The Story of $\pi$

And of its friend $e$
Wheels

Colin Adams in the video “The Great $\pi/e$ Debate” mentions the wheel “arguably the greatest invention of all times,” as an example of how $\pi$ appeared already in prehistoric times. But square wheels are possible. If the surface of the earth were not flat but rilled in a very special way.
For this to work each little rill (arc) has to have the shape of a hyperbolic cosine, a so called cosh curve. By definition, the hyperbolic cosine is

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Another appearance by e. As we shall see, π and e are closely related, even though the relationship is far from being well understood.
In these notes:

- $r =$ radius of a circle
- $d =$ its diameter, $d = 2r$
- $C =$ its circumference (perimeter)
- $A =$ its area

When did people discover that

$$\frac{C}{d} = \frac{A}{r^2}$$
The Story Begins…

There are records (skeletons, skulls and other such cheerful remains) that indicate that the human species existed as long as 300,000 years ago. Most of this was prehistory. The following graph compares history to prehistory:

The green part is prehistory, the red history. The vertical black line indicates the time to which some of the oldest records of human activity were dated.
Ahmes, the scribe

Scribes were a very important class in ancient Egypt.

The picture shows a statue of a scribe. Ahmes, the scribe of the Rhind papyrus (c. 1650 BC) may have looked much like this guy.

In the Rhind papyrus, Ahmes writes: Cut off 1/9 of a diameter and construct a square upon the remainder; this has the same area of the circle.
As was to be discovered in Assignment 1. This leads to the equation

\[ A = \left( \frac{8}{9} D \right)^2 \]

Since

\[ A = \pi r^2 = \pi \left( \frac{D}{2} \right)^2 \]

equating we get a value of \( \pi = \frac{256}{81} = 3.16049 \ldots \)
The pyramid of Khufu (aka Cheops) in Giza is one of the seven wonders of the ancient world. Many stories circulate about it, and the weirdest conclusions have been reached about its dimensions, orientation, whatever. One of them is that the ratio of one of the sides of the base to the height is exactly $\pi/2$. ``Exactly” in this context is, of course, nonsense.

In *The Joy of Pi*, David Blattner writes: *The Great Pyramid at Giza has a fascinating relationship inherent in its structure: The ratio of the length of one side to the height is approximately $\pi/2$. Egyptologists and followers of mysticism have supposed for centuries what this means and why it is, for the approximation is significantly better than the value for $\pi$ that the Egyptians are understood to have known. However, Herodotus writes that the pyramid was built so that the area of each lateral face would equal the area of a square that has one side as long as the pyramid was tall. It can be shown that any pyramid with this framework will automatically approximate $\pi$. 
Problem 1. Consider a pyramid of a square base. Suppose that the area of each lateral side equals the area of a square of side equal to the height. Let $h$ be the height and let $b$ be a side of the base. Compute $b/h$ and show that twice that value is indeed a better approximation to $\pi$ than the one given by Ahmes.

Incidentally, a lot of nonsense has been written about the pyramids, as well as about other ancient monuments. How could the Egyptians have built them? Some people even suggest they were the work of aliens. And if you take any structure, the FAU cafeteria for example, and take enough measurements, you are bound to find some strange connections.
The *Joy of Pi* is a very nice book, but it also may not be totally reliable. I happen to own a copy of Herodotus’ Histories. Herodotus of Halicarnassus (ca. 484-430 BCE) is known as the father of history. He travelled extensively, especially in Egypt. Here is what appears in my version of his book: *It is square at the base, its height equal to the length of each side.* But we’ll ignore this; we prefer the version that gives us $\pi$. 
The Babylonians

This is the name given to a culture that flourished for many centuries in the fertile area between the rivers Tigris and Euphrates, corresponding more or less to modern day Iraq. It is named for their principal city, Babylon. Two important civilizations precede the Babylonians in that area, the Sumerians and the Akkadians.

Officially, the Babylonian civilization seems to begin about 1800 BCE, with Hammurabi, who conquered most of the region and is famous for having written the first known legal code. It ends about 500 BCE, when it became part of the Persian empire.
Pi Tablets

The first picture shows a Babylonian tablet in the Yale collection. The second picture is the same tablet, with the numbers clarified. In the third picture, Arabic number equivalents are written beside the cuneiform ones. One interpretation is that 45 is the area of the circle and 3 the circumference. If so
It is not too hard to see that

$$\pi = \frac{C^2}{4A}$$

Problem 2: Establish this formula.
It is not too hard to see that \( \pi = \frac{C^2}{4A} \)

From this formula one would get \( \pi = \frac{9}{180} = 0.05 \), a really bad value. But in history, if things don’t work out as one wants, one can always re-interpret. Since our Babylonian friends had no zero, 45 could well be \( 0;45 \); that is 45/60. Using this value one gets

\[
\pi = \frac{9}{4 \left( \frac{45}{60} \right)} = \frac{540}{180} = 3
\]
The Babylonian value for $\pi$ seems to have been $3!$ Same as in the Bible: Also he made a molten see of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about. (1 Kings 7:23)
Blattner quotes the great medieval Jewish scholar Maimonides (1135-1204) as explaining the poor biblical approximation by: *The ratio of the diameter of a circle to its circumference cannot be known...but it is possible to approximate it...and the approximation used by scientists is the ratio of one to three and one seventh...since it is impossible to arrive at a perfectly accurate ratio...they assumed a round number and said, `Any [circle] which has a circumference of three fists, has a diameter of one fist.’ And they relied on this for all the measurements they needed.*

The picture shows a statue of Maimonides in (what once was) the Jewish quarter of his city of birth, Córdoba in Spain. However, he worked mostly in Egypt and Morocco, during the height of the Arab empire. His most famous book is, I think, *The Guide for the Perplexed.*
BUT

As we also learned in Assignment 1, there is another Babylonian tablet (from about 2000 BC, found in 1936 near the city of Susa) that states that the ratio of the perimeter of a regular hexagon to the circumference of the circumscribed circle is the number interpreted as

0; 57, 36

That would give a value of $\pi = 3.125$, a little bit better than the value given by Ahmes. In all events, for unpaved roads a value of $\pi = 3$ was probably good enough. The ancient inhabitants of the Mideast were not building watch mechanisms or airplane parts.
Somewhat after 500 BCE the city of Athens overthrew a tyrant and instituted a new form of government they called democracy.

But the real story begins in the Ionian (Athenian) colonies in Asia Minor, where Turkey is nowadays, with Thales at al. The Greeks seemed to understand that \( \pi \) could not be computed exactly and proceeded to estimate it. Around 440 BCE, Antiphon of Heraclea tried to estimate the area of a circle by calculating areas of inscribed polygons of more and more sides. Soon after Bryson, also of Heraclea, used inscribed and circumscribed polygons to get a better estimate. This primitive form of integration is known as the principle of exhaustion.
When Greek meets Greek, then comes the tug of war.

The Greeks invented (sort of) philosophy, mathematics, physics, theater. And were constantly at war with each other. Finally, Phillip II of Macedonia (382-336 BC) conquered all of Greece and imposed his power. After he was assassinated, his son Alexander (356-323 BCE) went out to conquer the world. He succeeded in creating the largest empire the west had yet seen and then died at age 32. The classical Greek period was over, the Hellenistic one just started. And the city of Alexandria in the north of Egypt became the intellectual center of the world.
Hippocrates’ Lunes

But still during classical times, Hippocrates of Chios (ca 440 BCE, not to be confused with the more famous Hippocrates of Cos, the Hippocratic oath guy) was among the first to write a treatise on geometry (lost) and to compute areas of circular figures exactly. These figures are known as his lunes, from the French word for moon.

Here is one of them. The lune is the shaded area, outside of circle of center C and inside the circle of diameter AB. Problem 3: Show that the area of the lune equals the area of the triangle ABC.
Archimedes

The Hellenistic period centered at Alexandria produced many brilliant scientists, but none as brilliant as Archimedes (287-212 BCE). His achievements still amaze us. He was perhaps the first human being to grasp the mathematical concept of infinity, and he applied it in his work.
There are many versions of this; I prefer the one in Howard Eves’ book, *An Introduction to the History of Mathematics*. What Archimedes found was a simple, and actually very beautiful relation between the perimeters of inscribed and circumscribed regular polygons of $n$ sides and their counterparts of $2n$ sides.
The Archimedean Method

I’ll introduce some notation because we are not as smart as Archimedes, so we need notation. I’ll write $p_n$ for the perimeter of a regular $n$-sided polygon inscribed in a circle of radius 1. I’ll write $Q_n$ for the perimeter of the corresponding circumscribed one. For example, the picture shows an inscribed and circumscribed hexagon.

One sees: $p_6 = 6$ (each side = radius = 1)

$Q_6 = 4\sqrt{3}$ (with a bit of effort)
The perimeter of the circle is $2\pi$ and it must be between $p_n$ and $Q_n$. That is

$$p_n < 2\pi < Q_n$$

For example, $6 = p_6 < 2\pi < Q_6 = 4\sqrt{3} \approx 6.928$; dividing by 2:

$$3 < \pi < 3.464$$

Not so wonderful, so far. So let’s double the number of sides, see what we get with $n = 12$.

One can hardly see the circle. With some effort one computes:

$$p_{12} = 12(2 - \sqrt{3})^{1/2} \approx 6.211657$$

$$Q_{12} = 24(2 - \sqrt{3}) \approx 6.43078$$

Dividing by 2, $3.1058 < \pi < 3.2154$
The Archimedean Algorithm

What Archimedes discovered (according to Eves and others) is that if you double the number of sides of the polygon, the perimeter of the new circumscribed polygon is the harmonic mean of the original perimeters, the perimeter of the new inscribed polygon is the geometric mean of the circumscribed polygon of twice the number of sides and the perimeter of the old inscribed polygon.

Let’s try to be more precise.
Means

The average or *arithmetic mean* of two numbers $a$, $b$ is the usual mean or average: $M_a = (a+b)/2$. The *geometric mean* of two positive numbers $a$, $b$ is the square root of their product: $M_g = \sqrt{ab}$. The *harmonic mean* is the reciprocal of the arithmetic mean of the reciprocals: $1/M_h = (1/a + 1/b)/2$, or $M_h = 2ab/(a+b)$. 
The Formulas

The formulas in question are:

\[ Q_{2n} = \frac{2 p_n Q_n}{p_n + Q_n} \]
\[ p_{2n} = \sqrt{p_n Q_{2n}} \]

This allows for a very simple algorithm to compute \( \pi \) to any precision. For example suppose we want to know \( \pi \) up to 6 exact decimal digits. The following procedure works.
Let us see how this actually works.

Begin with $p = 6$ (perimeter of the inscribed hexagon) and $Q = 4\sqrt{3}$ (perimeter of the circumscribed hexagon). (You can also start with a square instead of a hexagon but Archimedes chose the hexagon). Then

1. Replace $Q$ by $2pQ/(p+Q)$, or as one writes in computer programs $Q = \frac{2pQ}{p+Q}$
2. Replace $p$ by $\sqrt{pQ}$; that is $p = \sqrt{pQ}$
3. Do $p/2$ and $Q/2$ have the same digits up to six places past the decimal point?

   (a) If YES, done; all the digits of the common initial part of $p/2$, $Q/2$ are exact digits of $\pi$.
   (b) If NO, go back to step 1.
The first column indicates the number of sides, the last the number of equal digits past the decimal point of \(Q/2, p/2\).

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<th>(Q)</th>
<th>(p)</th>
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Because \(p_n/2 < \pi < Q_n/2\) for all \(n\), we can be absolutely sure that the first six digits past the decimal point of \(\pi\) are 141592. But if the seventh digit is 5 or larger, we should really round off the last 2 to a 3. Computing the last values of \(Q/2, p/2\) to one further digit we see that the seventh digit must be between 5 and 9. Thus the correct rounded off version is \(\pi = 3.141593\)
This became the “classical method”

Problem 4 (Optional): Establish the formulas

\[ Q_{2n} = \frac{2p_n Q_n}{p_n + Q_n} \]

\[ p_{2n} = \sqrt{p_n Q_{2n}} \]

For several centuries, at least in the west, this method of Archimedes became the method for computing \( \pi \). It was slow and in those days preceding electronic calculation devices, each successive step took a lot of effort and time. Archimedes got as far as \( n = 96 \) and came up with the value 3.14 for \( \pi \). More precisely, he was a master at doing rational approximations, so he found

\[ \frac{223}{71} < \pi < \frac{22}{7}. \]

The important thing however was that for the first time an algorithm was developed allowing one to compute \( \pi \) to any desired precision.
And the Greeks Leave

Alexandria still played an important role for some centuries after Archimedes. But I don’t think much more happens involving $\pi$ after the year 150 of our era. That was about the time that Claudius Ptolemy’s *Syntaxis Mathematica* appeared, better known now by its Arab name *Almagest*. In it $\pi$ is given in sexagesimal notation as $3 \ 8' \ 30''$. This works out to

$$3 + \frac{8}{60} + \frac{30}{3600} = \frac{377}{120} \approx 3.1417$$
China and India

The achievements of Chinese scientists and mathematicians during the period that in Europe coincides with the middle ages, are nothing short of astounding. Around 480 a Chinese mathematician and engineer, Tsu Ch’ung-chi (the name is also transcribed as Zu Chongzhi) calculated $\pi = \frac{355}{113} = 3.1415929$.

This fact appears in many sources, as well as the fact that he was also using Archimedes’ method, having probably discovered it independently on his own. Blattner in the Joy of $\pi$ states that he used polygons with as much as 24,576 sides; starting with 6 and doubling twelve times. That would be two more steps than in our calculations a few slides back.
This value of $\pi$, as well as the fact that $3.1415926 < \pi < 3.1415927$, appears in a book Tsu Ch’ung-chi wrote with his son; the book is unfortunately lost. It took Europe some 900 years to come up with the same approximation. He also did some impressive astronomical calculations, and discovered what in the west became known as Cavalieri’s principle, once Bonaventura Cavalieri discovered it some 1000 years later.
Further Results from the Far East

- ca. 530 The Indian mathematician Aryabhata gives
  \[ \pi \approx \frac{62,832}{20,000} = 3.1416. \]
- ca. 1150 Bhaskara gave several approximations, none better than 3.1416. He liked \( \sqrt{10} = 3.162277... \) For some reason, several Chinese and Indian mathematicians thought that the square root of 10 was a good approximation.
- Al-Kashi, astronomer royal to Ulugh-Beg, the grandson of the conqueror Tamerlane, computes \( \pi \) to sixteen decimal places by the classical method.
Back to Europe. Viète (1540-1603).

The great French mathematician and cryptographer François Viète, used the classical method, doubling the number of sides 16 times so that he got to \( n = 393,216 \). With this he got \( \pi \) correct up to sixteen decimal places. But more importantly, in 1579, he got the very interesting infinite product for \( \frac{2}{\pi} \):

\[
\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2+\sqrt{2}}}{2} \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \ldots
\]
Ludolph

- And several other people did more or less precise calculations of $\pi$ as the 1500’s slid into the 1600’s. And then came Ludolph van Ceulen (1540-1610).

- He lived during some very turbulent times, but most times are quite turbulent.
Ludolph

He was born in Germany, but lived most of his life in the Netherlands. The Netherlands had been under Spanish control, but were now in rebellion. The Dutch had become protestant, Spain was fiercely catholic and Spain established a reign of terror in the part it still controlled, and tried to subdue the rest. As a good Calvinist, Ludolph had to flee from one part of the Netherlands to another. He sustained himself as a mathematics teacher and as a fencing instructor. In the late 1580’s he learned of Archimedes’ method for $\pi$; not reading Greek (he came from a family of modest means and was not that well educated), he had a friend translate Archimedes for him. And then Ludolph found his calling: Calculate $\pi$ accurately to a higher precision than anybody had done before him!

It took him 23 years. And he did it! He calculated $\pi$ up to 35 correct decimal places.
Ludolph

He got

\[3.14159265358979323846264338327950288\]

as a lower bound and

\[3.14159265358979323846264338327950289\]

As an upper bound.

And then he died.

So far \(\pi\) had no name. It was called, for example, the ratio of the circumference to its diameter. But now it became to be called Ludolph’s number or the Ludolphine number, specially in Germany. Ludolph’s results were published posthumously, by his widow.
Willebrord Snell (1580-1626) was a Dutch mathematician, physicist (in those days there was no major difference between math and physics), astronomer who is most famous for Snell’s law. When a wave, like light, goes from one medium to another it bends due to a change in velocity. Each medium has an index of refraction traditionally denoted by $n$ which is proportional to the velocity of the wave in the medium. Snell’s law states (see picture on the right for meaning of the terms): $n_1 \sin \theta_1 = n_2 \sin \theta_2$.

This has little or nothing to do with $\pi$. Having studied with Ludolph, Snell also set out to compute $\pi$, improving the classical method so that from each pair of bounds for $\pi$ coming out of this method, he got a better estimate. Ludolph needed to get to polygons of $2^{62}$ sides to get 35 decimals, Snell only needed $2^{30}$ sides.
Calculus has been invented! And the world would never be the same.
The Invention of Calculus

The invention of calculus is usually credited to two great minds of the 17th/18th century: Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1647-1716). Newton was the better mathematician, the greater scientist. Leibniz was more a man of universal interests.

The dispute between followers of one and the other grew quite bitter. As a consequence of this hatred, England decided to shun continental mathematics and this had the effect of stunting England’s mathematical growth. Eventually the silliness of it all became apparent and the official consensus today is that both are to be given equal credit.
Wallis

Many beautiful formulas involving $\pi$ have been developed since Calculus was invented (and some, like Viète’s, even earlier). A famous one is, like Viète’s, an infinite product, bit of a much simpler nature. It is due to the English mathematician John Wallis (1616-1703):

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$
Things are changing

In 1671 the Scottish mathematician James Gregory (1638-1675) came up with a series for the inverse tangent function, the arctan function. If \( x \) is any real number, then \( y = \arctan x \) is a number in the interval from \(-\pi/2\) to \(\pi/2\) such that the tangent of \( y \) (or, if you prefer, the tangent of an angle measuring \( y \) radians) is \( x \). It seems that by 1668, Gregory was familiar with the series expansions of \( \sin x \), \( \cos x \) and \( \tan x \), as well as with some important integrals, as for example

\[
\int \sec x \, dx = \ln(\sec x + \tan x)
\]

The series for \( \arctan \) is

\[
\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\]

Valid for \(-1 \leq x \leq 1\).
More on Gregory

- Gregory anticipated Newton in discovering both the interpolation formula and the general binomial theorem as early as 1670; he discovered Taylor expansions more than 40 years before Taylor; he solved Kepler's famous problem of how to divide a semicircle by a straight line through a given point of the diameter in a given ratio (his method was to apply Taylor series to the general cycloid); he gives one of the earliest examples of a comparison test for convergence, essentially giving Cauchy's ratio test, together with an understanding of the remainder; he gave a definition of the integral which is essentially as general as that given by Riemann; his understanding of all solutions to a differential equation, including singular solutions, is impressive; he appears to be the first to attempt to prove that $\pi$ and $e$ are not the solution of algebraic equations; he knew how to express the sum of the $n$th powers of the roots of an algebraic equation in terms of the coefficients; and a remark in his last letter to Collins suggests that he had begun to realise that algebraic equations of degree greater than four could not be solved by radicals. (From MacTutor)

- He died at age 36.
The Gregory Series

The series $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ is known as Gregory's series.

Apparently Gregory never noticed that setting $x = 1$ one got a series that would allow a calculation of $\pi$. The tangent of an angle of 45 degrees; that is, an angle of $\pi/4$ radians, is 1; thus the arctan of 1 is $\pi/4$. One gets the very pretty (some say beautiful) expansion

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$
The series \[ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \]

is seen as beautiful because it shows this strange relation. On one side the ratio of a circumference to its diameter (divided by 4), a purely geometric entity. On the other side, the alternating sum of the reciprocals of the odd integers. Go figure! Where do these relations come from?, what does it all mean? (No questions will be answered here).

As a method for computing \( \pi \) it is not very good. The error one makes when one cuts the series of at the \( n \)-th term is approximately \( 1/n \). So if we want 6 exact decimals, an error of about \( 5 \cdot 10^{-7} \), one would have to add two million terms. Not to mention if we want to get to where Ludolph got. One says: The series converges very slowly.

Incidentally, this case of Gregory’s series may have been known much earlier to Chinese mathematicians. The whole series, as well as a whole bunch of other trigonometric series, was apparently also known to the Indian mathematician Sangamagrama Madhava (ca. 1350-1425).
But It’s Faster at the Center.

The speed at which the Gregory series

\[ \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \]

converges depends on how far away \( x \) is from 0 (zero). If \( x > 1 \) or \( x < -1 \), it diverges, so \( x = \pm 1 \) are the extremes of convergence. If \( x = 0 \), convergence is immediate, we get the highly unenlightening result that \( \arctan 0 = 0 \).

But if one can get a small value of \( x \) which has the property that one can substitute it into the series and calculations are not horribly difficult, and it is the tangent of an angle related to \( \pi \), one might get better results.
Gregory Series Calculations

We recall (from wherever we learned it) that the tangent of 30 degrees is $1/\sqrt{3}$. In radians, 30 degrees corresponds to $\pi/6$ and $1/\sqrt{3}$ is pretty small. In 1699 Abraham Sharp (1653-1742), who had worked as an assistant to the first Astronomer Royal, used $x = 1/\sqrt{3}$ in Gregory’s series to compute $\pi$ up to 72 correct decimal places.

In 1706 another English mathematician and astronomer, John Machin (1680-1751) had a better idea. He discovered the identity

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{238}$$

henceforth known as Machin’s formula. Gregory’s series converges quite fast if $x=1/5$, very fast for $x=1/238$. Machin obtained 100 decimal places for $\pi$ using this formula.
And the Calculations Go On

- Here is a brief chronology of computations (from H. Eves’ book)
- 1719, De Lagny, French, uses the Gregory series with $x = 1/\sqrt{3}$ to get 112 correct places
- 1841, William Rutherford, English, used Gregory’s series and the relation

\[
\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{70} + \arctan \frac{1}{99}
\]

- to find $\pi$ to 208 places, but only 152 turned out to be correct.
• 1844, Zacharias Dase, German, used Gregory’s series and the relation

\[
\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{5} + \arctan \frac{1}{8}
\]

to compute \( \pi \) correct to 200 decimal places. Dase, it seems, was a lightning calculator, perhaps the best ever.

• 1853, Rutherford returns to the problem, finds 400 places

• 1873, William Shanks, English, uses Machin’s formula to compute \( \pi \) to 707 places. For a long time this was ``state of the art.” But in 1946 and Englishman of name D. F. Ferguson found that there was an error in Shanks’ 528-th place.

• 1947, Ferguson gives a correct computation up to 710 places. In the same month, an American, J.W. Wrench, Jr. went all the way to 808 places, but Ferguson (again!) found an error in the 723 place.

• 1948, Ferguson and Wrench jointly publish the corrected value of \( \pi \) up to 808 places.
In the Days of the Digital Computer

- The ENIAC (Electronic Numerical Integrator and Computer) appeared in 1946, built at the University of Pennsylvania for the Army Ballistics Research Laboratory. It played a big role in the development of the H-bomb. And now one could really start calculating $\pi$ into the stratosphere.
All these calculations get boring. As the 20th century progressed (if that’s the word for a century that saw two world wars with millions of dead in each), faster and faster computers were built and (more importantly, perhaps) better and better algorithms were developed. One of the best early ones is due to the great Indian mathematician Srinivasa Ramanujan (1887-1920)

And so π was computed to 2037 places (1949), to 16,167 places (1959), to 100,265 places (1961), to 500,000 places (1967), to 1,000,000 places (1973), to 2,000,083 places (1981), to 29,360,000 places (1986).
After Which It Got Serious

- The late 20th century computations of π have been dominated by three names:
- The Borwein brothers, Peter and Jonathan. Born in Scotland, they are not just calculators but also very fine mathematicians who developed some of the new fast algorithms. Peter is a professor at Simon Fraser University in British Columbia (Canada), Jonathan used to also be in Canada, but now is a professor at the University of Newcastle in Australia.
- The Chudnovsky brothers, David and Gregory, born in Kiev (Ukrania); also very fine mathematicians.
- Yasumasa Kanada, a professor of mathematics at the University of Tokyo.
To the billions and trillions!

- The Borweins, the Chudnovskys, and Kanada have web pages, and you can read about all of their exploits. For the past 20 years or so, one of them came out with a value, then another one surpassed it, and so forth. And other people are entering into the game, frequently using algorithms developed by either one of the Borweins, one of the Chudnovskys, or by Kanada. Sources on the web abound. The current record seems to be into the trillion of digits.

Problem 5. How many pages would it take to write out a number one trillion of digits long? Use realistic estimates for digit size and page size.

All these calculations are very nice, but there is more to \( \pi \) than just digits.
How It Got Its Name

Let us recall that until Ludolph’s day $\pi$ had no name, no specific symbol. In a not very wide read publication of 1706, William Jones (English) used the symbol $\pi$ to denote the ratio of the circumference of circle to its diameter. This may have been completely forgotten, except that Euler adopted the symbol in 1737. And everybody read Euler!
The mathematics of π

It is time to bring in the number e. As Professor Colin Adams points out, that number was totally unknown in classical times. To look at the origin of e we have to go back a few years and discuss logarithms.
A logarithm is the inverse of a power. The idea behind logarithms as a computational device is, as many brilliant ideas, quite simple. Let’s select a base $b \neq 1$. If we want to multiply two numbers $x$ and $y$ we find the numbers $n$ and $m$ so that $x = b^n$, $y = b^m$, add $n+m$ and then find $z = b^{n+m}$. The number $z$ is the product $xy$. This is useful only if we don’t need to compute $n$, $m$ or $z$, at least not each time. The number $n$ is, of course, the logarithm of $x$ in base $b$.

What needs to be done is to set up a table of powers of $b$, and to find $m$, $n$ one reads it backwards. The only problem is that powers of $b$ increase very fast. For example if we take $b = 2$ and we want to multiply 32 times 64, we can read the table of powers of 2 backwards and see that 5 is the logarithm of 32 in base 2, while 6 is that of 64. Now $5+6 = 11$; looking again at the table of powers of 2, we see that 2 to the 11 is 2048, and we found 32 times 64 = 2048.
Logarithms

But what if we want to multiply 7 by 9, for example, we need more than just integer powers of 2. We would like to have a nice table where we can look up any number (within reason) and see what power of 2 (or whatever the base is) it is. John Napier (1550-1617), a Scottish laird (lord) who is considered the inventor of logarithms, decided to use $b = 1 - 10^{-7} = 0.9999999$ as a base. With a number so close to 1, the powers would be almost evenly spaced, at least for a while. But then, to avoid decimals, he multiplied each power by $10^7$. 
Napier’s Original Logarithms

So originally, Napier’s logarithm of a number $x$ was a number $L$ such that

$$x = 10^7 \left(1 - \frac{1}{10^7}\right)^L$$

If we introduce $\lambda = 10^7 L$, then $\lambda$ is the logarithm in base $10^{-7}$ of $x/10^7$. And $\left(1 - 10^{-7}\right)^{10^7}$ is very close to $1/e$.

In 1614 Napier published *Mirifici logarithmorum canonis descriptio* (A Description of the Wonderful Law of Logarithms). In 1615 he and Henry Briggs (1561-1631) decided that it would be good if log of 1 were 0, and of 10 were 1, creating the *Briggsian* or common logarithms.
Where is e?

The number e still had to be found. We now think, or should think, of logarithms as a function. This was not so in the 1600’s. A logarithm was a computational device, the base was not too important. A number of close calls occurred and then in 1683 Jakob Bernoulli (1654-1705) in a study of compound interest proved that

\[ \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \]

was a number between 2 and 3. He did not see any connection with logarithms.
According to MacTutor, the first appearance of $e$ as a specific number is in a letter Leibniz wrote to Christian Huygens in 1690. He used the letter $b$ to denote it. I don’t know what he said about it. A few decades later, when Euler threw his giant shadow upon the field of mathematics and of physics, the concept of function was sort of known, and it was known that the function $y = e^x$ equals its own derivative.
Leonhard Euler, born 1707 in the Swiss city of Basel, died 1783 in the Russian city of St. Petersburg, was one of the greatest mathematicians of all times. There are dividing lines in the history of mathematics, one of these is the creation of calculus. The work of Euler is another. There is hardly an area of mathematics or physics that he did not influence and advance. We already mentioned that it was thanks to him that we know the ratio of the circumference to its diameter by $\pi$. 
Euler was not afraid of complex numbers, so he used the (by then) well known power series for the exponential, namely

\[ e^x = 1 + x + \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2 \cdot 1} + \frac{x^4}{4 \cdot 3 \cdot 2 \cdot 1} + \cdots \]

with an imaginary value of \( x \); that is, with \( x \) replaced by \( i \cdot x \), where \( i = \sqrt{-1} \). This gave him the formula

\[ e^{ix} = \cos x + i \sin x \]

Setting \( x = \pi \) in that formula he got
The five most important numbers in mathematics, 0, 1, i, e, and π, appear in one simple formula. Euler was stunned. The formula suggests an intimate relationship between e and π, but I doubt anybody understands it completely.
How Irrational it all is

Proving that $e$ is irrational is actually quite easy, it was done by Euler. Proving $\pi$ irrational was a bit more difficult. Johann Lambert (1728-1777) was the first to accomplish it. Lambert was born in Alsace, a region between France and Germany that has changed owners a number of times; it is French since at least WWI. Lambert proved that if $x$ is rational but $x \neq 0$, neither $e^x$ nor $\tan x$ can be rational. Since $\tan (\pi/4) = 1$ rational, $\pi/4$ cannot be rational, nor can $\pi = 4(\pi/4)$ be rational (1768).
The Circle Squarers

- Attempts to square the circle, specifically: To construct a square having the same area as a given circle using only a compass and a straightedge, go all the way back to antiquity.
The great problem which has baffled the greatest philosophers and the brightest minds of ancient and modern times has now been solved by a humble American citizen of the city of Brooklyn.

This is from a book appearing early 1900’s or late 1800’s (According to Dan Cohen)
The first of these two misguided visionaries filled me with a great ambition to do a feat I have never heard of as accomplished by man, namely to convince a circle squarer of his error! The value my friend selected for Pi was 3.2: the enormous error tempted me with the idea that it could be easily demonstrated to BE an error. More than a score of letters were interchanged before I became sadly convinced that I had no chance.

Charles Dodgson (Lewis Carroll, 1855)
One of the unnoticed good effects of television is that people now watch it instead of producing pamphlets squaring the circle.
Underwood Dudley, Contemporary Mathematician, Textbook Author.

By the mid 1700’s there had already been so many erroneous attempts to square the circle, that in 1755 the French Academy of Sciences decided that it would not examine anymore “solutions” of the problem.

But would the circle be squared one day?
The Circle Cannot be Squared!

In 1873 the French mathematician Charles Hermite (1822-1901) proved that $e$ was transcendental. He could have proved a bit more. Extending the techniques used by Hermite, the German mathematician Ferdinand von Lindemann (1852-1939) proved in 1882 that if $z$ was a complex algebraic number other than 0, then $e^z$ was transcendental.

If we take $z = 1$, clearly an algebraic number, we get Hermite’s result; that $e = e^1$ is transcendental. But if $\pi$ were algebraic, then $i\pi$, where $i$ is the square root of $-1$, would be also algebraic; being non-zero, $e^{i\pi}$ would have to be transcendental. But, as we know, $e^{i\pi} = -1$, algebraic. Thus $\pi$ must be transcendental.

The circle cannot be squared.
A real or complex number is *algebraic* if (and only if) it is the root of an algebraic equation with integer coefficients; that is, of an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

Where the coefficients $a_0, a_1, \ldots, a_n$ are integers $n \geq 1, a_n \neq 0$. 

**Algebraic and Transcendental Numbers**
Examples of Algebraic Numbers

- All rational numbers are algebraic. If $x = \frac{m}{n}$ is rational, $m, n$ integers; it solves $nx + (-m) = 0$.

- Square roots, cubic roots, etc. of rational numbers are algebraic. For example, $x = \sqrt{2}$ is algebraic; it solves $x^2 - 2 = 0$.

- The imaginary unit $i$, the square root of $-1$, is algebraic, it satisfies $x^2 + 1 = 0$.

- Sums, products, differences, quotients, and roots of algebraic numbers are algebraic.
Transcendental Numbers

- By definition, a real or complex number is *transcendental* if (and only if) it is not algebraic.
- In 1851 the French mathematician Joseph Liouville (1809-1882) constructed the first transcendental numbers, proving they do exist!
- In 1874, the German mathematician Georg Cantor (1845-1918) proved in a quite non-constructive way, that most real numbers are transcendental: If you select a real number at random, the probability of it being algebraic is zero. Liouville’s numbers were still the only concrete examples.
Relation With Constructions

The three classical problems of Greek geometry can be rephrased as follows:

1. Doubling of the cube: Given a segment of length 1, construct one of length $\sqrt[3]{2}$.
2. Trisecting the angle: Given a segment of length 1 and one of length $\sin \theta$, construct one of length $\sin \theta/3$. In particular, because $\sin 30^\circ = 1/2$, trisecting the angle of $30^\circ$ is equivalent to: Given a segment of unit length, construct one of length $\sin 10^\circ$.
3. Squaring the circle: Given a segment of unit length, construct one of length $\pi$. 
A number $x$ is said to be *constructible* if given a segment of unit length, we can construct a segment of length $x$ using only a compass and a straightedge. Thus the three problems are equivalent to the questions:

1. Is $\sqrt[3]{2}$ constructible?

2. Is $\sin \theta/3$ constructible every time $\sin \theta$ is constructible?

3. Is $\pi$ constructible?

It was shown (Gauss et al., ca 1800) that a number is constructible if and only if it can be obtained by solving a sequence of quadratic and linear equations, with the first equation having integer coefficients, coefficients of subsequent equations being either also integers, or solutions of previously solved equations. Two consequences follow in the if direction:

a. If a number is constructible, it is algebraic.

b. If a number is constructible the degree of the algebraic equation with integer coefficients of minimum degree that it satisfies, must be a power of 2.

The second condition immediately showed that neither the cubic root of 2 nor the sine of 10 degrees were constructible.
Both are algebraic, but the equation of minimum degree they satisfy (with integer coefficients) is 3, and 3 is not a power of 2.

π refused to surrender for a while longer. But, by proving that it was transcendental, Lindemann finally closed the book on the three problems.

The End (for now)

**Partial Bibliography**

