Isolated Singularities.

These notes supplement the material at the beginning of Chapter 3 of Stein-Shakarchi.

I begin with our (slightly stronger) version of Riemann’s Theorem on removable singularities, that appears as Theorem 3.1 in Stein-Shakarchi. Before stating and proving our version, I remind you of a result from the notes titled “A general version of Cauchy’s Theorem. Runge’s Theorem.” Because we may have occasion to use it again in the future (the only time one can use something again), I’ll state it as a Lemma, and add the proof.

**Lemma 1** Let $z_0 \in \mathbb{C}$ and let $\Omega = \{ z \in \mathbb{C} : r_2 < \lvert z - z_0 \rvert < R \}$ be an annulus, $0 \leq r < R$ and assume $f : \Omega \to \mathbb{C}$ is holomorphic. Let $r < r_1 < r_2 < R$. Then

$$f(z) = \frac{1}{2\pi i} \int_{|z-z_0|=r_2} \frac{f(\zeta)}{\zeta-z} \, d\zeta - \frac{1}{2\pi i} \int_{|z-z_0|=r_1} \frac{f(\zeta)}{\zeta-z} \, d\zeta$$

for $r_1 < |z - z_0| < r_2$.

**Proof.** We let $\gamma_2$ be the positively oriented circle of radius $r_2$ and $\gamma_1$ the negatively oriented circle of radius $r_1$, both centered at $z_0$ and consider the cycle $\Gamma = \gamma_1 + \gamma_2$. Because

$$W_{\gamma_1}(z) = W_{\gamma_2}(z) = 0 \quad \text{if} \quad |z - z_0| \geq R; \quad W_{\gamma_1}(z) = W_{\gamma_2}(z) = 1 \quad \text{if} \quad \lvert z - z_0 \rvert < r,$$

it follows that $W_\Gamma(z) = W_{\gamma_1 + \gamma_2}(z) = 0$ for all $z \notin \Omega$. Moreover, because

$$W_{\gamma_1}(z) = 0, \ W_{\gamma_2}(z) = 1 \quad \text{if} \quad r_1 < |z - z_0| < r_2,$$

it follows that $W_\Gamma(z) = 1$ for all $z$ with $r_1 < |z - z_0| < r_2$. By (2) of Theorem 7 of the aforementioned “A general version...” notes,

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta-z} \, d\zeta = \frac{1}{2\pi i} \int_{|z-z_0|=r_2} \frac{f(\zeta)}{\zeta-z} \, d\zeta - \frac{1}{2\pi i} \int_{|z-z_0|=r_1} \frac{f(\zeta)}{\zeta-z} \, d\zeta$$

for $r_2 < |z| < r_1$, as desired.

Here is our version of Riemann’s theorem.

**Theorem 2** Assume $\Omega$ is open in $\mathbb{C}$, let $z_0 \in \Omega$ and assume $f : \Omega \setminus \{z_0 \} \to \mathbb{C}$ is holomorphic. If $\lim_{z \to z_0} (z-z_0) f(z) = 0$, then $f$ extends to a function also holomorphic at $z_0$. To be precise, $\lim_{z \to z_0} f(z)$ exists and defining $f$ at $z_0$ to be that limit, $f$ is holomorphic at $z_0$.

**Proof.** Select $R > 0$ so that $D_R(z_0) \subset \Omega$ and apply Lemma 1 with $r = 0$. Since we don’t need any additional symbols for 0, we are free to use $r$ with a different meaning. Assume $0 < \epsilon < r < R$. By Lemma 1 we have

$$f(z) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(\zeta)}{\zeta-z} \, d\zeta - \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f(\zeta)}{\zeta-z} \, d\zeta$$

for $\epsilon < |z - z_0| < r$. We now fix $z$ such that $0 < |z - z_0| < R$ and take $r$ so $|z - z_0| < r < R$, $\epsilon > 0$ such that $\epsilon < \frac{1}{2} |z - z_0| < |z - z_0|$. Then (1) is valid. Another consequence of this assumption on $\epsilon$ is that if $|\zeta - z_0| = \epsilon$, then

$$|\zeta - z| = |z - \zeta| = |(z - z_0) - (z_0 - \zeta)| \geq |z - z_0| - |z_0 - \zeta| = \lvert z-z_0 \rvert - \epsilon \geq \frac{1}{2} \lvert z - z_0 \rvert.$$

By our hypothesis on $f$, given $\eta > 0$, there is $\delta > 0$ such that

$$|f(\zeta)| \leq \frac{\eta}{|\zeta - z_0|}.$$
if $0 < |\zeta - z_0| < \delta$. Assuming now $\epsilon < \delta$ (in addition to $\epsilon < \frac{1}{2}|z - z_0|$) we have if $|\zeta - z_0| = \epsilon$, 
\[
\left| \frac{f(\zeta)}{\zeta - z} \right| = \frac{|f(\zeta)|}{|\zeta - z|} \leq \frac{\eta}{2|z - z_0|} = \frac{2\eta}{\epsilon|z - z_0|}.
\]
Thus (with $\gamma_\epsilon$ denoting the circle of radius $\epsilon$)
\[
\left| \frac{1}{2\pi i} \int_{|\zeta - z_0| = \epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta \right| \leq \frac{2\eta}{2\pi \epsilon|z - z_0|} \Lambda(\gamma_\epsilon) = \frac{2\eta}{2\pi \epsilon|z - z_0|} 2\pi \epsilon = \frac{2\eta}{|z - z_0|} \to 0
\]
as $\epsilon \to 0+$. Since we can take $\epsilon > 0$ arbitrarily small, we can let $\epsilon \to 0+$ in (2) to conclude that
\[
f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} d\zeta
\]
if $0 < |z - z_0| < r$. By Theorem 5.4 in Chapter 2 of Stein-Shakarchi, the map
\[
z \mapsto \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} d\zeta
\]
is holomorphic in all of $D_r(z_0)$; since it coincides with $f$ in $D'_r(z_0)$, it extends $f$ holomorphically to all of the disc. That is, defining
\[
f(z_0) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0} d\zeta,
\]
it follows that $f$ is holomorphic at $z_0$.

Concerning isolated singularities here are a few results, all of them to be proved in class (and in the text, though perhaps not in the same order).

We assume $z_0 \in \mathbb{C}$ and there is $r > 0$ such that $f : D'_r(z_0) \to \mathbb{C}$ is holomorphic.

• One and precisely one of the following three possibilities occurs.

1. $\lim_{z \to z_0} f(z) \in \mathbb{C}$ exists. In this case we will always assume that $f$ is defined at $z_0$ by $f(z_0) = \lim_{z \to z_0} f(z)$ and by Theorem 2 $f$ is holomorphic at $z_0$.

2. $\lim_{z \to z_0} |f(z)| = \infty$. In this case we say $f$ has a pole at $z_0$.

3. None of the above. However, none of the above is also equivalent to these three other conditions:

   (a) $f(z)$ does not remain bounded near $z_0$, however, $\lim_{z \to z_0} |f(z)| = \infty$ is false. (It stays bounded by some approaches, not by others).

   (b) For each $k \in \mathbb{N}$, it is false that $\lim_{z \to z_0} |z - z_0|^k |f(z)| = 0$. That is, $\limsup_{z \to z_0} |z - z_0|^k |f(z)| > 0$ for all $k \in \mathbb{N}$.

   (c) For each $k \in \mathbb{N}$, it is false that $\lim_{z \to z_0} |z - z_0|^{-k} |f(z)| = \infty$. That is, $\liminf_{z \to z_0} |z - z_0|^{-k} |f(z)| < \infty$ for all $k \in \mathbb{N}$.

We call such an isolated singularity an essential singularity. The equivalence of these three conditions and their equivalence with “none of the above” should be fairly self-evident; given Theorem 2.

• We say $f$ has a zero at $z_0$ if, what else?, $f(z_0) = \lim_{z \to z_0} f(z) = 0$. In this case, of course, $f$ is holomorphic at $z_0$ and therefore has a series expansion converging in all of $D_r(z_0)$; that is $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $z \in D_r(z_0)$. The fact that $f(z_0) = 0$ is equivalent to $a_0 = 0$. Assuming $f$ is not identically zero in $D_r(z_0)$, there is $k \geq 1$ with $a_k \neq 0$. We then say that $f$ has a zero of order $k$ at $z_0$. We can then write
\[
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = (z - z_0)^k \sum_{n=0}^{\infty} a_{n+k} (z - z_0)^n.
\]
The series $\sum_{n=0}^{\infty} a_{n+k} (z - z_0)^n$ has the same radius of convergence as the series for $f$, thus defines a holomorphic function $g(z) = \sum_{n=0}^{\infty} a_{n+k} (z - z_0)^n$ in $D_r(z_0)$; $g(z_0) = a_k \neq 0$. This proves most of the following simple Lemma.
The function $f$ has a zero of order $k \in \mathbb{N}$ at $z_0$ if and only if there exists a function $g$ holomorphic at $z_0$ such that $g(z_0) \neq 0$, and $f(z) = (z - z_0)^k g(z)$.

- If $f$ has a zero of order $k$ at $z_0$, then there has to be some deleted neighborhood of $z_0$ where $f(z) \neq 0$ and $1/f$ will be holomorphic in that neighborhood. Clearly, the fact that $\lim_{z \to z_0} f(z) = 0$ implies that $\lim_{z \to z_0} |1/f(z)| = \infty$, thus $1/f$ has a pole at $z_0$. Conversely, if $f$ has a pole at $z_0$, then $f$ cannot possibly be zero if we are close enough to $z_0$, so $1/f$ is holomorphic in a deleted neighborhood of $z_0$. But we will have $\lim_{z \to z_0} 1/f(z) = 0$, so that $1/f$ extends to holomorphic at $z_0$ and has a zero at $z_0$. This zero has an order, say $k$. We say that $f$ has a pole of order $k$ at $z_0$. To summarize, $f$ has a pole of order $k$ at $z_0$ if and only if $1/f$ has a zero of order $k$ at $z_0$. In this case we can write $1/f(z) = (z - z_0)^k g(z)$ with $g(z_0) \neq 0$. Then $h = 1/g$ is holomorphic and non-zero at $z_0$. We get the following counterpart to Lemma 3

**Lemma 4** The function $f$ has a pole of order $k \in \mathbb{N}$ at $z_0$ if and only if there exists a function $h$ holomorphic at $z_0$ such that $h(z_0) \neq 0$, and $f(z) = (z - z_0)^{-k} h(z)$.

- A slightly different characterization of poles is given by the next lemma.

**Lemma 5** The function $f$ has a pole of order $k$ at $z_0$ if and only if there exist complex numbers $a_1, a_{-2}, \ldots, a_{-k}$ such that $a_{-k} \neq 0$ and

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \cdots + \frac{a_{-1}}{z - z_0}$$

is holomorphic at $z_0$ (i.e., extends continuously to a holomorphic function at $z_0$).

*Proof.* Assume first $f$ has a pole of order $k$ at $z_0$ and write $f(z) = (z - z_0)^{-k} h(z)$, where $h$ is holomorphic at $z_0$, $h(z_0) \neq 0$. Write $h(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$; this series converging in some disc of positive radius around $z_0$. Then

$$f(z) = (z - z_0)^{-k} \sum_{n=0}^{\infty} b_n(z - z_0)^n = \sum_{n=0}^{k} b_n(z - z_0)^{n-k} = \frac{b_0}{(z - z_0)^k} + \cdots + \frac{b_{k-1}}{z - z_0} + g(z),$$

where $g(z) = \sum_{n=0}^{k} b_{n+k}(z - z_0)^n$ is holomorphic at $z_0$.

Conversely, assuming that

$$g(z) = f(z) - \left( \frac{a_{-k}}{(z - z_0)^k} + \cdots + \frac{a_{-1}}{z - z_0} \right)$$

is holomorphic at $z_0$, with $a_{-k} \neq 0$, multiplying by $(z - z_0)^k$ we get

$$(z - z_0)^k f(z) = a_{-k} + \cdots + a_{-1}(z - z_0)^{k-1} + (z - z_0)^k g(z)$$

is holomorphic at $z_0$ and $a_{-k} + \cdots + a_{-1}(z - z_0)^{k-1} + (z - z_0)^k g(z) \bigg|_{z=z_0} = a_{-k} \neq 0$. \hfill \blacksquare

- We see that $f$ has a pole of order $k \in \mathbb{N}$ at $z_0$ if and only if we can write

$$f(z) = \sum_{n=-k}^{\infty} a_n(z - z_0)^n = \frac{a_{-k}}{(z - z_0)^k} + \cdots + \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

with $a_{-k} \neq 0$ and the power series converging (to $f(z)$, of course) in some disc of positive radius. A series in positive and negative powers of $z - z_0$ is called a *Laurent series*, the part of negative powers; namely,

$$\frac{a_{-k}}{(z - z_0)^k} + \cdots + \frac{a_{-1}}{z - z_0}$$

is called the principal part of the series, and the coefficient of the first negative power; namely $a_{-1}$, is called the *residue* of $f$ at $z_0$, and will be denoted by $\text{res}_{z_0} f$. 

We will probably see eventually the following theorem:

Let $0 < r < R$ and assume $f$ is holomorphic in the open annulus $A_{r,R}(z_0) = \{ z \in \mathbb{C} : r < |z - z_0| < R \}$. Then there exist $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$ such that

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n
\]

for $r < |z - z_0| < R$. More precisely, the non-negative part of this series, $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, converges absolutely for $|z - z_0| < R$, while the negative part (a.k.a the principal part) converges absolutely for $|z - z_0| > r$. An important case is the case $r = 0$. In this case $f$ is holomorphic at $z_0$ if and only if the principal part vanishes. It has a pole at $z_0$ if and only if there is a non vanishing principal part, but it consists of only a finite number of terms.

The series in (2) is known as the Laurent series of $f$. In case of $r = 0$; that is, when $f$ is holomorphic in $D'_R(z_0)$, we now have the following classification:

1. If $a_n = 0$ for all $n \in \mathbb{N}$, then $f$ is holomorphic (or extends as such) at $z_0$. Moreover, if then $k \in \mathbb{N}$ is such that $a_k \neq 0$ but $a_n = 0$ for $n < k$, then $f$ has a zero of order $k$ at $z_0$.
2. If there is only a finite but non zero number of negative terms; that is, if there is $k \in \mathbb{N}$ such that $a_{-k} \neq 0$, but $a_n = 0$ if $n < -k$, then $f$ has a pole of order $k$ at $z_0$.
3. If the set of $n \in \mathbb{N}$ such that $a_{-n} \neq 0$ is infinite, then $f$ has an essential singularity at $z_0$. In this case we still call the coefficient $a_{-1}$ the residue of $f$ at $z_0$. 