1 The One Dimensional Case

We recall the one-dimensional Sturm-Liouville problem: Assume $p, q, r$ are continuous functions in the interval $[0, L]$. Assume $p(x) > 0, r(x) > 0$ for $0 < x < L$. For what values of $\lambda$ is there a nonzero solution $y = y(x)$ of the following problem:

$$
\frac{d}{dx} \left( p \frac{dy}{dx} \right) + qy + \lambda ry = 0,
$$

satisfying the boundary conditions:

$$
\alpha_0 y(0) + \alpha_1 y'(0) = 0, \\
\beta_0 y(L) + \beta_1 y'(L) = 0,
$$

where $\alpha_0, \beta_0, \alpha_1, \beta_1$ are given constants and at least one of $\alpha_0, \alpha_1$ is not zero, at least one of $\beta_0, \beta_1$ is not zero.

Numbers $\lambda$ for which a nonzero solution of the ODE satisfying the boundary conditions exists are called eigenvalues of the problem; the non-zero solutions are called eigenfunctions.

Assume $\lambda$ is an eigenvalue, $y = \varphi(x)$ is an eigenfunction, of the Sturm-Liouville problem. That is, assume that $\varphi$ is a twice differentiable function in $[0, L]$ such that

$$
(1) \quad \frac{d}{dx} (p(x)\varphi'(x)) + q(x)\varphi(x) + \lambda r(x)\varphi(x) = 0 \quad \text{for all } x \in [0, L],
$$

and

$$
(2) \quad \alpha_0 \varphi(0) + \alpha_1 \varphi'(0) = 0, \\
(3) \quad \beta_0 \varphi(L) + \beta_1 \varphi'(L) = 0,
$$

We multiply both sides of equation (1) by $\varphi(x)$ and integrate from $0$ to $L$, by parts. We get the following chain of equalities:

$$
0 = \int_0^L \left( \frac{d}{dx} (p(x)\varphi'(x)) + q(x)\varphi(x) + \lambda r(x)\varphi(x) \right) \varphi(x) \, dx \\
= \int_0^L \left( \frac{d}{dx} (p(x)\varphi'(x)) \right) \varphi(x) \, dx + \int_0^L q(x)\varphi(x)^2 \, dx + \lambda \int_0^L r(x)\varphi(x)^2 \, dx \\
= p(x)\varphi'(x)\varphi(x) \bigg|_0^L - \int_0^L p(x)\varphi'(x)^2 \, dx + \int_0^L q(x)\varphi(x)^2 \, dx + \lambda \int_0^L r(x)\varphi(x)^2 \, dx
$$

We now solve for $\lambda$ to get

$$
\lambda = \frac{-p(x)\varphi'(x)\varphi(x) \bigg|_0^L + \int_0^L p(x)\varphi'(x)^2 \, dx - \int_0^L q(x)\varphi(x)^2 \, dx}{\int_0^L r(x)\varphi(x)^2 \, dx}.
$$
This formula, the one dimensional version of Ryaleigh’s quotient, allows us to compute \( \lambda \) assuming we know a corresponding eigenfunction, which we usually don’t. That is, at least in our case, finding \( \lambda \) and the eigenfunction happened simultaneously. But the formula has other uses. Let us first see an example of it working. Consider for example the problem

\[
X'' + \lambda X = 0, \quad X(0) = X(1) = 0,
\]

so \( L = 1, \quad \alpha_0 = \beta_0 = 1, \quad \alpha_1 = \beta_1 = 0, \quad p(x) \equiv 1 \equiv r(x), \quad q(x) \equiv 0. \) Now, in this case,

\[
p(x)\varphi'(x)\varphi(x)\bigg|_0^L = p(L)\varphi'(L)\varphi(L) - p(0)\varphi'(0)\varphi(0) = 0,
\]

and Rayleigh’s quotient becomes

\[
(5) \quad \lambda = \frac{\int_0^1 \varphi'(x)^2 \, dx}{\int_0^1 \varphi(x)^2 \, dx}.
\]

Knowing that the \( \varphi(x) = \sin n\pi x \) is an eigenfunction, we get

\[
\lambda = \frac{\int_0^1 n^2 \pi^2 \cos^2 n\pi x \, dx}{\int_0^1 \sin^2 n\pi x \, dx} = \frac{n^2 \pi^2 \frac{1}{2}}{\frac{1}{2}} = n^2 \pi^2.
\]

If this were all, the quotient would not be too interesting. It has however other applications. For example, if

\[
p(x)\varphi'(x)\varphi(x)\bigg|_0^L = p(L)\varphi'(L)\varphi(L) - p(0)\varphi'(0)\varphi(0) = 0,
\]

as happens quite frequently, and if \( q = 0 \), we can see at once from the quotient

\[
\lambda = \frac{\int_0^1 p(x)\varphi'(x)^2 \, dx}{\int_0^1 r(x)\varphi(x)^2 \, dx}
\]

we can conclude that there are no negative eigenvalues. But there is more; One\(^{1}\) can show that in general

\[
\lambda_1 = \min \left( \frac{\int_0^L -p(x)\psi'(x)\psi(x) \, dx + \int_0^L p(x)\psi'(x)^2 \, dx - \int_0^L q(x)\psi(x)^2 \, dx}{\int_0^L r(x)\psi(x)^2 \, dx} \right),
\]

where the minimum is taken over all once differentiable functions \( \psi \) defined in the interval \([0, 1]\), satisfying the boundary conditions \((2), (3)\). This allows one to at least estimate the first eigenvalue in cases where computing it can be quite hard. In words: If you take any function \( \psi \) that is at least once differentiable in \([0, L]\), that satisfies the BC’s, and you evaluate the quotient

\[
\frac{\int_0^L -p(x)\psi'(x)\psi(x) \, dx + \int_0^L p(x)\psi'(x)^2 \, dx - \int_0^L q(x)\psi(x)^2 \, dx}{\int_0^L r(x)\psi(x)^2 \, dx},
\]

what you get is going to be larger or at most equal to the first eigenvalue \( \lambda_1 \) of the Sturm Liouville problem with those BC’s (and functions \( p, q, r \)). The eigenvalue is, in fact and precisely, the smallest value you can get if you do this computation. For example, if for the problem \( X'' + \lambda X = 0, \quad X(0) = X(1) = 0 \) we try \( \psi(x) = x(1-x) \), so \( \psi'(x) = 1-2x \) and

\[
\frac{\int_0^1 \psi'(x)^2 \, dx}{\int_0^1 \psi(x)^2 \, dx} = \frac{\int_0^1 (1-2x)^2 \, dx}{\int_0^1 x^2(1-x)^2 \, dx} = \frac{1/3}{1/30} = 10.
\]

This is not so terribly far from the actual eigenvalue, namely \( \pi^2 \approx 9.87 \).

\(^{1}\)It has been mentioned that “One” is the greatest mathematician of all times, he or she can prove almost anything.
2 Higher Dimensions

In this section $D$ is a “nice” region in either 2 or 3 space. By “nice” I mean the following:

- $D$ is a connected, open set. The “open” part means that every point of $D$ can be surrounded by disk if we are in two dimensions, a ball if in 3, that stays inside $D$. Connected means that $D$ is in one piece; any two points of $D$ can be joined by a curve staying entirely within $D$.

- The boundary $\partial D$ of $D$ is mostly smooth. That is, the boundary may have a few corners, but there is a well defined tangent line (2 dimensions)/plane (3 dimensions) at most points of the boundary.

To avoid repetitions, I will work in three space. Just replace all triple integrals by double ones, eliminate the last variable to get the corresponding two dimensional result. The problem I will consider will be one of the following three possible problems:

1. (Dirichlet)
   \[ \nabla^2 w(x, y, z) + \lambda w(x, y, z) = 0 \quad \text{for} \quad (x, y, z) \in D, \]
   \[ w(x, y, z) = 0 \quad \text{for} \quad (x, y, z) \in \partial D. \]

2. (Neumann)
   \[ \nabla^2 w(x, y, z) + \lambda w(x, y, z) = 0 \quad \text{for} \quad (x, y, z) \in D, \]
   \[ \frac{\partial w}{\partial \nu}(x, y, z) = 0 \quad \text{for} \quad (x, y, z) \in \partial D. \]
   Here $\frac{\partial w}{\partial \nu}$ is the derivative with respect to the outer normal $\nu$, $\frac{\partial w}{\partial \nu} = \nabla w \cdot \nu$.

3. (mixed)
   \[ \nabla^2 w(x, y, z) + \lambda w(x, y, z) = 0 \quad \text{for} \quad (x, y, z) \in D, \]
   the boundary $\partial D$ is the disjoint union of two parts; that is $\partial D = A \cup B$ with $A \cap B$ empty, and
   \[ w(x, y, z) = 0 \quad \text{for} \quad (x, y, z) \in A, \quad \frac{\partial w}{\partial \nu}(x, y, z) = 0 \quad \text{for} \quad (x, y, z) \in B. \]

The PDE is the same in all three, only the boundary conditions are different. But all boundary conditions have one feature in common: $w \frac{\partial w}{\partial \nu} = 0$ at all boundary points. In one dimension we multiplied the ODE by $y$ and integrated using integration by parts. Here we will multiply the PDE by $w$ and integrate using a higher dimensional integration by parts analog; Gauss’ divergence theorem. We notice first that $\text{div} \left( f \nabla g \right) = \nabla f \cdot \nabla g + f \nabla^2 g$; in particular, $\text{div} \left( w \nabla w \right) = \| \nabla w \|^2 + w \nabla^2 w$, which we use in the form $w \nabla^2 w = \text{div} \left( w \nabla w \right) - \| \nabla w \|^2$. Multiplying the PDE by $w$ ad integrating over $D$ we get

\[
\int_D \left( w \nabla^2 w \lambda w \right) dV = \int_D \text{div} \left( w \nabla^2 w \right) dV - \int_D \left( \| \nabla w \|^2 + \lambda \| w \|^2 \right) dV.
\]

Using that $w \frac{\partial w}{\partial \nu} = 0$ on $\partial D$, the integral over $\partial D$ is zero and we can solve for $\lambda$ to get

\[ \lambda = \frac{\int_D \| \nabla w \|^2 dV}{\int_D \| w \|^2 dV}. \]

Equation (\ref{eq:lambda}) is the higher dimensional analog of (\ref{eq:lambda_1D}).

First consequence: The eigenvalues of the three problems mentioned above are always non negative. Concerning 0 being a possible eigenvalue, we have

Exercise 1 (Could be an exam exercise?) Show that in the problems labeled 1 and 3, 0 is not an eigenvalue. Show that, however, 0 is an eigenvalue in the problem labeled 2.
As in the case of one dimension one has the following results for problems 1, 2, and 3:

- There exists a sequence $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ of eigenvalues with $\lim_{n \to \infty} \lambda_n = \infty$.

- For each eigenvalue there exists at most a finite number of linearly independent eigenfunctions. Specifically, if $\lambda$ is an eigenvalue, one can find a finite number $\varphi_1, \ldots, \varphi_m$ of eigenfunctions such that every eigenfunction $\varphi$ of $\lambda$ can be written in the form $\varphi = a_1\varphi_1 + \cdots + a_m\varphi_m$ where $a_1, \ldots, a_m$ are uniquely determined real numbers.

- Thenumber of linearly independent eigenfunctions corresponding to the first eigenvalue $\lambda_1$ is one; that is, if $\varphi$ is an eigenfunction corresponding to $\lambda_1$, then every other eigenfunction is of the form $c\varphi$ for some nonzero constant $c$.

$$\lambda_a = \min \frac{\iiint_D |\nabla \varphi|^2 dV}{\iiint_D |\varphi|^2 dV}$$

where the minimum is taken over all differentiable functions $\varphi$ satisfying the appropriate boundary conditions.