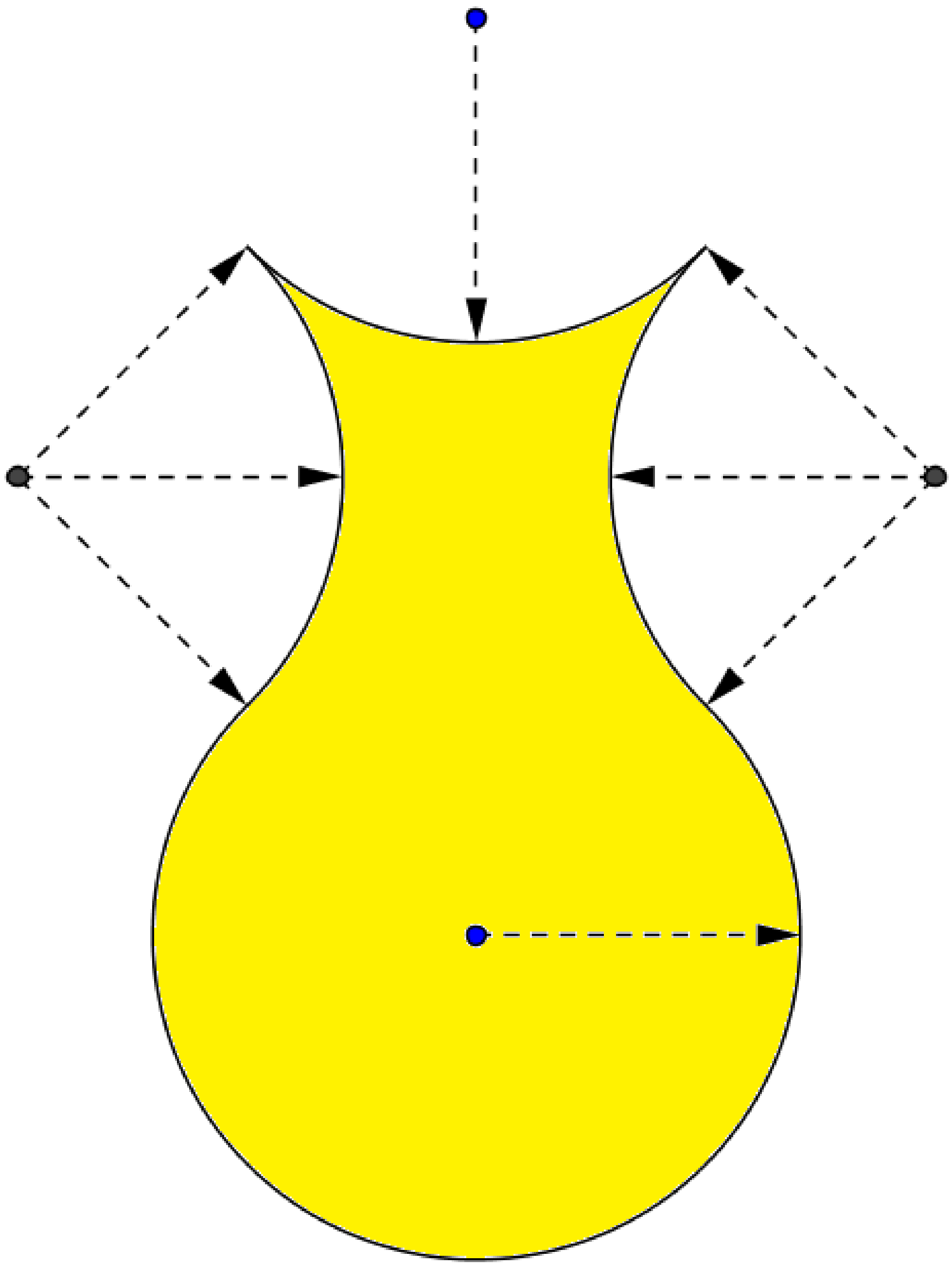
FAU Math Circle
2022/10/8
SOLUTIONS

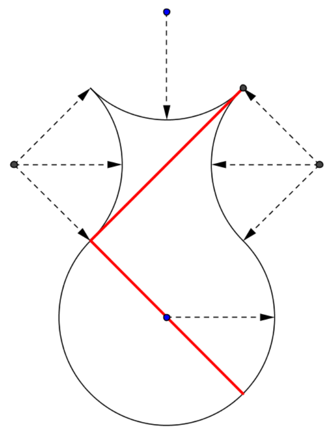
Math Warm Up

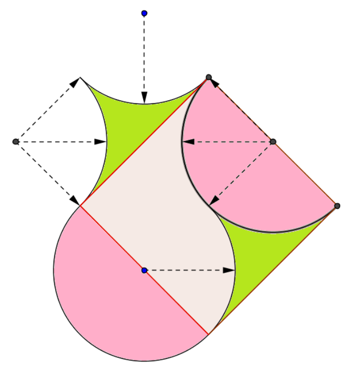
Cut the jug into 3 pieces by two **straight** cuts, and form a square out of the parts. The dashed arrows could be a nuisance, or of some help. We have additional pictures available for experimentation, and to take home.



Solution.

The first figure shows the two cuts in red; the second how to get the square.





Today's Problems

We will be dealing with numbers followed by a little bit of geometry.

1. Find a three digit number such that when you subtract 7 from it, the result is divisible by 7; if 8, the result is divisible by 8; and if 9, by 9.

   Solution. If the number is divisible by 7 after 7 was subtracted from it, it must have been divisible by 7 all along. The same goes for 8 and 9. Also, because 7, 8, 9 have no common factors, a number divisible by 7, 8, 9 must be divisible by $7 \times 8 \times 9 = 504$. The smallest such number is 504. The next one is $2 \times 504 = 1008$, and it has already four digits. The answer is **504.**

2. Four cruise ships leave the port of Miami at noon, January 2, 2014. The first ship returns to this port every 4 days, the second every 8 days, the third every 12, and the fourth every 16. When did all four ships meet again in this port?

   Solution. The number of days when they first come again into the port together is the least common multiple of 4, 8, 12, and 16. That number is 48. So one answer is 48 days after January 2. Alternatively, February 19.

3. Find all two digit numbers (are there any?) such that the number equals the product of its digits plus the sum of its digits. Be sure you find all of them.

   Solution. If we write a number in the form ab where a and b are digits; a digit being an integer in the range 0–9, it means that the number equals $10 \times a + b$. For example $56 = 10 \times 5 + 6$; the product of its digit is 30 the sum of its digits is 11; $30 + 11 = 44 \neq 56$, so 56 does not equal the product of its digits plus the sum of its digits. For another example, $72 = 10 \times 7 + 2$; here the product plus the sum of is digits works out to 23; so 72 is not one of the numbers we are looking for. I

If a two digit number equals the product of its digits plus the sum of its digits it is of the form ab with

$$10 \times a + b = a \times b + a + b.$$

We can subtract $a + b$ from both sides. On the left b cancels and we are left with $10 \times a - a = 9 \times a$ or $9a$. On the right we are left with a times b or ab. So the number has to satisfy that $9a = ab$, so $b = 9$. The second digit must be 9. We see that a cn be any digit. The numbers are

$$19, \quad 29, \quad 39, \quad 49, \quad 59, \quad 69, \quad 79, \quad 89, \text{ and } \quad 99.$$

4. When Peter broke his piggy bank it contained no more than 100 coins. He divided the coins into piles of 3 coins each, but was left with one extra coin. The same happened when Peter divided the coins into piles of 3 coins, piles of 4 coins, and piles of 5 coins. Each time he was left with an extra coin. How many coins were in the piggy bank?

   Solution. If we borrow from Peter the coin that is always left over, the

   remaining coins can be nicely arranged in piles of 2, 3, 4, or 5 coins. Since 3, 4, and 5 have no common factors, the number of coins left must be divisible by $3 \times 4 \times 5 = 60$. In the range 0 to 99 the only numbers divisible by 60 are 0 and 60. We may assume Peter had more than one coin and that the piles were not empty piles. The answer is: Peter had 61 coins in his piggy bank.

5. Together Winnie the Pooh, Owl, Rabbit and Piglet ate 70 bananas. Each ate a whole number of bananas and each ate at least one. Pooh ate more than each of the others. Owl and Rabbit together ate 45 bananas. How many bananas did Piglet eat?

   Solution. Owl and Rabbit ate 45 Bananas together so one of them ate at least 23 bananas. So Pooh ate at east 24 bananas. o between Pooh, Rabbit and Owl ate least 24+45 = 69 bananas are gone. Since everybody ate at least the answer is that Piglet ate one banana.

6. While walking in a park, Jack and Jill come upon a large round clearing surrounded by a ring of palm trees. They decide to count the trees. Both walk around the clearing and count all the trees, but beginning at different trees. Jill's 20th tree was Jack's 7th, while Jill's 7th tree was Jack's 94th. How many trees were growing around the clearing?

   sol The answer is 100. Since Jill's 20th tree is Jack's 7th, we can conclude that Jack's starting tree was Jill's 13th. Jill is at her 7th tree when Jack is at 94l 6 more trees and Jill is at 13, which is Jack's starting point. So at 94 trees Jack is 6 trees away from being de, the number of trees is $94 + 6 = 100$.

7. Which is greater:

$$\frac{12345678765432199987}{12345678765432199988} \quad \text{or} \quad \frac{12345678765432199988}{12345678765432199989}?$$

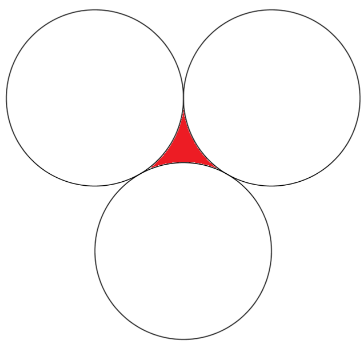
Solution. The second one is greater. The fractions have the form $\dfrac{a-1}{a}$ and $\dfrac{a}{a+1}$. We can use that for fractions $\dfrac{c}{d}, \dfrac{e}{f}$ we have $\dfrac{c}{d} < \dfrac{e}{f}$ if and only if $cf < ed$. Then

$$\frac{a-1}{a} < \frac{a}{a+1} \quad \text{if and only if} \quad (a-1)(a+1) < a \cdot a = a^2.$$

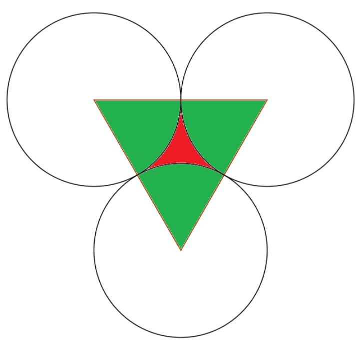
How does $(a-1)(a+1)$ compare with a^2? Multiplying out, $(a-1)(a+1) = a^2 - 1$ which is $< a^2$.

A Bit of Geometry

8. Three equal circles of radius 3 are tangent to each other, as shown. Find the area of the shaded region.



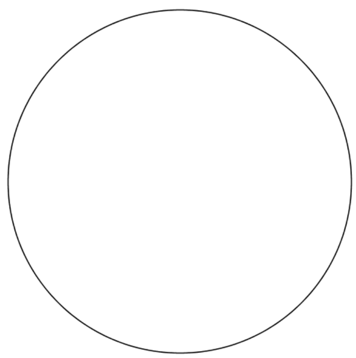
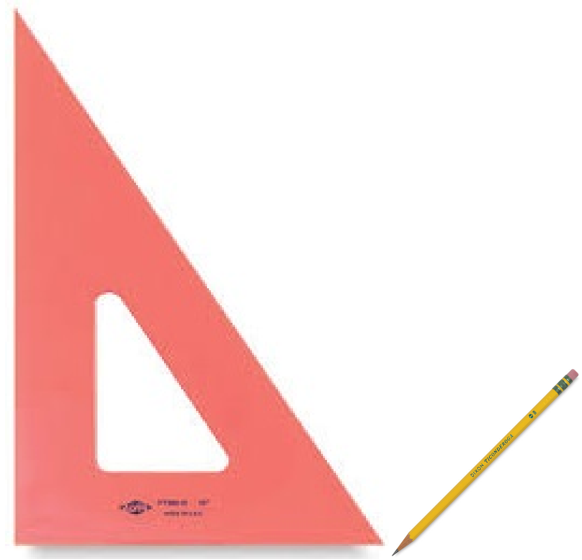
Solution. Connect the centers of the circles to form a triangle as shown below.



The triangle is equilateral and its sides are of length $6 = 3 + 3$ (each side is made up out of two radii). Using the theorem of Pythagoras one can see that the altitude of an equilateral triangle is $(\sqrt{3}/2)$ times the length of the sides, equal to $3\sqrt{3}$ if the side length is 6. The area of the triangle in the picture is thus $\frac{1}{2}(3\sqrt{3}) \times 6 = 9\sqrt{3}$. The shaded region equals what remains of the triangle once one removes the three green circular sectors. Each one of these is equal to one sixth of the full circle. This is because the angles of an equilateral triangle all equal 60 degrees, and the full circle corresponds to an angle of $360 = 6 \times 60$ degrees. The area of each one of the circles is 9π (πr^2 when $r = 3$). Putting it all together, the shaded region has an area of

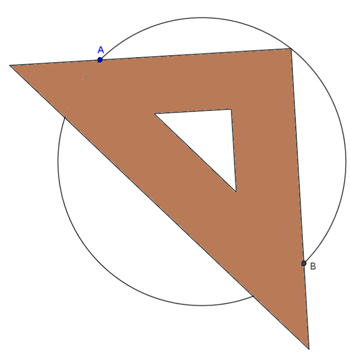
$$9\sqrt{3} - 3 \times \frac{9\pi}{6} = 9\sqrt{3} - \frac{9}{2}\pi.$$

9. You visited the far away country of Noplacia, and now you are not allowed to leave. Emperor Lunaticus will let you go if you can do one thing for him, find the center of a circle. All he allows you to use is a drafting triangle and a pencil. How can you do it?

	
The Circle	Drafting Triangle and pencil

(A couple of very primitive drafting triangles are available if you decide to try)

Solution. Use the fact that the diameter of the circle subtends a 90 degree angle, and a 90 degree angle is subtended by a diameter. Place the right angle corner of the drafting triangle at any point of the circle and mark the points where the other two sides intersect the circle:



Draw the line from A to B; that's a diameter. Repeat placing the right angle corner at some other point to get a second diameter. The two diameters intersect at the center.