The Italian Connection

THE CUBIC CHRONICLES
Cast of Characters

**Omar Khayyam** (1048-1131). A great Persian mathematician working in the Seljuk (Turkish) empire. Solved cubic equations numerically by intersection of conic sections. Stated that some of these equations could not be solved using only straightedge and compass, a result proved some 750 years later.

**Luca Pacioli** (1445-1509) His Suma (1494) summarized all that was known on equations. He discusses quartic equations stating what in modern notation would be: $x^4 = a + bx^2$ can be solved as a quadratic equation but $x^4 + ax^2 = b$ and $X^4 +a = bx^2$ are impossible at the present state of science.
There may not be any reliable portrait of Scipione dal Ferro on the web. One that pops up is really Tartaglia.

Scipione dal Ferro (1465-1526). Professor at Bologna. Around 1515 figured out how to solve the equation \( x^3 + px = q \) by radicals. Kept his work a complete secret until just before his death, the revealed it to his student Antonio Fior.

Antonio Fior (1506-?). Very little information seems to be available about Fior. His main claim to fame seems to be his challenging Tartaglia to a public equation ‘‘solvathon,’’ and losing the challenge.

No portrait of Antonio Fior seems to be available.
Nicolo of Brescia who adopted the name Tartaglia (Stutterer) (1499-1557). Hearing a rumor that cubic equations had been solved, figured out how to solve equations of the form $x^3 + mx^2 = n$, and made it public. This made Fior think that Tartaglia would not know how to deal with the equations del Ferro knew how to solve and he challenged Tartaglia to a public duel. But Tartaglia figured out what to do with del Ferro’s equation and won the contest.

Girolamo Cardano (1501-1576). Mathematician, physician, gambler (which led him to study probability), a genius and a celebrity in his day. Hearing of Tartaglia’s triumph over Fior convinced Tartaglia to reveal the secret of the cubic, swearing solemnly not to publish before Tartaglia had done so. Then he published first. But there are some possible excuses for this behavior.
More Characters

**Lodovico Ferrari** (1522-1565). A protégé of Cardano, he discovered how to solve the quartic equation, by reducing it to a cubic. But his result could not be published before making public how to solve cubic equations. Tartaglia still not publishing, and the discovery that Scipione dal Ferro had already solved the cubic, where part of the reasons why Cardano published first.

**Rafael Bombelli** (1526-1572). Engineer and self-taught mathematician, began writing in 1557 his masterwork, *Algebra*. It was to be five volumes long, but only three were ready for publication in 1572, the year he died. Made sense of the complex expressions that appeared as solutions to cubic equations. Because of that, MacTutor calls him the inventor of complex numbers.
Important general fact: For all of the mathematicians trying to solve cubic equations, negative numbers and 0 were mysterious not well understood concepts. For us, the cubic equation is

\[ x^3 + ax^2 + bx + c = 0. \]

For us there is little difference, if any, between the equations \( x^3 + ax = b \) and \( x^3 = ax + b \). For Scipione dal Ferro, Tartaglia et al, the difference was essential because they could only understand the equation if \( a, b \) were positive numbers. And, of course, the equation was never written in the form \( x^3 + ax + b = 0 \); setting to 0 just didn’t make any sense.

1 Omar Khayyam.

Large Pictures are essential to understanding this material. They will be provided in class. You may also consult the following online articles, which come with pictures.

[Omar Khayyam and a Geometric Solution of the Cubic] and [Omar Khayyam and the cubic equation]

Khayyam was led to cubic equations by trying to solve the following problem:

*Find a right triangle with the property that the hypotenuse equals the sum of one leg plus the altitude on the hypotenuse.*

It led him to trying to solve the cubic \( x^3 + 200x = 20x^2 + 2000 \). Omar Khayyam classified cubics into 14 different types (remember: all coefficients have to be positive!) and explained how to solve them intersecting conic sections. Because it corresponds closest (or exactly) to the case of second web page quoted above, I’ll explain how he solved an equation of the form

The main behind all of his method is that if you draw a circle and a conic (a parabola or branch of a hyperbola) they will have two points of intersection. By choosing the first one conveniently, we may get the second point of intersection to have an \( x \)-coordinate (of course, Khayyam would have not expressed it in this way) that solves our cubic equation.

For the purpose of explaining this in modern terms, it might be convenient to notice that the equation of a circle can be written in the form

\[ (x - a)(c - x) = (y - b)^2. \]

This is the equation of a circle through the points \((a, b)\) and \((c, b)\). A bit of algebra shows it has center at \(((a + c)/2, b)\), the radius is \(|a - b|/2\), in other
words it is the equation of a circle whose diameter is the segment joining \((a, b)\) to \((c, b)\). If we now draw a parabola of equation \(xy = ab\), it will intersect the circle at \((a, b)\). To figure out the second intersection, we substitute \(y = ab/x\) in the equation of the circle, and get successively

\[
(x - a)(c - x) = \left(\frac{ab}{x} - b\right)^2
\]

\[
(x - a)(c - x) = \frac{b^2}{x^2}(a - x)^2 = \frac{b^2}{x^2}(x - a)^2
\]

In the last equation we can cancel \(x - a\) on both sides. This loses a root, the root \(x = a\), but we know this root already. We get

\[c - x = \frac{b^2}{x^2}(x - a).\]

Multiplying by \(x^2\) this rearranges to \(x^3 + b^2x = cx^2 + ab^2\).

We can now transform this into a method for solving an equation \(x^3 + b^2x = cx^2 + ab^2\):

- Mark of the points of coordinates \((a, b)\) and \((c, b)\) and draw the line segment joining them.
- Draw the circle whose diameter is that line segment.
- Draw the first quarter branch branch of the hyperbola \(xy = ab\).
- Notice the hyperbola intersects the circle in two points. One is \((a, b)\). The coordinate of the second one is the root of the equation. Estimate its value.

### 2 The Main Players.

Scipione del Ferro (ca. 1515) figured out how to solve the equation \(x^3 + px = q\). If one can do this, one can do it all, but only if one really understands negative numbers. And negative numbers made people feel queasy in those days. Del Ferro never published his results but shortly before his death in 1526 revealed the secret to his student Antonio Maria Fior. By all accounts, Fior was a very mediocre mathematician, who never achieved much on his own.

When Nicolo of Brescia was 12 years old his city was invaded by the French and a French soldier hit him in the face with a sword, splitting his jaw. He recovered, but stammered for the rest of his life and adopted the name Tartalea, later Tartaglia. All his surviving writings are signed as either Nicolo Tartalea or Tartaglia. A lot of texts call him Nicolo Fontana, even Niccolò (with two c’s, as
Tartaglia claimed to have discovered the formula for expansion of a cube, namely

\[(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3\].

If one notices that \(-3u^2v + 3uv^2 = -3uv(u - v)\) and writes this formula in the form

\[(u - v)^3 + 3uv(u - v) = u^3 - v^3\],

one sees a similarity with the equation \(x^3 + px = q\): If one can find \(u, v\) so that

\[3uv = p\] and \(u^3 - v^3 = q\), then \(x = u - v\) solves the equation.

Finding \(u, v\) reduces to solving a quadratic equation. In fact, substituting \(v = p/3u\) into the equation \(u^3 - v^3 = q\) and multiplying by \(u^3\) leads to

\[27u^6 - 27quv^3 - p^3 = 0\].

This equation is quadratic in \(z = u^3\), with \(z = u^3\) it becomes \(27z^2 - 27qz - p^3 = 0\). Solving:

\[z = \frac{27q \pm \sqrt{(27q)^2 + 4 \cdot 27p^3}}{54} = \frac{1}{2} \left( q \pm \sqrt{q^2 + \left(\frac{p}{3}\right)^3} \right)\].

Choosing the + sign we get

\[u = \sqrt[3]{\frac{1}{2} \left( q + \sqrt{q^2 + \left(\frac{p}{3}\right)^3} \right)}\].

Solving for \(v\) we get an expression

\[v = \frac{p}{3u} = \frac{p}{3 \left( \frac{1}{2} \left( q + \sqrt{q^2 + \left(\frac{p}{3}\right)^3} \right) \right)^{1/3}}\]

which can be simplified a bit if we multiply numerator and denominator by \(\left(\sqrt{q^2 + (p/3)^2}\right)^{1/3}\). One ends with

\[v = \left(\frac{2}{9} \sqrt{q^2 + (p/3)^2}\right)^{1/3}\].

...
Tartaglia's solution is then
\[ x = \sqrt[3]{\frac{1}{2} \left( q + \sqrt{q^2 + \left(\frac{p}{3}\right)^3} \right)} - \sqrt[3]{\frac{2}{5}} \sqrt{q^2 + \left(\frac{p}{3}\right)^2}. \]

But one should try to use the last formula for the solution. Instead one should try to find \( u, v \) as solutions to \( uv = \frac{p}{3}, \ u^3 - v^3 = q \) in all cases.

Reducing a general equation to the depressed equation solved by del Ferro and Tartaglia is done by a simple substitution. To solve \( x^3 + bx^2 + cx + d = 0 \) substitute unknowns by \( y = x - \frac{b}{3} \). The equation becomes
\[ y^3 + \left( c - \frac{b^2}{3} \right) y = \frac{1}{2} bc - \frac{b^2}{9} - d \]
which is depressed. Solve for \( y \), then set \( x = y + \frac{b}{3} \).

In 1535 Antonio Fior challenged Tartaglia to a public problem solving session. Fior would give Tartaglia 30 problems, Tartaglia would give Fior another thirty, and whoever solved all first would win. Tartaglia totally humiliated Fior, of whom not much more is heard.

Girolamo Cardano was certainly one of the most colorful figures of his time. He had an innate ability to make enemies, so when he got his medical doctorate in 1525 and applied to join the College of Physicians of Milan, they rejected him on the grounds that he was an illegitimate child. From then on his life had many ups and downs. An inveterate gambler he was sometimes very wealthy, other times broke. He served some time in jail having written the horoscope of Jesus Christ. He loved one of his sons, that son was executed because he murdered his wife. He disliked his younger son who was also a gambler. Cardano wrote that the four great sadnesses of his life were:

\textit{The first was my marriage; the second, the bitter death of my son; the third, imprisonment; the fourth, the base character of my youngest son.}

In addition to Cardan's major contributions to algebra he also made important contributions to probability, hydrodynamics, mechanics and geology. Because he loved to gamble, he may have been the first to study with care the laws of chance, initiating the study of probability theory. His most important work is \textit{Ars Magna} published in 1545. In it one has the first published appearance of the formula to solve the cubic equation. Tartaglia revealed the secret to him and, according to Tartaglia, Cardano swore:

\textit{I swear to you, by God's holy Gospels, and as a true man of honour, not only never to publish your discoveries, if you teach me them, but I also promise you, and I pledge my faith as a true Christian, to note them down in code, so that after my death no one will be able to understand them.}
But one should not be too rash in judging. Cardano’s protégé and assistant, Ferrari, had figured out how to reduce a quartic equation to a cubic, and he was eager to publish. But to show that his method was useful, one had to show that cubics could be solved, otherwise it was just an exercise in futility. Moreover, Cardano discovered that Tartaglia had not really been the first to have the formula. As Ferrari wrote a little after Ars magna was publ

*Four years ago when Cardano was going to Florence and I accompanied him, we saw at Bologna Hannibal Della Nave, a clever and humane man who showed us a little book in the hand of Scipione del Ferro, his father-in-law, written a long time ago, in which that discovery was elegantly and learnedly presented.*

Lodovico Ferrari’s reduction of the quartic to the cubic is very ingenious and was much admired. I’ll let you discover it on your own (if you are so inclined)