Calculus III (MAC 2313-17772) 
Homework #1 
Solutions.

1. Calculate the given quantity if 
   \[ a = i + j - 2k \quad b = 3i - 2j + k \quad c = j - 5k \]

(a) \( a \cdot b \).
(b) \( a \times b \).
(c) \( a \cdot (b \times c) \).
(d) \( a \times (b \times c) \).
(e) \( (a \times b) \times c \).
(f) The angle between \( a \) and \( b \) (in radians, correct to 2 places after the decimal point).

SOLUTION.

(a) \( a \cdot b = -1 \).

(b) 
\[
\begin{vmatrix}
i & j & k \\
1 & 1 & -2 \\
3 & -2 & 1 \\
\end{vmatrix} = -3i - 7j - 5k
\]

(c) 
\[
\begin{vmatrix}
i & j & k \\
1 & 1 & -2 \\
3 & -2 & 1 \\
0 & 1 & -5 \\
\end{vmatrix} = 18
\]

(d) 
\[
\begin{vmatrix}
i & j & k \\
3 & -2 & 1 \\
0 & 1 & -5 \\
\end{vmatrix} = 9i + 15j + 3k;
\]

thus
\[
\begin{vmatrix}
i & j & k \\
1 & 1 & -2 \\
9 & 15 & 3 \\
\end{vmatrix} = 33i - 21j + 6k
\]

(e) We use the value of \( a \times b \) calculated in part (b) to get

\[
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \begin{vmatrix}
i & j & k \\
-3 & -7 & -5 \\
0 & 1 & -5 \\
\end{vmatrix} = 40i - 15j - 3k
\]

(f) We have 
\[
\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{-1}{\sqrt{6}\sqrt{14}} = -\frac{1}{2\sqrt{21}}
\]

Applying the arccos function we get \( \theta = 1.68 \) radians (a little bit over 91 degrees).
2. Let \( \mathbf{u}, \mathbf{v} \) be in \( V_3 \); that is, vectors of 3 components. Show: \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal if and only if \( |\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 \).

**SOLUTION.** We have

\[
|\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = |\mathbf{u}|^2 + 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2.
\]

From the equality of the first and last expression it is clear that \( |\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 \) if and only if \( \mathbf{u} \cdot \mathbf{v} = 0 \); that is, if and only if \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal.

3. Given the points \( A(1, 0, 1), B(2, 3, 0), C(-1, 1, 4), \) and \( D(0, 3, 2) \), find the volume of the parallelepiped with adjacent edges \( AB, AC, AD \).

**SOLUTION.** We think of \( AB, AC, AD \) as vectors with origin at \( A \) going to \( B, C, D \), respectively. Then

\[
AB = (1, 3, -1), AC = (-2, 1, 3), AD = (-1, 3, 1).
\]

The volume of the parallelepiped is the absolute value of \( AB \cdot (AC \times AD) \).

Now

\[
AB \cdot (AC \times AD) = \begin{vmatrix} 1 & 3 & -1 \\ -2 & 1 & 3 \\ 1 & 3 & 1 \end{vmatrix} = -6.
\]

The volume is thus \( V = |-6| = 6 \).

4. Find the vector equation, parametric equations and the symmetric equations for the following lines.

(a) The line through the points \((-1, 2, 3)\) and \((4, -3, 2)\).

(b) The line through \((1, 1, -3)\) and perpendicular to the plane \(x - 2y + 3z = 5\).

**SOLUTION.**

(a) As a direction vector for the line we can take any vector parallel to the vector joining the two points; the simplest one is obtained subtracting the coordinates of one point from those of the other. Basing things on the first point, we get the vector equation

\[
\mathbf{r} = (-1, 2, 3) + t(5, -5, -1).
\]

From this equation, the others follow at once. 

**Parametric Equations:**

\[
\begin{align*}
x &= -1 + 5t, \\
y &= 2 - 5t, \\
z &= 3 - t,
\end{align*}
\]

\(-\infty < t < \infty\).

Solving for \( t \) we get the symmetric equations. 

**Symmetric Equations:**

\[
\frac{x + 1}{5} = -\frac{y - 2}{5} = -(z - 3).
\]
(b) The vector $\langle 1, -2, 3 \rangle$ is perpendicular to the plane and can be used as a direction vector for the line.

Parametric Equations:

$$\begin{align*}
  x &= 1 + t, \\
  y &= 1 - 2t, \\
  z &= -3 + 3t,
\end{align*}$$

$-\infty < t < \infty$.

Symmetric Equations:

$$x - 1 = -\frac{y - 1}{2} = \frac{z + 3}{3}.$$ 

5. For the following lines, determine if they are parallel, skew or intersecting. If intersecting, determine their intersection point.

(a) The lines of equations

$$\mathbf{r} = \langle 3, 1, 5 \rangle + t\langle 2, -1, 2 \rangle$$

and

$$\mathbf{r} = \langle -2, 2, 4 \rangle + t\langle 3, 0, -1 \rangle$$

(b) The lines of equations

$$\mathbf{r} = \langle 1, 2, 3 \rangle + t\langle 2, 3, 4 \rangle$$

and

$$\mathbf{r} = \langle -1, 3, -5 \rangle + t\langle 6, -1, 2 \rangle$$

SOLUTION.

(a) Lines are parallel if their direction vectors are parallel, which means that one vector has to be a scalar times the other one. The direction vectors of the lines are $\langle 2, -1, 2 \rangle$ and $\langle 3, 0, -1 \rangle$. They are clearly not parallel; there is no $c$ solving

$$c\langle 2, -1, 2 \rangle = \langle 3, 0, -1 \rangle; \text{ i.e., } \langle 2c, -c, 2c \rangle = \langle 3, 0, -1 \rangle.$$

(This strange scalar $c$ would have to satisfy simultaneously $2c = 3$, $-c = 0$ and $2c = -1$, more than can be expected from a single number.) Do they intersect? If they intersect there is no reason why they should do so for the same value of the parameter $t$, so we will write out the parametric equations of the lines, using different symbols for the parameters. The first line is given by

$$x = 3 + 2t, y = 1 - t, z = 5 + 2t;$$

the second one by

$$x = -2 + 3s, y = 2, z = 4 - s;$$

We now ask whether there are values $s, t$ for which

$$\begin{align*}
  3 + 2t &= -2 + 3s \\
  1 - t &= 2 \\
  5 + 2t &= 4 - s
\end{align*}$$
The second equation gives \( t = -1 \). Using this in the first equation, we get \( s = 1 \). But there is still a third equation to satisfy; we see that with \( t = -1 \), \( s = 1 \) the third equation also holds.

The two lines intersect. The point of intersection is the point corresponding to \( t = -1 \) on the first line, to \( s = 1 \) on the second line; namely \( (1, 2, 3) \).

(b) Proceeding as in the previous exercise, we first see that the lines are not parallel. To check for an intersection, this time we have to solve

\[
\begin{align*}
1 + 2t &= -1 + 6s \\
2 + 3t &= 3 - s \\
3 + 4t &= 5 + 2s
\end{align*}
\]

Solving the first two equations, we get \( t = 1/5 \), \( s = 2/5 \). But these values do not satisfy the third equation; in fact, then

\[3 + 4t = \frac{19}{5} \neq \frac{29}{5} = 5 + 2s.\]

The lines are skew.

6. Find the equation of the following planes.

(a) The plane through \((2, 1, 0)\) and perpendicular to the vector \(\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}\).

(b) The plane through the points \((3, 2, 2)\), \((2, -1, -1)\), and \((1, 2, -1)\).

(c) The plane through \((1, 2, -2)\) that contains the line \(x = 2t\), \(y = 3 - t\), \(z = 1 + 3t\).

SOLUTION.

(a) \[\frac{(x - 2)}{1} + \frac{4(y - 1)}{2} - \frac{3z}{3} = 0\] or \[x + 4y - 3z = 6\]

(b) We take any one of the three points as the base point; say we take the last on, namely \((1, 2, -1)\). The vectors with origin at this point and ending at the two other points are \(\langle 2, 0, 3 \rangle\) and \(\langle 1, -3, 0 \rangle\). Their cross product is thus a vector perpendicular to the plane:

\[
\langle 2, 0, 3 \rangle \times \langle 1, -3, 0 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{vmatrix} = \langle 9, 3, -6 \rangle.
\]

This gives the equation \[9(x - 1) + 3(y - 2) - 6((z + 1)\) or \[9x + 3y - 6z = 21\]

or \[3x + y - 2z = 7\]

(c) (Incidentally, if the point happens to be on the line, there would be no solution). To get the equation of the plane we must know a vector perpendicular to the plane. We are given one vector paralell to the plane, namely the direction vector of the line; that is, the vector \(\langle 2, -1, 3 \rangle\). We need a second such vector, then their cross product will give us a vector perpendicular to the plane. To get the second vector, select any point of the line and take the vector from the given point to that point on the line. For example, setting \(t = 0\) (the simplest choice, but any other value of \(t\) would do) we get the point
(0, 3, 1) on the line; the vector from the given point to this one is \((-1, 1, 3)\). Now

\[
(2, -1, 3) \times (-1, 1, 3) = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ -1 & 1 & 3 \end{vmatrix} = (-6, 9, 1).
\]

The equating works out to \(-6(x - 1) - 9(y - 2) + (z + 2) = 0\) or \(6x + 9y - z = 26\).

7. Coordinate changes

(a) The rectangular (Cartesian) coordinates of a point are \((2, 2, -1)\). Find the cylindrical and spherical coordinates of the point.

**SOLUTION.**

- **Cylindrical:** \(r = \sqrt{8}, \theta = \frac{\pi}{4}, z = -1\).
- **Spherical:** \(\rho = 3, \theta = \frac{\pi}{4}, \phi = \arccos\left(-\frac{1}{3}\right)\).

(b) The cylindrical coordinates of a point are \((2\sqrt{5}, \pi/6, 4)\). Find the rectangular and spherical coordinates of the point.

**SOLUTION.**

- **Rectangular:** \(x = \sqrt{15}, y = \sqrt{5}, z = 4\).
- **Spherical:** \(\rho = 6, \theta = \frac{\pi}{6}, \phi = \arccos\left(\frac{2}{3}\right)\).

(c) The spherical coordinates of a point are \((6, \pi/4, \pi/3)\). Find the rectangular and cylindrical coordinates of the point.

**SOLUTION.**

- **Rectangular:** \(x = 3\sqrt{6}/2, y = 3\sqrt{6}/2, z = 3\).
- **Cylindrical:** \(r = 3\sqrt{3}, \theta = \frac{\pi}{4}, z = 3\).

8. Find the domain of the vector function.

\[
r(t) = \frac{t - 2}{t + 2}i + \sin tj + \ln(9 - t^2)k.
\]

**SOLUTION.**  All components need to be defined, thus we must have \(t \neq -2\) and \(9 - t^2 > 0\). Acceptable answers are: \(|t| < 3\) and \(t \neq -2\) or \((-3, -2) \cup (2, 3)\).

9. Find the limit

\[
\lim_{t \to \infty} \left( \arctan t, \left(1 + \frac{2}{t}\right)^t, te^{-3t} \right).
\]

I hope it is known (if not, please know it!) that

\[
\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a.
\]

The limit is \((\frac{\pi}{2}, e^2, 0)\).

10. Show that the curve with parametric equation \(x = t \cos t, y = t \sin t, z = t^2\) lies on the paraboloid \(z = x^2 + y^2\) and use this fact to help sketch the curve.

**SOLUTION.**  If \((x, y, z)\) are on the curve, then for some \(t\) we have \(x = t \cos t, y = t \sin t, z = t^2\), hence

\[
x^2 + y^2 = (t \cos t)^2 + (t \sin t)^2 = t^2 = z.
\]
The curve lies indeed on the paraboloid. I’ll omit the sketch, but here is a description in words. For \( t = 0 \) the curve is at the lowest point of the paraboloid; as \( t \) increases (or decreases) the curve spirals upward around the paraboloid.

11. Two particles travel along space curves

\[ r_1(t) = (t, t^2, t^3) \quad r_2(t) = (1 + 2t, 1 + 6t, 1 + 14t). \]

Do the particles collide? Do their paths intersect?

**SOLUTION.** Do the particles collide? That is the same as asking if there is some \( t \) for which the particles are at the same point; i.e., a \( t \) satisfying

\[ t = 1 + 2t, t^2 = 1 + 6t, t^3 = 1 + 14t. \]

The first equation is only satisfied for \( t = -1 \); this \( t \) does not satisfy the second equation (nor the third one). Thus the particles do **not** collide.

Do their paths intersect? That is the same as asking if there are \( t, s \) satisfying

\[ t = 1 + 2s, t^2 = 1 + 6s, t^3 = 1 + 14s. \]

Solving the first two equations we get the solutions \( s = 0 \) and \( t = 1 \), and \( s = 1/2 \) and \( t = 2 \). Both choices actually satisfy the third equation. Taking \( s = 0, t = 1 \) we get the point \((1,1,1)\) which lies on both curves. With \( s = 1/2, t = 2 \), we get the point \((2,4,8)\), also on both curves. The paths intersect.

12. Consider the plane curve of vector equation

\[ r(t) = e^t i + e^{-t} j. \]

(a) Sketch the curve. It might help to figure out what equation is satisfied by the points \((x, y)\) on the curve; i.e., by the points \((x, y)\) such that \( x = e^t, y = e^{-t} \) for some \( t \). One can do this by eliminating \( t \) from the equations, getting a relation between \( x \) and \( y \).

(b) Find \( r'(t) \).

(c) Sketch the position vector \( r(t) \) and the tangent vector \( r'(t) \) for \( t = 0 \). (Add this to the sketch of the curve).

**SOLUTION.**

(a) The curve is sketched below, in part c. Notice \( xy = e^t e^{-t} = 1 \); since we also have \( x = e^t > 0, y = e^{-t} > 0 \), the curve is one branch of the hyperbola \( xy = 1 \); specifically, the graph of \( y = 1/x, x > 0 \).

(b) \[ r'(t) = e^t i - e^{-t} j. \]
13. Find the derivative of the following vector functions.

(a) \( \mathbf{r}(t) = \langle \cos 3t, t, \sin 3t \rangle \).
(b) \( \mathbf{r}(t) = 2\mathbf{i} + \mathbf{j} + e^{-2t}\mathbf{k} \).
(c) \( \mathbf{r}(t) = t\mathbf{a} \times (\mathbf{b} + t\mathbf{c}) \), where \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) are constant vectors.

**SOLUTION.**

(a) \( \mathbf{r}'(t) = \langle -3 \sin 3t, 1, 3 \cos 3t \rangle \).
(b) \( \mathbf{r}'(t) = -2e^{-2t}\mathbf{k} \).
(c) \( \mathbf{r}'(t) = \mathbf{a} \times (\mathbf{b} + t\mathbf{c}) + t\mathbf{a} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} + 2t\mathbf{a} \times \mathbf{c} \).

14. Find the parametric and symmetric equations for the tangent line to the curve given by

\[
\begin{align*}
x &= t^2 - 1, \\
y &= t^2 + 1, \\
z &= t + 1
\end{align*}
\]

at the point \((0, 2, 0)\).

**SOLUTION.** The point \((0, 2, 0)\) corresponds to \(t = -1\). The derivative of the position vector of the curve for \(t = 1\) is \(\langle -2, -2, 1 \rangle\). The parametric equations of the tangent line are

\[
\begin{align*}
x &= -2t, \\
y &= 2 - 2t, \\
z &= t
\end{align*}
\]

The symmetric equations are

\[
\begin{align*}
-\frac{x}{2} &= -\frac{y - 2}{2} = z.
\end{align*}
\]

15. If \( \mathbf{r}(t) = \langle t \cos t, t \sin t, e^t \rangle \), find \( \mathbf{r}'(t) \), \( \mathbf{T}(t) \), \( \mathbf{r}''(t) \), and \( \mathbf{r}'(t) \times \mathbf{r}''(t) \).

**SOLUTION.**

\[
\mathbf{r}'(t) = \langle \cos t - t \sin t, \sin t + t \cos t, e^t \rangle
\]

Then

\[
|\mathbf{r}'(t)| = |\langle \cos t - t \sin t, \sin t + t \cos t, e^t \rangle| = \sqrt{1 + t^2 + e^{2t}};
\]
thus
\[ T(t) = \frac{1}{\sqrt{1 + t^2 + e^{2t}}} (\cos t - t \sin t, \sin t + t \cos t, e^t) \]

Next
\[ r''(t) = \langle -2 \sin t - t \cos t, 2 \cos t - t \sin t, e^t \rangle \]

Finally,
\[
\begin{vmatrix} 
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos t - t \sin t & \sin t + t \cos t & e^t \\
-2 \sin t - t \cos t & 2 \cos t - t \sin t & e^t \\
\end{vmatrix}
\]
\[ = \langle (\sin t - 2 \cos t + t \cos t + t \sin t) e^t, (t \sin t - \cos t - t \cos t - 2 \sin t) e^t, 2 + t^2 \rangle \]

16. Find \( r(t) \) if \( r'(t) = \sin t \mathbf{i} - \cos t \mathbf{j} + 2t \mathbf{k} \) and \( r(0) = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \).

**SOLUTION.**
\[
r(t) = \int (\sin t \mathbf{i} - \cos t \mathbf{j} + 2t \mathbf{k}) \, dt = (-\cos t + A) \mathbf{i} - (\sin t + B) \mathbf{j} + (t^2 + C) \mathbf{k},
\]
where \( A, B, C \) are constants. We have to determine these constants so that \( r(0) \) is as given; i.e., so that \((-1 + A) \mathbf{i} - B \mathbf{j} + C \mathbf{k} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}\). Thus \( A = 2, B = -1, C = 2 \). The answer is
\[
r(t) = (-\cos t + 2) \mathbf{i} - (\sin t - 1) \mathbf{j} + (t^2 + 2) \mathbf{k}.
\]