Escape Velocity

Suppose we have a planet (or any large near to spherical heavenly body) of radius $R$ and acceleration of gravity at the surface of $g$. We launch a rocket straight into the air. The rocket has no motor, so once it is launched the only force we assume acting on it is the gravitational force due to the attraction of the planet. What velocity must it be given initially so that it never returns to the planet? We denote this velocity by $v_e$ and call it the escape velocity. Of course, in a real life situation, once the rocket is far enough from the planet it may well be strongly influenced by other masses in space. We will denote by $y(t)$ the distance the rocket is from the planet at time $t$; time 0 is the launch time so $y(0) = 0$. The picture is the same as in the first lecture; I repeat it here.

As long as the rocket moves away from the planet we can think of its speed as being a function of $y$, so $v(y)$ denotes the speed of the rocket at distance $y$ from the planet. Newton’s second law states that mass $\times$ acceleration = force. At distance $y$ the force acting on the rocket (with a negative sign because it goes against the direction of motion) is

$$f = -\frac{GMm}{(y+R)^2}.$$  

This is just Newton’s law of universal gravitation that states that the force of attraction between two masses is proportional to the product of the masses and
inversely proportional to the distance between the centers of mass of the masses. In the equation for \( f \), \( m \) is the mass of the rocket, \( M \) of the planet, and \( G \) is the universal gravitational constant. The acceleration is the derivative of velocity with respect to time; in this case, since velocity is positive, it coincides with the derivative of speed with respect to time. However, we want to see all as function of \( y \), thus

\[
\frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = \frac{dv}{dy} v.
\]

Newton’s second law can thus be stated as follows:

\[
mv \frac{dv}{dy} = - \frac{GMm}{(y + R)^2}.
\]

Canceling \( m \),

\[
v \frac{dv}{dy} = - \frac{GM}{(y + R)^2}.
\]

Before we solve this separable equation, we notice that on the planet’s surface the force with which the planet attracts the mass \( m \) is \(-mg\). It also is \(-\frac{GMm}{(y + R)^2}\) with \( y = 0 \). Setting \( y = 0 \) and equating we get \( mg = GMm/R^2 \) so that \( GM = gR^2 \) and the equation satisfied by \( v(y) \) is

\[
v \frac{dv}{dy} = - \frac{gR^2}{(y + R)^2}.
\]

The variables separate to \( v dv = - \frac{gR^2}{(y + R)^2} \, dy \); integrating we get

\[
\frac{1}{2} v^2 = \frac{gR^2}{y + R} + C.
\]

Setting \( y = 0 \) we want \( v(0) = v_e \), the escape velocity; solving for \( C \) we get \( C = \frac{1}{2} v_e^2 - gR \) so that the velocity \( v \) satisfies

\[
\frac{1}{2} v^2 = \frac{gR^2}{y + R} + \frac{1}{2} v_e^2 - gR.
\]

As \( y \) increases, if there comes a moment at which \( v(y) = 0 \); this means that the rocket has reached peak height and will start falling after that. For escape velocity we should have \( v(y) > 0 \) as long as \( y > 0 \) with \( \lim_{y \to \infty} v(y) = 0 \) (if the limit is negative, \( v(y) \) must have become 0 at some point; if the limit is positive, we left the planet faster than necessary). Now, we will have \( \lim_{y \to \infty} v(y) = 0 \) if and only if

\[
0 = \lim_{y \to \infty} \frac{1}{2} v^2 = \lim_{y \to \infty} \left( \frac{gR^2}{y + R} + \frac{1}{2} v_e^2 - gR \right) = \frac{1}{2} v_e^2 - gR.
\]

It follows that \( v_e = \sqrt{2gR} \).

For the earth, \( g \approx 9.8 \times 10^{-3} \) km/s\(^2 \), \( R \approx 6,371 \) km, so that

\[
v_e \approx \sqrt{2 \times 9.8 \times 10^{-3} \times 6371} \approx 11.17 \text{ km/s}
\]

or (approximately) 6.98 mps.
The Brachistochrone

Here is one formulation of the problem. We have points $A, B$ in the same vertical plane with $A$ a bit higher than $B$. We want to join $A$ to $B$ by a very thin wire on which a bead can slide down frictionlessly, subject only to the acceleration of gravity. What shape should we give the wire so as to minimize the time it will take the bead to slide from $A$ to $B$?

As a warmup exercise we might calculate the time it takes the bead to do the trip if we keep the wire straight. We introduce coordinates placing $A$ at the origin $(0, 0)$ of the plane, $B$ at $(x_0, y_0)$, with $y_0 < 0$. For the purpose of the pictures we will assume $x_0 \geq 0$ so $B$ is to the right of $A$.

The force acting on the bead in the direction of the wire is (in intensity)
$$f = mg \cos \beta = \frac{mg|y_0|}{\sqrt{x_0^2 + y_0^2}}.$$ If $s(t)$ is the distance the bead is from $A$ at time $t$, Newton’s second law gives
$$ms''(t) = \frac{mg|y_0|}{\sqrt{x_0^2 + y_0^2}},$$
canceling $m$, 
$$s''(t) = \frac{g|y_0|}{\sqrt{x_0^2 + y_0^2}}.$$

We can integrate this equation twice, using $s(0) = 0$ and $s'(0) = 0$ to get
$$s(t) = \frac{g|y_0|}{2\sqrt{x_0^2 + y_0^2}} t^2.$$

The bead will be at $B$ once $s(t) = \sqrt{x_0^2 + y_0^2}$. Equating
$$\frac{g|y_0|}{2\sqrt{x_0^2 + y_0^2}} t^2 = \sqrt{x_0^2 + y_0^2}$$
and solving for $t$ we get
$$t = \frac{2(x_0^2 + y_0^2)}{g|y_0|}.$$

Many authors worked on the problem of finding the curve, among them Newton and the Bernoulli brothers, who also solved the problem. What I am presenting below is one of the ways Johann Bernoulli solved it, according to Wikipedia. Bernoulli imagines how a ray of light would go from $A$ to $B$. Light always takes the path that will take it from $A$ to $B$ in minimum time. All things being equal, that path is the line segment joining $A$ to $B$. However, if somewhere along the road the speed of light changes, a straight line might
not be the best possible path from a point of view of minimizing time. That phenomenon is known as refraction, in its simplest manifestation light passes from a medium where it has velocity $v_1$ to one of velocity $v_2$. The picture below shows how a ray of light will travel from $A$ to $B$ if $A$ is in a medium where the speed of light is $v_1$ at $A$, $B$ in one with speed $v_2$; the green line denotes the separation of the media; the red line segments show the path of a ray of light.

It is a simple calculus exercise to see\(^*\) that the angles $\theta_1, \theta_2$ the light ray forms with the perpendicular to the surface separating the two media has to satisfy Snell’s law

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

if the time it takes the ray to go from $A$ to $B$ is to be minimal.

If the ray moves from a higher to a lower point in a non-homogeneous substance, and the different media are separated by horizontal planes, then Snell’s law becomes

\[(2) \quad \frac{\sin \theta}{v} = k\]

for some constant $k$, where $v$ is the velocity and $\theta$ is the angle the light ray forms with the vertical.

Here is how Bernoulli reasoned. Let us suppose we have a light ray and its velocity at each point is the one that would be had if it were a particle under the force of gravity. Such a particle, trying to minimize the time to get from $A$ to $B$, would follow exactly the path given by the curve one is looking for. Energy would be preserved. The kinetic energy of the particle is $\frac{1}{2}mv^2$, where $v$ is the speed; its potential energy under the force of gravity is $mgy$, where $y$ is its vertical coordinate so that $\frac{1}{2}mv^2 + mgy$ is constant, or (canceling $m$) we see that\(^{\dagger}\) $v^2 + 2gy = C$, for some constant $C$. If we consider that at $A$ both $y$ and $v$ are 0, we see that $C = 0$ and

\[(3) \quad v = \sqrt{-2gy}.

For a curve of equation $y = y(x)$ the sine of the angle $\theta$ at any point with the vertical is given by

\[(4) \quad \sin \theta = \frac{1}{\sqrt{1 + (y')^2}}.\]

\(^*\)Assuming light always minimizes time.

\(^{\dagger}\)I show how one gets this from Newton’s second law in an Appendix.
In fact, consider the following diagram.

The tangent line forms an angle \( \theta \) with the vertical, thus an angle of \( \varphi = \frac{\pi}{2} - \theta \) with the horizontal. The slope of the line is easily seen to be \( -\tan \varphi \); now \( \sin \theta = \cos \varphi = 1/\sqrt{1 + \tan^2 \varphi} \); considering that \( \tan \varphi = y' \), the formula follows. So now combining the results from (2), (3), (4), we get

\[
k = \frac{\sin \theta}{v} = \frac{1}{\sqrt{1 + (y')^2}} \cdot \frac{1}{\sqrt{-2gy}}
\]

From this we get \( (-y)(1 + y'^2) = \frac{1}{2gk^2} \). Relabeling, writing \( k \) for \( 1/(2gk^2) \) (a constant is a constant is a constant) we get (as Johann Bernoulli did) the ODE

\[
y(1 + y'^2) = -k.
\]

Notice: \( k > 0 \).

At this point things go pretty much as in the textbook. Solving for \( y' \) we get

\[
y' = \pm \sqrt{-\frac{k}{y} - 1}.
\]

Since \( y \) will decrease from 0 to a negative value, we choose the negative sign;

\[
y' = -\sqrt{-\frac{k}{y} - 1}.
\]

Since \( k > 0 \), the quantity inside the square root will be positive as long as \( y > -k \).

The variables separate to

\[
\frac{1}{\sqrt{-\frac{k}{y} - 1}} dy = -dx
\]

It is possible to integrate this and get the solution in the form \( x = x(y) \), but the following trick simplifies things. We introduce a new variable \( \theta \) by \( y = -k \sin^2 \theta \). For \( 0 \leq \theta \leq \pi/2 \), \( y \) will range from 0 to \(-k\). Then

\[
dy = -2k \sin \theta \cos \theta d\theta
\]

and

\[
\frac{1}{\sqrt{-\frac{k}{y} - 1}} = \frac{1}{\sqrt{\frac{1}{\sin^2 \theta} - 1}} = \frac{\sin \theta}{\cos \theta}.
\]
The separated ODE becomes
\[ 2k \sin^2 \theta \, d\theta = dx, \]
which integrates to
\[ k(\theta - \sin \theta \cos \theta) = x + C. \]
The range of \( \theta \) is \( 0 \leq \theta \leq \pi \); for \( \theta = 0 \) we want not only \( y = 0 \) but also \( x = 0 \); this forces \( C = 0 \). We can now think of \( \theta \) as a parameter and have the curve we are looking for given parametrically by
\[ x = k(\theta - \sin \theta \cos \theta), \quad y = -k \sin^2 \theta, \quad 0 \leq \theta \leq \pi. \]
This is the equation of an arc of an inverted cycloid. It is not the most common form of the equation. For the common form we use that
\[ \sin \theta \cos \theta = \frac{1}{2} \sin(2\theta), \quad \sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta)), \]
change variables by \( t = 2\theta \) (so \( t \) ranges from 0 to \( 2\pi \)) and get the equation of an arc of a cycloid in the form
\[ x = \frac{1}{2}k(t - \sin t), \quad y = \frac{1}{2}k(1 - \cos t), \quad 0 \leq t \leq 2\pi. \]
Our curve, the inverted arc, is then given parametrically by
\[ x = \frac{1}{2}k(t - \sin t), \quad y = -\frac{1}{2}k(1 - \cos t), \quad 0 \leq t \leq 2\pi. \]
Of course for \( t = 2\pi \) we are at \((k\pi, 0)\); we need to determine \( k > 0 \) and \( t_0 \in [0, 2\pi] \) such that
\[ x_0 = \frac{1}{2}k(t_0 - \sin t_0), \quad y_0 = -\frac{1}{2}k(1 - \cos t_0), \]
so that the path is described parametrically by
\[ x = \frac{1}{2}k(t - \sin t), \quad y = -\frac{1}{2}k(1 - \cos t), \quad 0 \leq t \leq t_0. \]
It is not easy to solve the equation for \( t_0 \) explicitly, get the solution in closed form, except in particular cases. However, it is not too hard to show that the arcs given parametrically by equation \((6)\) will fill the positive lower half-plane and that no two arcs, corresponding to different values of \( k \), will intersect. Since the proof I have of this fact is a bit complicated (there might be easier proofs around) I relegate it to an Appendix 2. For those who prefer pictures to analytic proofs, the picture below sketches an arc of the inverted cycloid for values of \( k \) ranging from 0.01 to 2 in steps of 0.01.

![Arc of inverted cycloid](image)

You can see that if we keep on increasing \( k \) every point of the fourth quadrant will be reached.
For simplicity I assume that we have a particle of mass $m$ moving in a plane we call the $(x, y)$-plane. At time $t$ the particle is at $(x(t), y(t))$. I will use the physics notation for time derivatives (invented by Newton) in which $\frac{dx}{dt} = \dot{x}$, $\frac{d^2x}{dt^2} = \ddot{x}$; same for $y$. Assume that when the particle is at point $(x, y)$ it is subject to a force that decomposes into a force $f_1(x, y)$ in the $x$-direction, $f_2(x, y)$ in the $y$ direction. Again, for simplicity, I assume the forces depend only on the position; not on time or velocity. Newton’s second law becomes the following system of ODE’s:

$$\begin{cases}
m\ddot{x} = f_1(x, y) \\
m\ddot{y} = f_2(x, y)
\end{cases}$$

The force field $(f_1, f_2)$ is said to be conservative if there exists a function $V(x, y)$, known as the potential, such that

$$f_1 = -\frac{\partial V}{\partial x}, \quad f_2 = -\frac{\partial V}{\partial y}.$$ 

If you see any relation between this and exact ODE’s, you are right. Then one proves easily:

The quantity

$$E(t) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + V(x(t), y(t))$$

remains constant.

In fact, by the chain rule

$$E'(t) = \frac{1}{2}m(2\dot{x}\ddot{x} + 2\dot{y}\ddot{y}) + \frac{\partial x}{\partial V}(x, y)\dot{x} + \frac{\partial y}{\partial V}(x, y)\dot{y}$$

$$= \dot{x} \left( m\ddot{x} + \frac{\partial x}{\partial V}(x, y) \right) + \dot{y} \left( m\ddot{y} + \frac{\partial y}{\partial V}(x, y) \right)$$

$$= \dot{x}(m\ddot{x} - f_1(x, y)) + \dot{y}(m\ddot{y} - f_2(x, y)) = 0.$$ 

This constant is called the energy of the system. If we set $v = \sqrt{\dot{x}^2 + \dot{y}^2}$, then $v$ is the speed of the particle and $E = \frac{1}{2}mv^2 - V(x, y)$. One calls $\frac{1}{2}mv^2$ the kinetic energy, $V(x, y)$ is the potential energy. The potential energy $V$ is determined (if it exists) up to an additive constant. One usually sets $V$ more or less conveniently equal to 0 at some point; in this way the additive constant gets determined.

Assume now the $(x, y)$-plane is a vertical plane and the particle moving in it is subject to the force due to acceleration of gravity. This force decomposes into a vertical force of $f_2(x, y) = -mg$, and zero horizontal force: $f_1(x, y) = 0$. We see that if $V(x, y) = -mgx$, then

$$\frac{\partial V}{\partial x} = 0 = f_1, \quad \frac{\partial V}{\partial y} = -mg = f_2.$$ 

The gravitational field is thus conservative and the corresponding energy is $E = \frac{1}{2}mv^2 - gy$.

**Appendix 2-Solving for $t_0, k$**

Since it is nicer to work with positive quantities than with negative ones, I show here that the following result is true.
Let \( x, y \) be positive real numbers, so \((x, y)\) is in the first quadrant. There is a unique \( k > 0, t \in (0, 2\pi) \) such that

\[
(7) \quad x = \frac{1}{2} k(t - \sin t), \quad y = \frac{1}{2} k(1 - \cos t).
\]

For the proof we will consider the function \( \phi(t) = \frac{1 - \cos t}{t - \sin t} \). This function is well defined for \( t > 0 \); we only need it for \( 0 < t < 2\pi \). We notice that

\[
\phi'(t) = \frac{\sin(t(t - \sin t) - (1 - \cos t)^2}{(t - \sin t)^2} = \frac{t \sin t - 2(1 - \cos t)}{(t - \sin t)^2}.
\]

The numerator \( N(t) = t \sin t - 2(1 - \cos t) \) of \( \phi'(t) \) is negative in \((0, 2\pi)\) thus so is \( \phi'(t) \). To see this we can use the so called half angle formulas,

\[
\sin t = 2 \sin \frac{t}{2} \cos \frac{t}{2}, \quad 1 - \cos t = 2 \sin^2 \frac{t}{2}
\]

to get

\[
N(t) = 2 \sin \frac{t}{2} \left( t \cos \frac{t}{2} - \sin \frac{t}{2} \right).
\]

If there is a point where \( N(t) \) is not negative it has to happen for \( 0 < t < \pi \). This is best seen by the first expression for \( N \); if \( \pi < t < 2\pi \) then \( \sin t < 0 \) and so is \(-2(1 - \cos t)\). So if \( N(t) \) is not always negative, since it is negative at \( \pi \), there has to exist a point \( t \in (0, \pi) \) where \( N(t) = 0 \). Now \( \sin(t/2) > 0 \) for \( 0 < t < \pi \) so that we conclude that there would have to be \( t \) such that

\[
t \cos \frac{t}{2} - \sin \frac{t}{2} = 0.
\]

Setting \( \theta = t/2 \), there has to exist \( 0 < \theta < \pi/2 \) such that \( \tan \theta = \theta \). It is not hard to see (graphically or analytically) that \( \tan \theta > \theta \) if \( 0 < \theta < \pi/2 \), so no such \( \theta \) exists. We did prove that \( \phi'(t) < 0 \) in \((0, 2\pi)\). So \( \phi(t) \) is strictly decreasing in that interval, assumes no value twice. One also sees that

\[
\lim_{t \to 0} \phi(t) = \infty \quad \text{(Use L'Hôpital)}, \quad \lim_{t \to 2\pi} \phi(t) = \phi(2\pi) = 0,
\]

so that all values between 0 and \( \infty \) must be assumed by \( \phi \). There exists thus a unique \( t \in (0, 2\pi) \) such that

\[
\phi(t) = \frac{y}{x}; \quad \text{that is} \quad \frac{1 - \cos t}{t - \sin t} = \frac{y}{x}.
\]

Since \( \cos t < 1 \) in \((0, 2\pi)\) we can define \( k \) by

\[
k = \frac{2y}{1 - \cos t}.
\]

Then \( k > 0 \) and \( y = \frac{1}{2} k(1 - \cos t) \). But also,

\[
\frac{1}{2} k(t - \sin t) = \frac{t - \sin t}{1 - \cos t} = \frac{x}{y} = x.
\]

We are done showing that for every choice of \((x, y)\) there exists a unique pair \((k, t), k > 0, t \in (0, 2\pi)\) such that (7) holds. (The uniqueness of \( k \) is sort of obvious, once one sees that \( t \) is unique.)
Appendix 3-Time along the brachistochrone

Generally speaking, consider a curve given parametrically by \( x = x(\theta), y = y(\theta) \), \( a \leq \theta \leq b \); assume \( y(\theta) < 0 \) for \( \theta > a \) and interpret the \( y \) direction as being a vertical direction and interpret the curve as the trajectory of a particle falling subject only to the acceleration due to gravity. How long does it take to get from \( \theta = a \) to \( \theta = b \)? Assume its initial speed at \( \theta = a \) was 0. Let us reason like people from the eighteenth century, think of the particle being at a generic point \((x(\theta), y(\theta))\) of the curve. As we did for the brachistochrone, energy conservation says that its speed at this point will be \( v = \sqrt{-gy(\theta)} \). Using very elementary physics, namely time=distance/speed, we can say that as the particle traverses an infinitesimal portion of the curve of length \( ds \), it will do so in time \( dt = ds/v \).

Now \( ds = \sqrt{x'(\theta)^2 + y'(\theta)^2} \ d\theta \), so the total time will be

\[
T = \int_a^b dt = \int_a^b \frac{\sqrt{x'(\theta)^2 + y'(\theta)^2}}{-2gy(\theta)} \ d\theta.
\]

Let us apply this formula for the brachistochrone. To avoid confusing the parameter with time, I'll write the equations in the form

\[
x = \frac{1}{2} k(\theta - \sin \theta), \quad y = -\frac{1}{2} k(1 - \cos \theta)
\]

and I will take \( a = 0, b = \theta_0 \), where \( 0 < \theta_0 < 2\pi \). Now

\[
x'(\theta)^2 + y'(\theta)^2 = \frac{1}{4} k^2 [(1 - \cos \theta)^2 + \sin^2 \theta] = \frac{1}{2} k^2 (1 - \cos \theta) = k^2 \sin^2(\theta/2);
\]

writing also \( y(\theta) = -\frac{1}{2} k(1 - \cos \theta) = -k \sin^2(\theta/2) \), we get

\[
T = \int_0^{\theta_0} \frac{\sqrt{x'(\theta)^2 + y'(\theta)^2}}{-2gy(\theta)} \ d\theta = \frac{k}{\sqrt{g}} \int_0^{\theta_0} \ d\theta = \frac{\theta_0 \sqrt{k}}{\sqrt{2g}}
\]

Let us consider the arc that ends at \((\pi, -2)\), for which we need \( k = 2, \theta_0 = \pi \). The time it takes the particle is thus

\[
\frac{\pi \sqrt{2}}{\sqrt{2g}} = \frac{\pi}{\sqrt{g}}.
\]

On the other hand, if we go by a straight line, the time it takes is given by (1) with \( x_0 = \pi, y_0 = -2 \) and it works out to \( \frac{\sqrt{\pi^2 + 4}}{\sqrt{g}} \), a little bit longer.

Incidentally, along the brachistochrone it would take time \( \frac{2\pi \sqrt{k}}{\sqrt{2g}} \) for the particle to go from \((0, 0)\) to \((k\pi, 0)\). It would take infinity time for the particle to this along a straight line segment.