Calculus III (MAC 2313-17772)  
Exam #1 – October 1, 2004  
SOLUTIONS

1. Determine the indicated equations. (5 points each)

(a) The symmetric equations of the line through the point (1, 2, 0), perpendicular to the plane 5x – 3y + z = 0.

**SOLUTION.**

\[
\frac{x - 1}{5} = \frac{y - 2}{-3} = z.
\]

(b) The equation of the plane through the points (1, 1, 0), (1, 0, 1) and (0, 1, 1).

**SOLUTION.** Basing things on the first point, the vectors \( \langle 0, -1, 1 \rangle \) and \( \langle -1, 0, 1 \rangle \) are parallel to the plane. Thus

\[
\langle 0, -1, 1 \rangle \times \langle -1, 0, 1 \rangle = \begin{vmatrix} i & j & k \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = (-1, -1, -1)
\]

is perpendicular to the plane. Any of the following three answers is acceptable, though the first one is not very nice looking.

\[
-(x - 1) - (y - 1) - z = 0 \quad \text{or} \quad (x - 1) + (y - 1) + z = 0 \quad \text{or} \quad x + y + z = 2
\]

2. (5 points) Find the limit

\[
\lim_{t \to \infty} (t^2 e^{-t}, \frac{\ln t}{\sqrt{t}}, \left(1 + \frac{1}{t}\right)^t).
\]

**SOLUTION.**

\[
\lim_{t \to \infty} (t^2 e^{-t}, \frac{\ln t}{\sqrt{t}}, \left(1 + \frac{1}{t}\right)^t) = (0, 0, e).
\]

3. The position vector of a particle moving in space is given by

\[
r(t) = e^t i + e^{-t} j + t^2 k.
\]

Answer the following questions.

(a) (5 points) What is the velocity and acceleration at time \( t = 0 \)?

**SOLUTION.** We have

\[
v(t) = r'(t) = e^t i - e^{-t} j + 2t k, \quad a(t) = r''(t) = e^t i + e^{-t} j + 2k.
\]

Thus

\[
v(0) = i - j, \quad a(0) = i + j + 2k.
\]
(b) (5 points) What are the coordinates of the point at which it crosses the plane \( z = 4 \)?

**SOLUTION.** The third component of the curve is 4 when \( t \) satisfies \( t^2 = 4 \); that is, \( t = \pm 2 \). The curve crosses the plane \( z = 4 \) twice, once for \( t = -2 \) at the point \((e^{-2}, e^2, 4)\), the second time for \( t = 2 \) at the point \((e^2, e^{-2}, 4)\).

4. (10 points) Compute the arc length of the curve given by
\[
r(t) = t^2 \mathbf{i} + 2t \mathbf{j} + \ln t \mathbf{k}
\]
for \( 1 \leq t \leq e \).

**SOLUTION.**
\[
r'(t) = 2t \mathbf{i} + 2 \mathbf{j} + \frac{1}{t} \mathbf{k},
\]
thus
\[
|r'(t)| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} = \left| \frac{2t^2 + 1}{t} \right|.
\]
In the interval \([1, e]\) both \( t \) and (of course) \( 2t^2 + 1 \) are positive, so we don’t need the absolute value and the arc length is
\[
L = \int_{1}^{e} \frac{2t^2 + 1}{t} \, dt = \int_{1}^{e} \left( 2t + \frac{1}{t} \right) \, dt = [\text{C}].
\]

5. (15 points) Find the vectors \( \mathbf{T}, \mathbf{N}, \mathbf{B} \) for the curve
\[
r(t) = \left\langle t^2, \frac{1}{3} t^3, t \right\rangle
\]
at the point \((1, -\frac{1}{3}, -1)\).

**SOLUTION.** The point \((1, -\frac{1}{3}, -1)\) corresponds to \( t = -1 \). We have
\[
r'(t) = \langle 2t, t^2, 1 \rangle,
\]
\[
|r'(t)| = \sqrt{4t^2 + t^4 + 1}.
\]
It might be best (though not necessary) to order the powers in the square root and write the tangent vector in the form
\[
\mathbf{T}(t) = (t^4 + 4t^2 + 1)^{-1/2}(2t, t^2, 1).
\]
Setting \( t = -1 \) we get one third of the required objects:
\[
\mathbf{T}(-1) = \frac{1}{\sqrt{6}} \langle -2, 1, 1 \rangle.
\]
Next,
\[ T'(t) = -\frac{1}{2}(t^4 + 4t^2 + 1)^{-3/2}(4t^3 + 8t)(2t, t^2, 1) + (t^4 + 4t^2 + 1)^{-1/2}(2, 2t, 0). \]

This is, of course, quite nasty, and if we had to differentiate it again we could have some sort of nervous breakdown. But we don’t have to differentiate it any more, so we can just set \( t = -1 \) now, before doing anything else. Things get better. One gets (believe it or not!):
\[ T'(-1) = \frac{1}{\sqrt{6}} (0, -1, 1). \]

Since
\[ |T'(-1)| = \frac{1}{\sqrt{6}} \cdot \sqrt{2} = \frac{1}{\sqrt{3}}, \]
we get the second required object in the form
\[ N(-1) = \frac{1}{\sqrt{2}} (0, -1, 1). \]

Finally
\[ T(-1) \times N(-1) = \frac{1}{\sqrt{6}\sqrt{2}} (-2, 1, 1) \times (0, -1, 1) = \frac{1}{2\sqrt{3}} (2, 2, 2), \]
thus (factoring out a 2 and canceling with the 2 in the denominator)
\[ B(-1) = \frac{1}{\sqrt{3}} (1, 1, 1). \]

6. (10 points each) Find the first partial derivatives of the following functions.

(a) \( f(x, y, z) = x^2e^{yz}. \)

\[ f_x = 2xe^{yz}, \quad f_y = x^2ze^{yz}, \quad f_z = x^2ye^{yz}. \]

(b) \( g(x, y) = xe^{y^2} + ye^{x^2}. \)

\[ g_x = e^{y^2} + 2xye^{x^2}, \quad g_y = 2xe^{y^2} + e^{x^2}. \]

(c) \( h(x, y) = \int_x^y e^{-t^2} \, dt. \)

\[ h_x = -e^{-x^2}, \quad h_y = e^{-y^2}. \]
7. (10 points each) Find the indicated partial derivatives.

(a) \( f(x, y) = 3xy^4 + x^3y^2 \), find \( f_{xxy}, f_{yyy}. \) **SOLUTION.**

\[
\begin{align*}
f_{xxy} &= 12xy, \\
f_{yyy} &= 72xy.
\end{align*}
\]

(b) \( w = \frac{x}{y + z} \), find \( \frac{\partial^3 w}{\partial z \partial y \partial x}, \frac{\partial^3 w}{\partial^2 x \partial y}. \) **SOLUTION.**

\[
\begin{align*}
\frac{\partial^3 w}{\partial z \partial y \partial x} &= 2(y + z)^{-3}, \\
\frac{\partial^3 w}{\partial^2 x \partial y} &= 0.
\end{align*}
\]