Adding Complexity to life

1 The Complex Fourier Series

The complex Fourier series is in some ways a superior product, at least for those people who are not terrified by complex numbers. Suppose \( f(x) \) is a piecewise smooth function in the interval \([-L, L]\); then with \( a_0, a_1, a_2, \ldots, b_1, b_2, \ldots \), defined by

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \quad n = 0, 1, 2, \ldots, \tag{1}
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 0, 1, 2, \ldots, \tag{2}
\]

one has

\[
f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}), \tag{3}
\]

the series converging (I hope it is known by now what this means) for all \( x \in (-\infty, \infty) \) to a periodic function of period \( 2L \) that in the interval \([-L, L]\) coincides with \( f(x) \) at all points \( x \) where \( f \) is continuous; with \( (f(x+) + f(x-))/2 \) where \( x \) fails to be continuous. Given that

\[
f(x+) + f(x-) = 2\frac{f(x)}{1},
\]

where \( f \) is continuous, we can state (3) in a more precise way as

\[
f(x+) + f(x-) = 2\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}), \quad -L < x < L. \tag{4}
\]

Equation (4) is also true for \( x = L \) and for \( x = -L \) if we interpret \( f(-L-) = f(L-) \), \( f(L+) = f(-L+) \).

To move over into a complex more satisfying (for some) vision of things, we have to recall Euler’s famous formula, namely

\[
e^{ix} = \cos x + i \sin x
\]

if \( x \) is a real number. One could take this as the definition of the complex exponential. From this formula we get

\[
\cos x = \frac{1}{2} (e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).
\]

Using these formulas in (3) or (4), and the definition of what it means for a series to converge we get (this course is case sensitive, \( N \) and \( n \) are different symbols!)

\[
\frac{f(x+) + f(x-)}{2} = \lim_{N \to \infty} \left( \frac{1}{2} a_0 + \sum_{n=1}^{N} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right)
\]

\[
= \lim_{N \to \infty} \left( \frac{1}{2} a_0 + \sum_{n=1}^{N} \left( \frac{1}{2} a_n (e^{i\frac{nx}{L}} + e^{-i\frac{nx}{L}}) + \frac{1}{2i} b_n (e^{i\frac{nx}{L}} - e^{-i\frac{nx}{L}}) \right) \right)
\]

\[
= \lim_{N \to \infty} \left( \frac{1}{2} a_0 + \sum_{n=1}^{N} \left( \frac{1}{2}(a_n + \frac{1}{i}b_n)e^{i\frac{nx}{L}} + \frac{1}{2}(a_n - \frac{1}{i}b_n)e^{-i\frac{nx}{L}} \right) \right)
\]

If we now consider that \( 1/i = -i \) and use that

\[
\sum_{n=1}^{N} \frac{1}{2}(a_n + ib_n)e^{-i\frac{nx}{L}} = \sum_{n=-N}^{-1} \frac{1}{2}(a_{-n} + ib_{-n})e^{i\frac{nx}{L}},
\]

with
we can continue our chain of equalities to get
\[
\frac{f(x+) + f(x-)}{2} = \lim_{N \to \infty} \left( \frac{1}{2} a_0 + \sum_{n=1}^{N} \frac{1}{2} (a_n - ib_n)e^{i \frac{\pi n x}{L}} + \sum_{n=-N}^{-1} \frac{1}{2} (a_{-n} + ib_{-n})e^{i \frac{\pi n x}{L}} \right).
\]

We now introduce the complex coefficients \( c_n \) defined for all integers \( n \) (positive, negative, and zero) by
\[
c_n = \begin{cases} 
\frac{1}{2} (a_n - ib_n), & n = 1, 2, 3, \ldots, \\
\frac{1}{2} a_0, & n = 0, \\
\frac{1}{2} (a_{-n} + ib_{-n}) = c_{-n}, & n = -1, -2, -3, \ldots.
\end{cases}
\]

With them, our last displayed equality can be written in the form
\[
\frac{f(x+) + f(x-)}{2} = \lim_{N \to \infty} \sum_{n=-N}^{N} c_n e^{i \frac{\pi n x}{L}}.
\]

(5)

The limit existed before; it still exists. From the definition of the coefficients \( c_n \) and (1), (2) one now has

**Theorem 1** Suppose \( f(x) \) is piecewise smooth in the interval \([-L, L]\). Then
\[
\frac{f(x+) + f(x-)}{2} = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{\pi n x}{L}}, \quad -L < x < L,
\]

(6)

where
\[
c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{\pi n x}{L}} dx, \quad n = 0, \pm 1, \pm 2, \ldots
\]

(7)

The series converges for all real numbers \( x \); within the interval \((-L, L)\) its sum is as indicated.

I will emphasize that we interpret
\[
s = \sum_{n=-\infty}^{\infty} a_n
\]

to mean
\[
s = \lim_{n \to \infty} (a_{-n} + a_{-n+1} + \cdots + a_{-1} + a_0 + a_1 + \cdots + a_n).
\]

This definition of what the sum of a double series is will suffice for us. In reality, in a slightly more precise environment, one declares divergent a series where the limit above exists only because the positive and negative parts cancel each other out and one considers convergent a series only if both the series of terms with negative indices and the series of the terms with positive indices converge.

We interrupt this presentation for a few words from our sponsor; complex valued functions.

It is useful to know certain facts about complex valued functions; that is functions of the form \( f(x) = u(x) + iv(x) \) defined for \( x \) in an interval \( I \) (open or closed, or half open, bounded or unbounded), where \( u, v \) are real valued. We say \( f \) is differentiable at \( x \) if and only both \( u, v \) are differentiable at \( x \) and then we define \( f'(x) = u'(x) + iv'(x) \).

Similarly, if \( u, v \) are continuous on \( I \), we define
\[
\int f(x) \, dx = \int u(x) \, dx + i \int v(x) \, dx,
\]

the constant of integration being now a complex constant. If \( a, b \in I \),
\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} u(x) \, dx + i \int_{a}^{b} v(x) \, dx;
\]

the result is a complex number.
Exercise 1 Use the definition (which means: use the definition!) to compute

1. \( \int_{0}^{1} (x^2 + 3ix)^2 \, dx \).

2. \( \int_{0}^{2} e^{-5ix} \, dx \).

Exercise 2 Show that if \( f, g \) are complex valued functions on an interval \( I \), say \( f = u_1 + iv_1 \), \( g = u_2 + iv_2 \), where \( u_1, u_2, v_1, v_2 \) are real valued functions, one still has

\[
(f + g)' = f' + g', \quad (fg)' = f'g + fg'.
\]

Concerning this last exercise, what needs to be done, for example, to show \( (fg)' = f'g + fg' \) is: Write \( f = u + iv \), \( g = p + iq \) where \( u, v, p, q \) are real valued functions on \( I \). Then \( fg = (up - vq) + i(uq + vp) \). Use the definition of derivative given above, and properties of the Calculus 1 derivative, to get a formula for \( (fg)' \). See that it coincides with what you get when you compute \( f'g + fg' \) following the same rules.

Because of the definition of derivative and integral, it should be clear that the fundamental theorem of calculus remains valid for complex valued functions; namely: If \( F'(x) = f(x) \) for all \( x \) in the interval \([a, b] \), then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]

We defined the complex exponential above when the argument is purely imaginary. The more general definition is: If \( z = x + iy \) is a complex number, \( x, y \) real, then

\[
e^{x+iy} = e^x \cos y + i e^x \sin y.
\]

Exercise 3 Using the definition (which, I remind you again, means using the definition) show: If \( \alpha = a + ib \) is a complex number, \( a, b \) real, then

\[
\frac{d}{dx} e^{\alpha x} = \alpha e^{\alpha x}.
\]

Explain why this result implies that

\[
\int e^{\alpha x} \, dx = \frac{1}{\alpha} e^{\alpha x} + C
\]

if \( \alpha \neq 0; C \) a complex constant. Then use this result to calculate again \( \int_{0}^{2} e^{-5ix} \, dx \).

We return now to our presentation on complex Fourier series and Fourier transform.

Exercise 4 Express the following functions in the form of a complex Fourier series in the indicated interval. In the first two of these expansions, try NOT to use the real Fourier series to do this, go directly to the complex coefficients. For the third one, use whatever way you think is easiest.

1. \( f(x) = x, \ -1 \leq x \leq 1 \).

2. \( f(x) = x^2, \ -1 \leq x \leq 1 \).

3. \( f(x) = 3 \cos 2x + (5 + 6i) \sin 7x, \ -\pi \leq x \leq \pi \).

Exercise 5 Suppose a function is defined by

\[
f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 2} e^{inx}.
\]

Compute

\[
\int_{-\pi}^{\pi} f(x) \cos(3x) \, dx.
\]
2 The Fourier Transform

Suppose you have a function \(f(x)\) defined and piecewise smooth in the interval \(-\infty < x < \infty\). To analyze it in terms of its frequencies (which is what the Fourier series does) we could start by taking a very large \(L\). Then we could take an even larger \(L\), finally letting \(L \to \infty\). At that point a strange thing happens, the series becomes an integral. For the sake of completeness here are the computations, done also in the textbook. We begin with (assume \(f\) is defined at all points of discontinuity as the average value of its left and right limits)

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2\pi}{L} nx}, \quad -L < x < L
\]

where

\[
c_n = \frac{1}{2L} \int_{-L}^{L} f(x)e^{-i \frac{2\pi}{L} nx} \, dx, \quad n = 0, \pm 1, \pm 2, \ldots
\]

Define now a new function of a variable we will call \(\hat{f}\) (no mathematical article is complete without some Greek letters) by

\[
\hat{f}_L(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(x)e^{-i \omega x} \, dx.
\]

Then

\[
c_n = \frac{\sqrt{\pi}}{L \sqrt{2}} \hat{f}_L\left(\frac{n\pi}{L}\right).
\]

As \(L\) gets large, the number \(\pi/L\) gets to be small; we’ll set \(\Delta \omega = \pi/L\) and set also \(\omega_n = n\pi/L = n\Delta \omega\) for \(n = 0, \pm 1, \pm 2, \ldots\). The Fourier series expansion for \(f\) can then be written in the form

\[
f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_L(\omega_n)e^{-i \omega_n x} = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}_L(\omega_n)e^{-i \omega_n x} \Delta \omega.
\]

This looks a lot like a Riemann sum, and it should be at least plausible that when we let \(L \to \infty\), which is the same as letting \(\Delta \omega \to 0\), the sum becomes an integral, with \(\hat{f}_L\) being replaced by its limit as \(L \to \infty\). That is, if we define

\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i \omega x} \, dx = \lim_{L \to \infty} \hat{f}_L(\omega)
\]

(assuming, of course, this limit exists), then

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i \omega x} \, d\omega.
\]

Formula (8) defines the Fourier transform of \(f\), formula (9) shows how to recover \(f\) from \(\hat{f}\). Formula (8) makes sense if \(\int_{-\infty}^{\infty} |f(x)| \, dx < \infty\); then it defines a nice continuous function of the variable \(\omega\). If IN ADDITION, \(\int_{-\infty}^{\infty} |\hat{f}(\omega)| \, d\omega < \infty\), then (9) makes sense and is valid. I put the “in addition,” in bold capitals because our textbook author says that the only thing needed for the validity of (9) is \(\int_{-\infty}^{\infty} |f(x)| \, dx < \infty\) which, in our context, is false.

I should mention that our author has signs switched around. Some people prefer it that way; I don’t. But to agree with our textbook author, I will do the same and from now on I will use:

\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i \omega x} \, dx
\]

for the definition of the Fourier transform; the inverse Fourier transform is then given by

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i \omega x} \, d\omega.
\]

We will also denote the Fourier transform of \(f\) by \(\mathcal{F}f\), so that

\[
\mathcal{F}f(\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i \omega x} \, dx.
\]
PROPERTIES OF THE FOURIER TRANSFORM.

The inverse Fourier transform is then denoted by $\mathcal{F}^{-1}$; it can be applied to any function, not only to functions that are Fourier transforms of known functions:

$$\mathcal{F}^{-1}g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)e^{-ix\omega} \, d\omega \quad (13)$$

$\mathcal{F}$ and $\mathcal{F}^{-1}$ cancel each other out; sort of (see below). Keep in mind, incidentally, that $x, \omega$ are just symbols for variables. For example formula (13) is totally the same as

$$\mathcal{F}^{-1}g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{-ix\omega} \, dx$$

Exercise 6 The author shows that if

$$f(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1, \end{cases}$$

then $\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega}$. Using this give a plausible argument proving that

$$\lim_{R \to \infty} \int_{0}^{R} \frac{\sin x}{x} \, dx = \frac{\pi}{2}.$$ 

3 Properties of the Fourier transform.

I mentioned above that the inverse Fourier transform recovers the original function, equivalently, if we apply $\mathcal{F}$ followed by $\mathcal{F}^{-1}$ (or vice-versa) we are back where we started. This is almost true. If the function to be transformed is discontinuous at some points, the Fourier transform may lose whatever value the function had at the jump. But except for isolated values, it is true. A very important fact is thus that no essential information is lost when transforming Fourier. This fact seems to be rarely emphasized in textbooks at our level, but I believe it is one of the most important properties; the fact that in a sense $f$ and $\hat{f}$ are the same object because they contain the same information. The captain Kirk on the Enterprise and the Captain Kirk that has just beamed down on the planet X are the same because they have the same information. Well, maybe a few skin cells of the first Captain Kirk got lost in the process, but that’s not essential. Even if the Fourier transform of $f$ turns out to be a function that does not satisfy our criteria for applying $\mathcal{F}^{-1}$, it still contains all the essential information of $f$.

For engineers and physicists, one of the very important properties of the Fourier transform is that it breaks up a signal into its frequencies. That is why they prefer to put the $2\pi$ into the exponent and define it as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i x \omega} f(x) \, dx.$$ 

The function $f$ can then be recovered by

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i x \omega} \hat{f}(\omega) \, d\omega \quad (\ast)$$

For a fixed $\omega$, consider the function $h_\omega(x) = Ae^{2\pi i x \omega} = A(\cos(2\pi x \omega) + i \sin(2\pi x \omega))$, where $A$ is a constant. Think of $x$ as being time, measured in seconds. This function is periodic of period $T = 1/\omega$, meaning that if it where a signal, the signal repeats every $1/\omega$ seconds. If it takes $T$ seconds for the same thing to happen again, then in one second the thing will happen $1/T$ times (the frequency doesn’t have to be an integer; for example if you have a beep periodically occurring every 5 seconds, its frequency will be 0.2). Returning back to the Fourier transform, the formula labeled by $(\ast)$, expresses $f$ as a superposition of signals of frequency $\omega$, amplitude $\hat{f}(\omega)$, for all $\omega$ in $(-\infty, \infty)$.

In this course, however, the Fourier transform is a somewhat less glamorous object; it is just a tool for solving differential equations. The $2\pi$ in the exponent would complicate formulas a bit, so we take it out from there and use formula (10) (or, what’s the same, (12)) as our definition.

Just for fun, perhaps, let us be precise for a while and make some definitions. A function $f$ is piecewise continuous on the infinite interval $(-\infty, \infty)$ if it satisfies all of the following properties:
1. There is at most a sequence of points at which the function fails to be defined or fails to be continuous. By at most, one means that there might be none (the function is everywhere defined and continuous), a finite number, or an infinite number that could be indexed by the integers. 

2. If the number of bad points; that is, points where $f$ is undefined or fails to be continuous, is infinite, then it is either a sequence of points going to infinity, a sequence of points going to $-\infty$, or it can be broken up into two sequences, one going to $\infty$ the other one to $-\infty$. 

3. The function $f$ has finite left and right limits at each bad point. 

Such a function can always be integrated over finite (bounded) intervals. There are much more general functions with this property, but we are keeping things simple here. 

A function $f$ is piecewise smooth on the infinite interval $(-\infty, \infty)$ if it satisfies all of the following properties: 

1. There is at most a sequence of points at which the function fails to be defined or fails to be differentiable. By at most, one means that there might be none (the function is everywhere defined and differentiable), a finite number, or an infinite number that could be indexed by the integers. 

2. If the number of bad points; that is, points where $f$ is undefined or fails to be differentiable, is infinite, then it is either a sequence of points going to infinity, a sequence of points going to $-\infty$, or it can be broken up into two sequences, one going to $\infty$ the other one to $-\infty$. 

3. The function $f$ and its derivative $f'$ have finite left and right limits at each bad point. 

Because a function that is differentiable at a point is also continuous at that point, if $f$ is piecewise smooth, it is piecewise continuous. 

Here is a fact, let’s call it a theorem so as to make it seem important. 

**Theorem 2** If $f$ is piecewise continuous in $(-\infty, \infty)$ and satisfies 

$$\int_{-\infty}^{\infty} |f(x)| \, dx = \lim_{R \to \infty} \int_{-R}^{R} |f(x)| \, dx < \infty,$$

then $\hat{f}$, as defined by (10), exists (in the sense that the formula makes sense for all $\omega$). Moreover, $\hat{f}$ is continuous at all points of $(-\infty, \infty)$ and $\lim_{\omega \to \pm \infty} \hat{f}(\omega) = 0$. 

I want to illustrate this theorem a bit with the piecewise continuous function $f$ defined by 

$$f(x) = \begin{cases} 
1, & |x| < a, \\
0, & |x| > a,
\end{cases}$$

where $a$ is a constant (the value of $f$ at $a$ is of no consequence). As seen in class and in the text, 

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin a\omega}{\omega}.$$ 

One could object and say that the theorem must be false, since this function $f$ is not (in principle) defined for $\omega = 0$. True, it has a limit at 0 and one could define for it $\omega = 0$ by setting it equal to its limit; but shouldn’t the theorem have said this. No, it isn’t the theorem that’s at fault; it is what we said that $\hat{f}$ was. If we compute $\hat{f}(0)$ directly by the formula we get (since $e^{i\omega} = 1$ if $\omega = 0$) 

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 \, dx = \frac{2a}{\sqrt{2\pi}} = a \sqrt{\frac{2}{\pi}}.$$ 

It follows that the Fourier transform of this function $f$ is given by 

$$\hat{f}(\omega) = \begin{cases} 
\sqrt{\frac{2}{\pi}} \frac{\sin a\omega}{\omega}, & \omega \neq 0 \\
a \sqrt{\frac{2}{\pi}}, & \omega = 0.
\end{cases}$$
In other words, \( \hat{f} \) is defined at 0, and one sees easily that it is continuous at 0. And it is also clear that
\[
\lim_{\omega \to \pm \infty} \hat{f}(\omega) = 0.
\]

By the theorem, if \( f \) is piecewise continuous and the integral of its absolute value from \(-\infty\) to \( \infty \) is finite, then \( \hat{f} \) not only is piecewise smooth, but also continuous. Because \( \hat{f}(\omega) \) goes to 0 as \( \omega \to \pm \infty \) one almost has that \( \int_{-\infty}^{\infty} |\hat{f}(\omega)| \, d\omega \) is finite. Almost, but not quite. That is why to be able to actually recover \( f \) at our level by (11) we have to require that \( \int_{-\infty}^{\infty} |\hat{f}(\omega)| \, d\omega < \infty \). But we will never really have to recover \( f \) by actually inverting; if we have \( \hat{f} \) and want to find \( f \), what we will do (sort of) is look at a table of Fourier transforms, and read it form right to left. We’ll even go a step further; we’ll just transform more or less freely, assuming all transforms exist. There is a more sophisticated approach to the transform, but it is way beyond our scope.

Another important property of the Fourier transform (thus also of the inverse Fourier transform) is **linearity**. If \( f, g \) are functions that have a Fourier transform, if \( a, b \) are constants (real or complex!), then
\[
\mathcal{F}(af) = a\hat{f},
\]
\[
\mathcal{F}(af + bg) = a\hat{f} + b\hat{g},
\]
in general, if \( c_1, c_2, \ldots \) are constants and \( f_1, f_2, \ldots \) functions, then
\[
\mathcal{F}(c_1f_1 + c_2f_2 + \cdots) = c_1\hat{f_1} + c_2\hat{f_2} + \cdots.
\]

Here is a simple, but useful property, which we will write out somewhat incorrectly in the form:
\[
\mathcal{F}\{f(x + y)\}(\omega) = e^{-i\omega y} \hat{f}(\omega).
\]

It is somewhat incorrectly written because on the left hand side it isn’t clear which is the main variable. A better way of writing is: If \( f \) is a function whose Fourier transform exists and \( y \in (\infty, \infty) \), define a new function\(^1\) \( f_y \) by \( f_y(x) = f(x + y) \). Then
\[
\hat{f}_y(\omega) = e^{-i\omega y} \hat{f}(\omega).
\]

We now come to the main reason why the Fourier transform is a useful tool for solving differential equations. I’ll state it as a theorem:

**Theorem 3** If \( f \) and \( f' \) have Fourier transforms, then
\[
\mathcal{F}(f')(\omega) = -i\omega \hat{f}(\omega).
\]

In class I sort of proved this theorem under the assumption that \( f \) was piecewise smooth and \( \int_{-\infty}^{\infty} |f(x)| \, dx < \infty \). It is really a simple application of integration by parts that does it. Because everything that’s true for \( f \) holds (mutatis mutandis) for \( \hat{f} \), one has the following counterpart to this theorem

**Theorem 4** If \( f(x) \) and \( xf(x) \) have Fourier transforms, then
\[
\mathcal{F}(xf)(\omega) = i\frac{df}{d\omega}(\omega).
\]

A thing you should consider is that \( x, \omega \) are just symbols for variables. We use one for the function, the other one for the transform. But they have no real existence past their symbolic value and it would have been quite possible to reverse their use and see \( f \) as a function of \( \omega \), \( \hat{f} \) as a function of \( x \). In other words, if we had defined
\[
\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega)e^{i\omega x} \, d\omega
\]
and
\[
f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(x)e^{-i\omega x} \, dx,
\]
nothing changes except the symbolism. This can sometimes be useful to know, because it means we can transform twice. Here is an exercise:

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\(^1\)Here, the subscript \( y \) is just a subscript, not a derivative!
Exercise 7 Show that (assuming everything exists) that

1. \( \hat{f}(x) = f(-x) \).

2. If \( f \) is even, then \( \hat{f} \) is even, and \( \mathcal{F}f = \mathcal{F}^{-1}f \).

Because the Fourier transform is linear, it changes sums of functions to sum of functions, products by a constant to products by a constant. Its action on products is also relatively easy to describe. But we need a definition first.

Suppose \( f(x), g(x) \) are defined for \(-\infty < x < \infty \). The convolution product, or simply the convolution of \( f, g \) is a new function defined by

\[
f \ast g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y) \, dy.
\]

This function is only defined if \( f, g \) are reasonably nice functions so that for every value of \( x \), the function of \( y \) given by \( f(x-y)g(y) \) can be integrated over finite intervals, and the improper integral in the definition converges. It is guaranteed to be defined for all values of \( x \) if \( f, g \) are piecewise continuous and

\[
\int_{-\infty}^{\infty} |f(x)| \, dx < \infty, \quad \int_{-\infty}^{\infty} |g(x)| \, dx < \infty.
\]

In this case we also have

\[
\int_{-\infty}^{\infty} |f \ast g(x)| \, dx < \infty.
\]

The main reason for considering this product here is:

Theorem 5 Assuming all functions involved exist,

\[
\mathcal{F}(f \ast g)(\omega) = \hat{f}(\omega)\hat{g}(\omega), \tag{21}
\]

\[
\mathcal{F}(fg)(\omega) = \hat{f} \ast \hat{g}(\omega). \tag{22}
\]

Usually, in defining the convolution product, one omits the factor of \( 1/\sqrt{2\pi} \). If one excludes it, then one has to include it in formulas such as (21) and (22), so it is possibly a good idea to have it and have a simple relation between the two products (the usual one and convolution).

Other properties of the convolution are:

1. For all \( x \),

\[
f \ast g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y) \, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)g(x-y) \, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-y)f(y) \, dy,
\]

briefly, \( f \ast g = g \ast f \). Convolution is commutative.

2. Convolution is associative,

\[
(f \ast g) \ast h = f \ast (h \ast g).
\]

3. Convolution behaves like a real product; if \( f, g, h \) are functions and \( c \) is a constant,

\[
f \ast (g+h) = f \ast g + f \ast h,
\]

\[
f \ast (ag) = (af) \ast g = a(f \ast g).
\]

All these properties are not hard to justify, but they require being able to manipulate double integrals with some confidence. W will just accept them, trusting the higher authority that revealed them to us.

I mentioned a condition under which the convolution product is defined. There are many others. You probably saw (if I were writing this ten years ago I would omit the word “probably”) a convolution product when studying the Laplace transform in Engineering Mathematics 1 or Differential Equations 1. It looked somewhat different, but the only real difference is the presence of the factor \( 1/\sqrt{2\pi} \) in front. Here is an exercise
Exercise 8 Suppose $f, g$ are piecewise continuous in $(-\infty, \infty)$ and $f(x) = g(x) = 0$ if $x < 0$. Show that

$$f \ast g(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{x} f(x-y)g(y) \, dy$$

for all $x > 0$; $f \ast g(x) = 0$ if $x < 0$.

This exercise shows that for functions that are 0 on the negative real axis the convolution is always defined as long as finite integrals are defined. You might also recognize it as the EM 1 or DE 1 convolution product, except for the factor of $1/\sqrt{2\pi}$.

4 Solving Partial Differential equations, using the (full) Fourier transform

The equations, rather problems, we will be solving will all be of one of the following types, where $a, b, c$ are given constants, and $q$ is a given function:

1. $u_t(x, t) = au_{xx}(x, t) + bu_x(x, t) + cu(x, t) + q(x, t), \quad -\infty < x < \infty, \quad t > 0,$

with initial condition

$$u(x, 0) = f(x), \quad -\infty < x < \infty.$$  

2. $u_{tt}(x, t) = au_{xx}(x, t) + bu_x(x, t) + cu(x, t) + q(x, t), \quad -\infty < x < \infty, \quad t > 0,$

with initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad -\infty < x < \infty.$$  

The idea behind both problems is the same. For a function $h = h(x, t)$ of the variables $x, t$ we define $\hat{h}(\omega, t)$ as the Fourier transform of $h$ with respect to the $x$ variable,

$$\hat{h}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} h(x, t) \, dx.$$  

We then transform Fourier both sides of the partial differential equation using the fact (mostly true) that a derivative with respect to $t$ goes in and out of an integral with respect to $x$, and using the formula in Theorem 3 to deal with partial derivatives with respect to $x$:

$$\hat{u}_t(\omega, t) = \hat{u}_t(\omega, t),$$

$$\hat{u}_x(\omega, t) = -i\omega \hat{u}(\omega, t),$$

$$\hat{u}_{xx}(\omega, t) = -\omega^2 \hat{u}(\omega, t).$$

By linearity and after a slight arranging, applying the Fourier transform to both sides of the equation in the first problem, and to the initial condition, yields

$$\hat{u}_t(\omega, t) + (a\omega^2 + bi\omega - c)\hat{u}(\omega, t) = \hat{q}(\omega, t),$$

$$\hat{u}(\omega, 0) = \hat{f}(\omega).$$

For each fixed value of $\omega$ this is a first order, linear ordinary differential equation in $t$, with an initial value given at $t = 0$. If $b$ is a non-zero real number, then $bi$ is imaginary, and the solution will be complex, but we should be by now mature enough to love and embrace complexity. Solving this equation gives us $\hat{u}(\omega, t)$. Applying the antitransform (with respect to $\omega$), if we can figure out what it is, provides us with the solution $u(x, t)$. There is little difference at the idea level with the second type problem. As before, by linearity and after a slight arranging, applying the Fourier transform to both sides of the equation in the first problem, and to the initial condition, yields

$$\hat{u}_{tt}(\omega, t) + (a\omega^2 + bi\omega - c)\hat{u}(\omega, t) = \hat{q}(\omega, t),$$

$$\hat{u}(\omega, 0) = \hat{f}(\omega).$$
\[ \hat{u}(\omega, 0) = \hat{f}(\omega), \quad \hat{u}_t(\omega, t) = \hat{g}(\omega). \]

For each fixed value of \( \omega \) this is now a second order, linear ordinary differential equation in \( t \), with initial values given at \( t = 0 \).

Here are some examples

1. Solve
   \[ u_t(x, t) = ku_{xx}(x, t), \quad -\infty < x < \infty, \quad t > 0, \]
   subject to
   \[ u(x, 0) = f(x). \]

   Having done this in class and it being done in the text, I’ll be brief. Transforming Fourier yields
   \[ \hat{u}_t + k\hat{u} = 0, \quad \hat{u}(\omega, 0) = \hat{f}(\omega); \]
   the solution of this ODE problem is
   \[ \hat{u}(\omega, t) = e^{-k\omega^2 t} \hat{f}(\omega). \]

   Using tables or otherwise we discover that
   \[ e^{-k\omega^2 t} = \hat{G}(\omega, t), \]
   where
   \[ G(x, t) = \frac{1}{\sqrt{2\pi kt}} e^{-\frac{x^2}{4kt}}, \]
   so that by (21),
   \[ u(x, t) = \hat{G}(\omega, t) \ast f(x) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y) \, dy. \]

   Whether this last integral can be evaluated in close form or not, depends on \( f \). Mostly, it can’t be so evaluated.

   One example in the textbook has a function of the form
   \[ f(x) = \begin{cases} c, & |x| < a, \\ 0, & |x| > a, \end{cases} \]
   where \( a, c \) are positive constants. In the book example, \( a = 1 \) and \( c = -3 \). Using formula (23) we get
   \[ u(x, t) = \frac{c}{\sqrt{4\pi kt}} \int_{-a}^{a} e^{-\frac{(x-y)^2}{4kt}} \, dy. \]

   This integral can be evaluated in terms of the so called error function defined as
   \[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-y^2} \, dy. \]

   If we make the substitution \( z = \frac{x-y}{\sqrt{4kt}} \) in the integral in (24), since \( y \) is the variable of integration and \( x \) is constant as far as integration is concerned, then \( dy = -\frac{1}{\sqrt{4kt}} \, dz \), the limits change to \( (x \pm a)/\sqrt{4kt} \), so that
   \[ u(x, t) = \frac{c}{\sqrt{\pi}} \int_{\frac{x-a}{\sqrt{4kt}}}^{\frac{x+a}{\sqrt{4kt}}} e^{-z^2} \, dz = \frac{c}{2} \left( \text{erf} \left( \frac{x+a}{\sqrt{4kt}} \right) - \text{erf} \left( \frac{x-a}{\sqrt{4kt}} \right) \right). \]

2. Solve
   \[ u_{tt} = c^2 u_{xx}(x, t), \quad -\infty < x < \infty, \quad t > 0, \]
   subject to
   \[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \]
Transforming Fourier this becomes the problem
\[ \hat{u}_{tt} + c^2 \omega^2 \hat{u} = 0, \quad \hat{u}(\omega, 0) = \hat{f}(\omega), \quad \hat{u}_t(\omega, 0) = \hat{g}(\omega). \]

The solution is
\[ \hat{u}(\omega, t) = \cos(c\omega t) \hat{f}(\omega) + \frac{\sin(c\omega t)}{c\omega} \hat{g}(\omega). \]

Let’s consider both terms of the solution separately. We have
\[ \cos(c\omega t) \hat{f}(\omega) = \frac{1}{2} (e^{ic\omega t} + e^{-ic\omega t}) \hat{f}(\omega). \]

By (17), or (18), we get
\[ \frac{1}{2} e^{ic\omega t} \hat{f}(\omega) = \mathcal{F}\{f(x + ct)\}, \]
\[ \frac{1}{2} e^{-ic\omega t} \hat{f}(\omega) = \mathcal{F}\{f(x - ct)\} \]
so that \( \cos(c\omega t) \hat{f}(\omega) \) is the Fourier transform of \( \frac{1}{2} (f(x + ct) + f(x - ct)) \). Turning to the second term, \( \sin(c\omega t)/\omega \) is just missing the factor \( \sqrt{2/\pi} \) to be the Fourier transform of
\[ h(x) = \begin{cases} 1, & |x| \leq ct, \\ 0, & |x| > ct, \end{cases} \]
so that
\[ \frac{\sin(c\omega t)}{c\omega} \hat{g}(\omega) = \frac{1}{c} \sqrt{\frac{\pi}{2}} h(\omega) \hat{g}(\omega) = \frac{1}{c} \sqrt{\frac{\pi}{2}} (h \ast g)(\omega); \]
now
\[ \frac{1}{c} \sqrt{\frac{\pi}{2}} h \ast g(x) = \frac{1}{c} \sqrt{\frac{\pi}{2}} \frac{1}{2\pi} \int_{-ct}^{ct} g(x - y) dy = \frac{1}{2c} \int_{-ct}^{ct} g(x - y) dy = \frac{1}{2c} \int_{x - ct}^{x + ct} g(s) ds. \]

Putting it all together we have obtained D’Alembert’s formula for the solution of the infinite string problem:
\[ u(x, t) = \frac{1}{2} (f(x + ct) + f(x - ct)) + \frac{1}{2c} \int_{x - ct}^{x + ct} g(s) ds. \] (25)