COMPLEX ANALYSIS–Spring 2014

Cauchy and Runge Under the Same Roof.

These notes can be used as an alternative to Section 5.5 of Chapter 2 in the textbook. They assume the theorem on winding numbers of the notes on Winding Numbers and Cauchy’s formula, so I begin by repeating this theorem (and consequences) here, but first, some remarks on notation.

A notation I’ll be using on occasion is to write
\[ \int_{|z-z_0|=r} f(z) \, dz \]
for the integral of \( f(z) \) over the positively oriented circle of radius \( r \), center \( z_0 \); that is,
\[ \int_{|z-z_0|=r} f(z) \, dz = \int_0^{2\pi} f(z_0 + re^{it}) \, rie^{it} \, dt. \]

Another notation I’ll use, to avoid confusion with complex conjugation is \( Cl(A) \) for the closure of a set \( A \subseteq \mathbb{C} \).

If \( a, b \in \mathbb{C} \), I will denote by \( \lambda_{a,b} \) the line segment from \( a \) to \( b \); that is,
\[ \lambda_{a,b}(t) = a + t(b-a), \quad 0 \leq t \leq 1. \]

Triangles being so ubiquitous, if we have a triangle of vertices \( a, b, c \in \mathbb{C} \), I will write \( T(a,b,c) \) for the closed triangle; that is, for the set
\[ T(a,b,c) = \{ ra + sb + tc : r, s, t \geq 0, r + s + t = 1 \}. \]

Here the order of the vertices does not matter. But I will denote the boundary oriented in the order of the vertices by \( \partial T(a,b,c) \); that is, \( \partial T(a,b,c) = \lambda_{a,b} + \lambda_{b,c} + \lambda_{c,a} \). If \( a, b, c \) are understood (or unimportant), I might just write \( T \) for the triangle, and \( \partial T \) for the boundary.

1 Winding Numbers

In this section we assume that \( \gamma : [a, b] \to \mathbb{C} \) is a closed path (recall that in this course, except if otherwise noted, all curves, paths are piecewise smooth). For \( z \notin \gamma^* \) we define
\[ W_\gamma(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{\zeta - z} \, d\zeta. \]

By Theorem 1 in the notes on Winding Numbers and Cauchy’s formula or, even better, by Theorem 5.4, Chapter 2, of our textbook* (with \( F(z, s) = \frac{\gamma'(s)}{\gamma(s) - z} \)),

*To be precise, the Theorem does not apply in an immediate way, since \( \gamma' \) is not necessarily continuous on all of \([a, b]\). However, the integral is a finite sum of integrals over intervals in which \( \gamma' \) is continuous, hence a finite sum of holomorphic functions.
Theorem 1 Let \( \gamma : [a, b] \to \mathbb{C} \) be a closed path and define \( W_\gamma : \gamma^* \to \mathbb{C} \) as above. Then \( W_\gamma \) is integer valued and constant on each connected component of the complement of \( \gamma^* \). Moreover, it is 0 on the unbounded component of \( \mathbb{C} \setminus \gamma^* \).

Proof. A continuous integer valued function on a connected open set has little choice but to be constant. So all that we need to prove is that \( W_\gamma \) is integer valued and zero on the unbounded component of the complement of the curve. For the former, let \( z \not\in \gamma^* \) (to be fixed for now) and consider the function \( F : [a, b] \to \mathbb{C} \) defined by

\[
F(t) = (\gamma(t) - z)e^{-\int_a^t \frac{\gamma'(s)ds}{\gamma(s)-z}}
\]

for \( t \in [a, b] \). It is easy to see that \( F \) is continuous and that \( F \) is differentiable wherever \( \gamma \) is differentiable with

\[
F'(t) = \left( \gamma'(t) - (\gamma(t) - z) \frac{\gamma'(t)}{\gamma(t) - z} \right) e^{-\int_a^t \frac{\gamma'(s)ds}{\gamma(s)-z}} = 0
\]

at such points. It follows that \( F \) is constant in all the intervals in which \( \gamma' \) is defined. However since \( F \) is continuous in \([a, b] \) it has to be constant in the whole interval; in particular \( F(a) = F(b) \); that is

\[
\gamma(a) - z = F(a) = F(b) = (\gamma(b) - z)e^{-\int_a^b \frac{\gamma'(s)ds}{\gamma(s)-z}} = (\gamma(b) - z)e^{-2\pi i W_\gamma(z)}.
\]

Since \( \gamma(a) - z = \gamma(b) - z \neq 0 \) (because \( \gamma \) is closed and \( z \not\in \gamma^* \)), we can cancel \( \gamma(a) - z \) to get \( e^{-2\pi i W_\gamma(z)} = 1 \). This implies that \( W_\gamma(z) \in \mathbb{Z} \).

As mentioned above, proving that \( W_\gamma(z) \in \mathbb{Z} \) also proves that it is constant in the connected components of the complement of \( \gamma^* \). Let \( U \) be the unbounded one of these components. Let \( z \in U \) and let \( R = \text{dist}(z, \gamma^*) \). Then

\[
\left| \frac{1}{\zeta - z} \right| \leq \frac{1}{R}
\]

for all \( \zeta \in \gamma^* \) and

\[
|W_\gamma(z)| \leq \frac{1}{2\pi} \frac{1}{R} \Lambda(\gamma).
\]

In \( U \) there is \( z \) at arbitrary large distance from the curve; we can find \( z \) with \( R \) as large as we please, for example \( R > \Lambda(\gamma)/\pi \) for which \( |W_\gamma m(z)| < 1/2 \). One may do some searching, but the only integer of absolute value less than 1/2 that one can find is 0, so \( W_\gamma(z) = 0 \) for \( z \) large enough, thus for all \( z \in U \). \( \blacksquare \)

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\footnote{With some perversity, I believe some textbooks refer to the components of the complement of \( \gamma^* \) as the components of \( \gamma \).}
Proposition 2  Let $C$ be the positively oriented circle of radius $r$, centered at $z_0 \in \Omega$. Then

$$W_C(z) = \begin{cases} 1, & \text{if } z \in D_r(z_0), \\ 0, & \text{if } |z - z_0| > r. \end{cases}$$

Proof. It is clear that the complement of $C^*$ has exactly two connected components; $D_r(z_0)$ and $U = \{z \in \mathbb{C} : |z - z_0| > r\}$. It is also clear that $U$ is the unbounded component, thus $W_C(z) = 0$ for all $z \in U$. We also have, writing $C(t) = z_0 + ie^{it}, 0 \leq t \leq 2\pi$,

$$W_C(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{rie^{it} dt}{(z_0 + ie^{it}) - z_0} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{rie^{it}}{re^{it}} = \frac{1}{2\pi i} \int_0^{2\pi} i dt = 1.$$

Thus $W_C(z) = 1$ for all $z \in D_r(z_0)$. 

2  A slightly stronger version of the Cauchy Theorem for a triangle

Theorem 3  Let $\Omega$ be an open subset of $\mathbb{C}$, let $p \in \Omega$ and assume $f : \Omega \to \mathbb{C}$ is continuous, and $f$ is holomorphic at all $z \in \Omega, z \neq p$. Let $T$ be a closed triangle entirely contained in $\Omega$, then

$$\int_{\partial T} f(z) \, dz = 0.$$

Proof. Let $T = T(A, B, C) \subset \Omega$. We consider four cases.

Case 1. $p \notin T$. Then the previous version of the theorem is valid.

Case 2. $p$ is a vertex of the triangle, say $p = C$. Select two points $Q, R$ on the sides $AC, BC$, respectively, and consider the boundaries of $T_1, T_2, T_3$ of the triangles of vertices $RCQ, ABR, ARQ$, respectively.

If we integrate over $\partial T_1 = \partial T(B, C, Q)$, $\partial T_2 = \partial T(A, B, R)$ and over $\partial T_3 = \partial T(A, R, Q)$, we see that

$$\int_{\partial T} f(z) \, dz = \int_{\partial T_1} f(z) \, dz + \int_{\partial T_2} f(z) \, dz + \int_{\partial T_3} f(z) \, dz.$$
By Case 1, \( \int_{\partial T_2} f(z) \, dz = \int_{\partial T_3} f(z) \, dz = 0 \) so that

\[
\left| \int_{\partial T} f(z) \, dz \right| = \left| \int_{\partial T_1} f(z) \, dz \right| \leq \sup_{z \in \partial T_1} |f(z)| \Lambda(\partial T_1).
\]

The sup in the last inequality above is finite, since \( f \) is continuous everywhere, while \( \Lambda(\partial T_1) \to 0 \) as \( Q, R \to C \). The result follows.

**Case 3.** \( p \) is on a side of \( T \), but not a vertex. Suppose \( p \) is on the side \( AC \). We then split \( T \) into the triangles \( T_1 = T(A, B, p) \) and \( T_2 = T(p, B, C) \).

\[
\int_{\partial T_1} f(z) \, dz = \int_{\partial T_2} f(z) \, dz = 0,
\]

thus

\[
\int_{\partial T} f(z) \, dz = \int_{\partial T_1} f(z) \, dz + \int_{\partial T_2} f(z) \, dz = 0.
\]

**Case 4.** The point \( p \) is in the interior of \( T \). The picture shows how to split \( T \) so we are in case 3.

The following theorem is now proved EXACTLY as in the case of a function holomorphic everywhere, so the proof is omitted.

**Theorem 4** Let \( \Omega \) be an open star shaped set and assume \( f : \Omega \to \mathbb{C} \) is holomorphic at all points of \( \Omega \) except, perhaps at one point \( p \in \Omega \). Assume, however, \( f \) is continuous at \( p \). Then \( f \) has a primitive in \( \Omega \); i.e., there is \( F : \Omega \to \mathbb{C} \) holomorphic at all points of \( \Omega \) such that \( F'(z) = f(z) \) for all \( z \in \Omega \). In particular,

\[
\int_{\gamma} f(z) \, dz = 0
\]
for all closed curves in Ω.

**Theorem 5** A somewhat more general Cauchy formula. Let Ω be a star shaped region. Let \( f : \Omega \rightarrow \mathbb{C} \) be holomorphic. Let \( \gamma : [a, b] \rightarrow \Omega \) be a closed path in Ω. Then, for every \( z \in \Omega \setminus \gamma^* \) we have

\[
W_\gamma(z)f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.
\]

In particular if \( Cl(D_r(z_0)) = \{ z \in \mathbb{C} : |z - z_0| \leq r \} \subset \Omega \ (r > 0) \), then

\[
f(z) = \frac{1}{2\pi i} \int_{|z-z_0|=r} C \frac{f(\zeta)}{\zeta - z} d\zeta
\]

for all \( z \in D_r(z_0) \).

**Proof.** The second formula is an immediate consequence of the first since \( W_C(z) = 1 \) for all \( z \in D_r(z_0) \). For the first one let \( z \in \Omega \setminus \gamma^* \) and define \( g : \Omega \rightarrow \mathbb{C} \) by

\[
g(\zeta) = \begin{cases}
\frac{f(\zeta) - f(z)}{\zeta - z}, & \text{if } \zeta \in \Omega \setminus \{z\}, \\
f'(z), & \text{if } \zeta = z.
\end{cases}
\]

This function is continuous in Ω and holomorphic everywhere except perhaps at \( z \). By Theorem 4 we have that

\[
0 = \int_\gamma g(\zeta) d\zeta.
\]

Because \( z \notin \gamma^* \), we can separate the integral above to get

\[
0 = \int_\gamma g(\zeta) d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta - f(z) \int_\gamma \frac{1}{\zeta - z} d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta - 2\pi i W_C(z)f(z).
\]

The theorem follows.

### 3 What comes next

At this point one can get all the theorems following Cauchy’s integral formula in Section 4, of Chapter 2 of our textbook, and all of Section 5 up to Section 5.5.

### 4 Morera Speaks!

We now can see that the exceptional point at which a holomorphic function is continuous but perhaps not holomorphic, cannot really exist.
Theorem 6 Let $\Omega$ be open in $\mathbb{C}$, let $p \in \Omega$ and assume that $f : \Omega \to \mathbb{C}$ is continuous and $f$ is holomorphic at all points of $\Omega \setminus \{p\}$. Then $f$ is also holomorphic at $p$.

Proof. Let $D$ be an open disc contained in $\Omega$. By Theorem 3 in these notes, $\int_{\partial T} f(z) \, dz = 0$ for all closed triangles $T \subseteq \Omega$. By Morera’s Theorem, $f$ is holomorphic in $\Omega$.

5 A quite general version of Cauchy’s Theorem and Formula

It is silly not to do this since it takes so little extra effort. And it can be quite useful. Let $\gamma_1, \ldots, \gamma_m$ be a finite number of closed curves in $\mathbb{C}$. We define $\Gamma = \gamma_1 + \cdots + \gamma_m$, as being simply something on which to integrate functions; a convenient notation for

$$\int_{\gamma_1} f(z) \, dz + \cdots + \int_{\gamma_m} f(z) \, dz.$$ 

That is, we define $\Gamma = \gamma_1 + \cdots + \gamma_m$ by

$$\int_{\Gamma} f(z) \, dz = \sum_{j=1}^{m} \int_{\gamma_j} f(z) \, dz.$$ 

The technical term for such an object is cycle. For us, it exists only as a subscript of an integral. At least, for now. But given such a cycle $\Gamma = \sum_{j=1}^{m} \gamma_j$, we can (and will) define the winding number of a point $z \notin \Gamma^* := \bigcup_{j=1}^{m} \gamma_j^*$ by

$$W_\Gamma(z) = \sum_{j=1}^{m} W_{\gamma_j}(z) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{\gamma_j} \frac{1}{\zeta - z} \, d\zeta.$$ 

In this section I will prove the following quite general version of Cauchy’s Theorem and formula. The quite beautiful proof is adapted (stolen?) from Rudin, op. cit.

Theorem 7 Let $\Omega$ be an open subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be holomorphic. If $\Gamma = \sum_{j=1}^{m} \gamma_j$ is a cycle in $\Omega$ such that $W_\Gamma(z) = 0$ for all $z \notin \Omega$, then

(1) \[ \int_{\Gamma} f(z) \, dz = 0, \]

and

(2) \[ W_\Gamma(z) f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta \]

for all $z \in \Omega$. 

6
Before getting into the proof, here is a simple application. Let \( \Omega = \{ z \in \mathbb{C} : r < |z| < R \} \) be an annulus, \( 0 \leq r < R \). Let \( r < r_1 < r_2 < R \). We let \( \gamma_1 \) be the positively oriented circle of radius \( r_1 \) and \( \gamma_2 \) be the negatively oriented circle of radius \( r_2 \); that is

\[
\gamma_1(t) = r_1 e^{it}, \quad \gamma_2(t) = r_2 e^{-it}, \quad 0 \leq t \leq 2\pi.
\]

Because

\[
W_{\gamma_1}(z) = W_{\gamma_2}(z) = 0 \quad \text{if} \quad |z| \geq R; \quad W_{\gamma_1}(z) = -1, W_{\gamma_2}(z) = 1 \quad \text{if} \quad |z| \leq r,
\]

it follows that \( W_{\gamma_1 + \gamma_2}(z) = 0 \) for all \( z \notin \Omega \). Moreover, because

\[
W_{\gamma_1}(z) = 0, W_{\gamma_2}(z) = 1 \quad \text{if} \quad r_1 < |z| < r_2,
\]

if \( f \) is holomorphic in the annulus, then

\[
f(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta
\]

for \( r_1 < |z| < r_2 \). More generally, if \( f \) is holomorphic in a disc with holes; that is if \( R > 0 \) and \( \text{Cl}(D_{r_i}(z_i)) \subset D_{R}(z_0) \) for \( i = 1, \ldots, m \), and \( f \) is holomorphic in \( D_R(z_0) \setminus \bigcup_{i=1}^m \text{Cl}(D_{r_i}(z_i)) \), then for \( 0 < \rho < R \) such that one still has \( \text{Cl}(D_{r_i}(z_i)) \subset D_{\rho}(z_0) \) for \( i = 1, \ldots, m \), one gets

\[
f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{\zeta - z} d\zeta - \sum_{i=1}^m \frac{1}{2\pi i} \int_{|z - z_i| = r_i} \frac{f(\zeta)}{\zeta - z} d\zeta
\]

for all \( z \in D_{\rho}(z_0) \setminus \bigcup_{i=1}^m \text{Cl}(D_{r_i}(z_i)) \).

**Proof. (Of Theorem 7)** For \( z, \zeta \in \Omega \times \Omega \) define

\[
F(z, \zeta) = \begin{cases} 
\frac{f(\zeta) - f(z)}{\zeta - z}, & \text{if} \ (z, \zeta) \in \Omega \times \Omega, \ \zeta \neq z, \\
f'(z), & \text{if} \ (z, \zeta) \in \Omega \times \Omega, \ \zeta = z.
\end{cases}
\]

It is a simple exercise to verify that \( F : \Omega \times \Omega \to \mathbb{C} \) is continuous thus, assuming from now on that \( \Gamma = \sum_{j=1}^m \gamma_j \) is a cycle in \( \Omega \) satisfying \( W_\Gamma(z) = 0 \) for all \( z \notin \Omega \) (even though this last condition is not needed yet) we can now define \( h : \Omega \to \mathbb{C} \) by

\[
h(z) = \frac{1}{2\pi i} \int_{\Gamma} F(z, \zeta) d\zeta = \frac{1}{2\pi i} \sum_{j=1}^m \int_0^1 F(z, \gamma_j(t)) \gamma'_j(t) dt;
\]

where we assume (as we may) that \( \gamma_j : [0, 1] \to \Omega, \ \gamma_j(0) = \gamma_j(1), \) for \( j = 1, \ldots, m \). Because \( F \) is continuous on all of \( \Omega \times \Omega \), \( h \) is defined everywhere
in Ω, including at points z on the cycle. But there is more. If we look at the functions \((z, t) \mapsto F(z, \gamma_j(t))\gamma_j'(t)\) for \(j = 1, \ldots, m\), they are continuous on \(\Omega \times [0, 1] \to \mathbb{C}\). For a fixed \(t \in [0, 1]\), the map \(z \mapsto F(z, \gamma_j(t))\gamma_j'(t)\) is continuous and holomorphic except perhaps at \(z = \gamma_j(t)\). But that is one point at which Theorem 6 applies (the theorem in the “Morera speaks” section) so the function in question is holomorphic in Ω. Theorem 5.4 in Stein-Shakarchi applies to prove that \(z \mapsto \int_0^1 F(z, \gamma_j(t))\gamma_j'(t) dt\) is holomorphic in Ω for \(j = 1, \ldots, m\), hence \(h\) is holomorphic in Ω. The objective is to prove \(h(z) = 0\) for all \(z \in \Omega\); then for \(z \notin \Gamma^*\) we can separate the integral defining \(h\) and (2) follows at once. To achieve this objective, define now a new set \(\Omega_1\) by

\[
\Omega_1 = \{z \in \mathbb{C} : z \notin \Gamma^*, W_{\Gamma}(z) = 0\}.
\]

This is an open set being a finite intersection of open sets. At first glance it might not be clear that \(W_{\Gamma}(z) = 0\) defines an open set, but let us recall that the continuous function \(W_{\Gamma}\) only takes integer values, thus

\[
\{z \in \mathbb{C}\setminus\Gamma^* : W_{\Gamma}(z) = 0\} = \{z \in \mathbb{C}\setminus\Gamma^* : |W_{\Gamma}(z)| < 1/2\}.
\]

Since none of the curves making up the cycle are in \(\Omega_1\), we can define \(h_1 : \Omega_1 \to \mathbb{C}\) by

\[
h_1(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta.
\]

That \(h_1\) is holomorphic in \(\Omega_1\) is proved similarly as we proved the holomorphicity of \(h\); easier because we don’t have the problem of potentially bad points. By the hypothesis on \(\Gamma\), \(\mathbb{C}\setminus\Omega \subset \Omega_1\), hence \(\mathbb{C} = \Omega \cup \Omega_1\). Notice that if \(z \in \Omega \cap \Omega_1\), then \(h(z) = h_1(z)\), so that we can define \(H : \mathbb{C} \to \mathbb{C}\) by \(H(z) = h(z)\) if \(z \in \Omega\), \(H(z) = h_1(z)\) if \(z \in \Omega_1\), and \(H\) is a well defined entire function. The set \(\Omega_1\) must contain all points outside of a disc containing \(\Gamma^*\) so that

\[
\lim_{|z| \to \infty} H(z) = \lim_{|z| \to \infty} h_1(z) = 0.
\]

By Liouville’s Theorem, \(H \equiv 0\), proving \(h(z) = 0\) for all \(z \in \Omega\) and (2). Concerning now (1), select any \(z \in \Omega\setminus\Gamma^*\) and apply (2) to the holomorphic function \(\zeta \mapsto (\zeta - z)f(\zeta)\). Since this function vanishes at \(z\), (1) follows.

And star-shaped regions might never be mentioned again in this course. Or maybe they will.

6 More on Curves

As mentioned frequently, the Jordan Curve Theorem states that a simple closed (not necessarily piecewise smooth, merely continuous) curve in the plane divides the plane into two connected regions; one bounded, one unbounded. If the curve is actually piecewise smooth, the proof is easier and is provided toward the end of our textbook. In this course we will accept this theorem, in fact we
will accept the following result that is not too hard to verify in a lot of cases: Assume $\gamma: [a,b] \to \mathbb{C}$ is a closed (piecewise smooth) simple curve. Then $\mathbb{C}\setminus\gamma^*$ is the union of two disjoint open connected sets; a bounded one known as the interior or inside of the curve, to be denoted by $In(\gamma)$, and an unbounded one (known as the unbounded component) (So far, all this also holds assuming only continuity.) Moreover, if $\gamma$ is positively oriented (intuitively, as you move along $\gamma$, $In(\gamma)$ is always to your left), then

$$W_\gamma(z) = \begin{cases} 1, & \text{if } z \in In(\gamma), \\ 0, & \text{if } z \text{ is in the unbounded component.} \end{cases}$$

It follows from (2) that if $f: \Omega \to \mathbb{C}$ is holomorphic and $\gamma$ is a positively oriented simple curve such that $In(\gamma) \cup \gamma \subset \Omega$, then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for all $z \in In(\gamma)$.

Theorem 7 applies to an open set with the property that $W_\gamma(z) = 0$ for all closed curves $\gamma$ in $\Omega$ and $z \notin \Omega$. If we recall the fact that $W_\gamma(z)$ measures how many times the curve winds about $z$, this says that a closed curve in $\Omega$ cannot wind around any point outside of $\Omega$. One can show that this is equivalent to the open being simply connected, and also equivalent to the open set being connected with a complement connected in $\mathbb{C} \cup \{\infty\}$.

7 A Construction

The purpose of this section is to prove the following theorem.

**Theorem 8** Let $\Omega \neq \emptyset$ be an open subset of $\mathbb{C}$ and let $K$ be a compact subset of $\Omega$. There exist a finite number of (piecewise-smooth) closed curves $\gamma_1, \ldots, \gamma_r$ such that if $\Gamma = \sum_{j=1}^{r} \gamma_j$ then

1. $\Gamma^* \subset \Omega \setminus K$.

2. For every holomorphic $f: \Omega \to \mathbb{C}$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for all $z \in K$.

This construction will take a while. I want to emphasize that this theorem is true for every non-empty open set.

Recall that if $z, w \in \mathbb{C}$, we denoted by $\lambda_{z,w}$ the line segment from $z$ to $w$, parameterized by $\lambda_{z,w}(t) = z + t(w - z), 0 \leq t \leq 1$. To emphasize that there
is a beginning \((z)\) and an end \((w)\), we’ll also refer to \(\lambda_{z,w}\) as an oriented line segment.

Let \(\Lambda\) be a finite family of oriented line segments in the plane. We will say it is balanced iff and only for each \(z \in \mathbb{C}\) there are as many segments in \(\Lambda\) starting at \(z\) as there are segments ending at \(z\). Of course, to decide if a family is balanced or not, we only need to consider points that are actual endpoints or starting points of segments in the family. Here are a few examples.

- A family consisting of a single line segment can only be balance if its only segment reduces to a point: \(\Lambda = \{\lambda_{z,z}\}\) for some \(z \in \mathbb{C}\).
- The only way a family of two line segments can be balanced is if one is the opposite of the other one: \(\Lambda = \{\lambda_{z,w}, \lambda_{w,z}\}\) for some \(z, w \in \mathbb{C}\).
- A set of segments forming a closed polygonal line is balanced: \(\{\lambda_{z_0,z_1}, \lambda_{z_1,z_2}, \ldots, \lambda_{z_{n-1},z_0}\}\) for some \(z_0, \ldots, z_{n-1} \in \mathbb{C}\).

A balanced set of oriented intervals can be organized into a cycle, as the following lemma proves.

**Lemma 9** Let \(\Lambda\) be a finite, non-empty, balanced, set of oriented line segments. These segments can be organized into a family of closed curves \(\gamma_1, \ldots, \gamma_r\) in the sense that each one of these closed curves is a polygonal curve whose links are elements of \(\Lambda\) each element of \(\Lambda\) appearing exactly once in exactly one of these curves.

**Proof.** The process is sort of double inductive. The first induction is on the size of \(\Lambda\). If \(\Lambda\) contains a single oriented line segment, it has to be of the form \(\lambda_{\alpha,\alpha}\) and we can consider this a (degenerate) closed curve. Suppose now \(\Lambda\) is of size \(M > 1\) and the result has been proved for all balanced sets of size \(< M\). We now have another induction. Select any segment from \(\Lambda\); say it is \(\lambda_{\alpha_0,\alpha_1}\). This is the first step of the induction. Now assume that for some \(n\) we have found segments \(\lambda_{\alpha_{i-1},\alpha_i} \in \Lambda\) for \(i = 1, \ldots, m\) for some \(m \geq 1\). This has, of course, been done for \(m = 1\). If \(\alpha_m = \alpha_0\), we set \(\gamma_1 = \lambda_{\alpha_0,\alpha_1} + \lambda_{\alpha_1,\alpha_2} + \cdots + \lambda_{\alpha_{m-1},\alpha_0}\); this is a closed curve. A vertex of one of these line segments could have appeared more than once in the chain (nobody is saying the these polygonals can’t self-intersect), but every one of them appears as many times as an origin as it does as an end-point of a segment. The conclusion is that \(\Lambda \setminus \{\lambda_{\alpha_0,\alpha_1}, \lambda_{\alpha_1,\alpha_2}, \ldots, \lambda_{\alpha_{m-1},\alpha_0}\}\) is still balanced; it has \(< M\) elements. By the induction hypothesis its segments can be organized into closed curves and we are done. Assume next that \(\alpha_m \neq \alpha_0\). It is possible that \(\alpha_i\) with \(i < m\) equals \(\alpha_m\). But any previous \(\alpha_i\), except for \(\alpha_0 \neq \alpha_m\), appeared both as origin and as endpoint of a segment. Due to balance, there has to be some segment left starting at \(\alpha_m\). Add this segment to the list. This increases \(m\) to \(m + 1\). There being only a finite number of segments, eventually we will add a segment ending in \(\alpha_0\).

This concludes the proof of the lemma. We can call a cycle \(\Gamma = \gamma_1 + \cdots + \gamma_m\) constructed from a balanced set \(\Lambda\) of oriented line segments a cycle determined...
by Λ. There is, of course, no claim made that the construction is unique.

Well, let us embark now into the journey of proving Theorem 8. Assume for this purpose that Ω is a non-empty open subset of \( \mathbb{C} \) and let \( K \) be a compact subset of Ω. Let \( δ > 0 \) satisfy \( δ \leq \text{dist}(K, \mathbb{C}\setminus Ω) \) and consider the grid formed by drawing horizontal and parallel lines in the plane such that the distance between two consecutive horizontal lines is \( δ \); and the same holds for vertical lines. The grid determines squares of sides of length \( δ \); there is only a finite number that intersect \( K \); specifically let us denote by \( Q_1, Q_2, \ldots, Q_N \) all the closed squares intersecting \( K \).

A square of sides of length \( δ \) is completely determined by its lower left vertex; let us denote by \( w_k = a_k + ib_k \) the lower left vertex of \( Q_k \) so that

\[
Q_k = \{ z = x + iy \in \mathbb{C} : a_k \leq x \leq a_k + δ, b_k \leq y \leq b_k + δ \}.
\]

For each square \( Q_k \) of lower left vertex \( w_k \), let \( γ_1^k, γ_2^k, γ_3^k, γ_4^k \) be the segments determining the boundary of \( Q_k \), positively oriented, namely

\[
γ_1^k = λ_{w_k, w_k + δ}, \quad γ_2^k = λ_{w_k + δ, w_k + δ + iδ}, \quad γ_3^k = λ_{w_k + δ + iδ, w_k + iδ}, \quad γ_4^k = λ_{w_k + iδ, w_k}.
\]

The set of all these segments is clearly balanced. We will denote by \( ∂Q_k \) the positively oriented boundary of \( Q_k \); that is,

\[
∂Q_k = γ_1^k + γ_2^k + γ_3^k + γ_4^k.
\]

For future reference we notice that since

\[
W_∂Q_k(z) = \begin{cases} 1, & \text{if } z \in Q_k^2, \\ 0, & \text{if } z \in \mathbb{C}\setminus Q_k, \end{cases}
\]

it follows that

\[
\sum_{k=1}^{N} W_∂Q_k(z) = 1 \quad \text{if } z \in K \setminus \bigcup_{k=1}^{N} ∂Q_k^*.
\]

This can also be broken up into

\[
\sum_{k=1}^{N} \sum_{j=1}^{4} \int_{γ_j^k} \frac{dζ}{ζ - z} = 2πi \quad \text{if } z \in K \setminus \bigcup_{k=1}^{N} ∂Q_k^*.
\]

In addition, since the complement of \( Ω \) is clearly included in the unbounded complement of each square, we also have

\[
\sum_{k=1}^{N} \sum_{j=1}^{4} \int_{γ_j^k} \frac{dζ}{ζ - z} = 0 \quad \text{if } z \notin Ω.
\]

Next we remove from the set of line segments any pair consisting of a line segment and its opposite. To be specific, suppose two of our squares share an edge. For illustration purposes assume they share a vertical edge, so one square has left lower vertex at say \( w = a + ib \), the other one at \( (a + δ) + ib \). Then
the oriented line segment \( \lambda_{a+\delta+ib,a+\delta+ib} \) is part of the boundary of the first square; \( \lambda_{a+\delta+ib,a+\delta+ib} \) is part of the boundary of the second; remove both segments. Notice that since we remove one segment beginning at \( a+\delta+ib \) and one ending at \( a+\delta+ib \), and the same goes with \( a+\delta+ib \), this process of removing segments does not destroy the balance. We now end with a balanced set \( \Lambda = \{\lambda_1, \ldots, \lambda_s\} \) of oriented line segments with the following properties:

- If \( \lambda_j \in \Lambda \) then \( \lambda_j^* \subset \Omega \setminus K \). In fact, the diameter of a cube \( Q \) of side length \( \delta \) is \( \sqrt{2}\delta \); if this square intersects \( K \) then it has to be contained in \( \Omega \) since \( \sqrt{2}\delta < 2\delta \leq \text{dist}(K, \mathbb{C}\setminus\Omega) \). It follows that all the line segments in \( \Lambda \) are contained in \( \Omega \). On the other hand, if one of them has a point in common with \( K \), then it is on the boundary of a square \( Q_k \) intersecting \( K \); it thus has to be at least on the boundary of one square adjacent to \( Q_k \), and on pairs of line segments that were removed (some points of \( K \) were vertices of the squares in the grid; they can be on four pairs of sides). In other words, there is no such point left.

\[
\sum_{\nu=1}^{M} \int_{\lambda_{\nu}} \frac{d\zeta}{\zeta - z} = \begin{cases} 
2\pi i & \text{if } z \in K \setminus \bigcup_{k=1}^{N} \partial Q_k^* , \\
0 & \text{if } z \notin \Omega.
\end{cases}
\]

In fact, this is an immediate consequence of (4), (5). The set \( \Lambda \) differs from the set \( \{\gamma_{k}^{(j)} : 1 \leq k \leq N, 1 \leq j \leq 4\} \) only by the removal of pairs of opposing segments. Such a removal will not change the sum of the integrals; as far as the integrals are concerned, we are removing 0.

By Lemma 9, the segments in \( \Lambda \) can be organized into a cycle \( \Gamma \) that consists of closed curves that are made up out of the segments \( \lambda_1, \ldots, \lambda_s \), thus contained in \( \Omega \setminus K \). We have:

\[
W_{\Gamma}(z) = \begin{cases} 
1 & \text{for all } z \in K , \\
0 & \text{if } z \notin \Omega.
\end{cases}
\]

Both equalities follow from (6). Since we can break up these last equalities into a sum of integrals over all the oriented line segments in \( \Lambda \), the second equality is immediate from the second equality in (6). If \( z \in K \), but \( z \) is not on the boundary of any of the original squares \( Q_k \), then the first equality is also immediate from (6); the line segments involved are the same. If \( z \in K \) and on the boundary of a \( Q_k \), it still is at a certain distance away from \( \Gamma^* \) and can be connected by a (short) line segment to a point in the interior of \( Q_k \), a line segment that does not cross any of the curves that make up the cycle \( \Gamma \). Because winding numbers are constant in the connected component of the complement of closed curves, it follows that \( W_{\Gamma}(z) = 1 \) in this case.

The proof of Theorem 8 is almost over. By the second equality in (7), and Theorem 7, Cauchy’s formula

\[
W_{\Gamma}(z) f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]
is valid for all \( z \in \Omega \setminus \Gamma^* \), in particular for all \( z \in K \) where, by the first equality in (7), \( W_T(z) = 1 \). The theorem follows.

8 Runge’s Theorem

We begin proving the following lemma

**Lemma 10** Let \( K \) be a compact subset of \( \mathbb{C} \), let \( U \) be an open, connected subset of \( \mathbb{C} \), assume \( U \cap K = \emptyset \). Let \( z_0, z_1 \in U \). Then \( z \mapsto \frac{1}{z-z_0} \) can be uniformly approximated on \( K \) by polynomials in \( 1/(z-z_1) \).

**Proof.** If \( K = \emptyset \) there is nothing to prove. Otherwise let \( \rho = \text{dist}(z_0, K) \). Assume first that \( |z_1 - z_0| < \rho \). Then we use

\[
\frac{1}{z-z_0} = \frac{1}{z-z_1} \frac{1}{(z-z_1)} = \frac{1}{z-z_1} \frac{1}{1-z_1} = \sum_{n=0}^{\infty} \frac{(z_0-z_1)^n}{(z-z_1)^{n+1}},
\]

for \( |z_0 - z_1| < |z - z_1| \) convergence being uniform for \( z \) in the outside of any disc of radius \( |z_0 - z_1| \) centered at \( z_0 \). The fact that \( |z-z_0| \geq \rho > |z_1 - z_0| \) for all \( z \in K \) implies uniform convergence on \( K \). That means that given \( \epsilon > 0 \), a finite sum \( \sum_{n=0}^{N} \frac{(z_0-z_1)^n}{(z-z_1)^{n+1}} \), which is a polynomial in \( 1/(z-z_1) \), is within \( \epsilon \) of \( 1/(z-z_0) \) for all \( z \in K \). If we can approximate \( 1/(z-z_0) \) by such polynomials, we can actually approximate a polynomial in powers of \( 1/(z-z_0) \) uniformly for \( z \in K \) by polynomials in \( 1/(z-z_1) \). That is, we have: Assume \( |z_0 - z_1| < \rho \). If

\[
P\left( \frac{1}{z-z_0} \right) = c_0 + \frac{c_1}{z-z_0} + \cdots + \frac{c_m}{(z-z_0)^m}
\]

is a polynomial in \( 1/(z-z_0) \), then for every \( \epsilon > 0 \) there is a polynomial

\[
Q\left( \frac{1}{z-z_1} \right) = d_0 + \frac{d_1}{z-z_1} + \cdots + \frac{c_N}{(z-z_0)^N}
\]

such that \( |P\left( \frac{1}{z-z_0} \right) - Q\left( \frac{1}{z-z_1} \right)| < \epsilon \) for all \( z \in K \).

Assume now \( z_1 \in U \) is arbitrary. Since \( U \) is connected, there is a path (piecewise smooth curve) \( \gamma \) from \( z_0 \) to \( z_1 \); say \( \gamma : [0, 1] \to U \), \( \gamma(0) = z_0 \), \( \gamma(1) = z_1 \). Let \( \rho = \text{dist}(\gamma^*, K) \), so \( \rho > 0 \). Since \( \gamma \) is continuous, there is \( \delta > 0 \) such that \( |\gamma(t) - \gamma(s)| < \rho \) for \( s, t \in [0, 1] \), \( |s-t| < \delta \). Let \( n \in \mathbb{N} \) satisfy \( n > 1/\delta \) and partition \( [0, 1] \) into \( n \) intervals \( t_0 < t_1 < t_2 = 2/n < \cdots < t_{n-1} = (n-1)/n < t_n = 1 \) of length \( 1/n < \delta \). By what we proved, every polynomial in powers of \( 1/(z - \gamma(t_{i-1})) \) can be approximated by polynomials in powers of \( 1/(z - \gamma(t_i)) \). Given \( \epsilon > 0 \), we can first find a polynomial \( P_1(1/(z - \gamma(t_1))) \) in powers of \( 1/(z - \gamma(t_1)) \) within \( \epsilon/n \) of \( 1/(z-z_0) \) for all \( z \in K \). Then we can find a polynomial \( P_2(1/(z - \gamma(t_2))) \) in powers of \( 1/(z - \gamma(t_2)) \) within \( \epsilon/n \) of \( P_1(1/(z - \gamma(t_1))) \) for all \( z \in K \). And so forth. The end result is a polynomial
in powers of $1/(z-z_1)$ within $\epsilon$ from $1/(z-z_0)$ for all $z \in K$. \hfill \blacksquare

The most useful case is actually the case proved in full in our textbook, when $U$ is unbounded and we can take $z_1 = \infty$. Polynomials in $1/(z-\infty)$ are to be interpreted simply as polynomials.

**Lemma 11** Let $K$ be a compact subset of $\mathbb{C}$, let $U$ be an open, connected, unbounded subset of $\mathbb{C}$, assume $U \cap K = \emptyset$. Let $z_0, z_1 \in U$. Then $z \mapsto \frac{1}{z-z_0}$ can be uniformly approximated on $K$ by polynomials.

**Proof.** Let $R > 0$ be such that $K \subset D_R(0)$. Because $U$ is unbounded, there is $z_1 \in U \setminus D_1(R)$. By Lemma 10, we can approximate $1/(z-z_0)$ by polynomials in powers of $1/(z-z_1)$; it suffices to prove $1/(z-z_1)$ can be approximated by polynomials. We write

$$\frac{1}{z-z_1} = \frac{1}{z_1} \cdot \frac{1}{1 - \frac{z}{z_1}} = \sum_{n=0}^{\infty} z_1^{-n-1} z^n,$$

the series converges for $|z/z_1| < 1$; uniformly for $|z/z_1| \leq r$ if $r < 1$. If $z \in K$ then $|z/z_1| < r$ with (for example) $r = R/|z_1| < 1$. The result follows. \hfill \blacksquare

We are almost ready to state and prove Runge’s Theorem. Just a few preliminary observations; if $U$ is an open subset of $\mathbb{C}$, $U$ has at most a countable number of connected components. In fact, in the first place the connected components are all open. Thus every connected component will contain at least one point of the form $x + iy$ with $x, y \in \mathbb{Q}$ (in fact, it will contain an infinite number of such points, but one is enough). There is only a countable number of such points to go around, so that limits the number of components to being at most countable.

A rational function whose denominator has zeros at $z_1, \ldots, z_r$ can be expressed as a sum of polynomials in powers of $z$, $1/(z-z_1), \ldots, 1/(z-z_r)$. This is just the partial fractions decomposition; powers of $z$ appear if the numerator has degree $\geq$ than the denominator.

**Theorem 12** (Runge) Let $K$ be a compact subset of $\mathbb{C}$, let $\{V_j\}_{j \in J}$ be the family of connected components of $\mathbb{C} \setminus K$, where $J = \mathbb{N}$ or $J = \{1, \ldots, m\}$, with $V_1$ being the unbounded connected component of $\mathbb{C} \setminus K$. For each $j \in J$ let $\alpha_j \in V_j$; for $j = 1$ (the unbounded case) we allow $\alpha_1 = \infty$. Let $S = \{a_j : j \in J\}$, so $S$ is a finite or countably infinite set.

Let $\Omega$ be an open subset of $\mathbb{C}$ such that $K \subset \Omega$. If $f : \Omega \to \mathbb{C}$ is holomorphic, $f$ can be approximated uniformly on $K$ by rational functions all of whose singularities are points of $S$.

**Proof.** What needs to be proved is: If $f : \Omega \to \mathbb{C}$, if $\epsilon > 0$, there exist polynomials $P, Q$ (powers of $z$) such that $Q(z) = 0$ implies $z \in S$ and $|f(z) - \frac{P(z)}{Q(z)}| < \epsilon$. In addition, if $\alpha_1 \neq \infty$, then the degree of $Q$ is $> \deg P$. If $\alpha_1 = \infty$, then $Q(z) \neq 0$ for all $z \in V_1$. An equivalent formulation, which
is how we will prove it, is that there exist ordinary polynomials $P_1, \ldots, P_r$ and indices $j_1, \ldots, j_r \in J$ such that

$$
\left| f(z) - \sum_{\nu=1}^{r} P_{\nu} \left( \frac{1}{z - \alpha_{j_{\nu}}} \right) \right| < \epsilon
$$

for all $z \in K$; if one of the $\alpha_{j_{\nu}}$’s is $\infty$ then $P_{\nu} \left( \frac{1}{z - \infty} \right) = P_{\nu}(z)$.

The actual proof begins exactly as in Stein-Shakarchi. But let us denote by $\mathcal{R}$ the set of all rational functions with singularities only at points of $S$ so as to avoid repeating too many times the same phrase. Moreover, we may assume $K \neq \emptyset$ to avoid wasting time on trivialities. Assume $\Omega$ is an open subset of $\mathbb{C}$ containing $K$ and let $f : \Omega \to \mathbb{C}$ be holomorphic. By Theorem 8, there is a cycle $\Gamma = \sum_{j=1}^{m} \gamma_j$, each $\gamma_j$ being a closed curve in $\Omega \setminus K$ such that

$$
f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta
$$

for all $z \in K$. It will suffice to prove that each $\int_{\gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta$ can be approximated by elements of $\mathcal{R}$, uniformly in $K$. As proved in Stein-Shakarchi, we can approximate a function given by an integral of the form

$$
\int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta,
$$

where $\gamma$ is a curve in the complement of $K$, uniformly on $K$ by rational functions with singularities on $\gamma^*$, thus in $\Omega \setminus K$. Each one of such rational functions has only a finite number of singularities and these will be in $\bigcup_{j \in J} V_j$. Using partial fractions, give $\epsilon > 0$ there is a finite number of points $z_1, \ldots, z_r \in \bigcup_{j \in J} V_j$, ordinary polynomials $P_1, \ldots, P_r$, such that

$$
\left| f(z) - \sum_{j=1}^{r} P_j \left( \frac{1}{z - z_j} \right) \right| < \epsilon/2
$$

for all $z \in K$. Of course, this number $r$ of points can be enormous. Each $z_j$ is in some $V_j$; by Lemmas 10, 11 then $P(1/(z - z_j))$ can be uniformly approximated (to within $\epsilon/2r$) on $K$ by a polynomial in $1/(z - \alpha_j)$. And we are done. ■

As mentioned, powers in $1/(z - \infty)$ are to be interpreted as polynomials; we thus have

**Corollary 13** Let $K$ be a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected. Every holomorphic function on an open set $\Omega$ containing $K$ can be approximated uniformly on $K$ by polynomials.