SERIES

We refer to the textbook, or to what was done in class, for the definition of series (a.k.a. infinite series). An important point is that when we are given the series denoted by

\[ \sum_{n=1}^{\infty} a_n, \]

nobody is telling us to add an infinite number of terms, or to add till we drop, or anything like it. When one writes

\[ \sum_{n=1}^{\infty} a_n = L \]

where \( L \) is a number, one means \textbf{PRECISELY} the following:

1. Form the \textit{sequence} (don’t confuse sequences and series)

\[ a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, \]

2. The limit of this sequence is \( L \).

\textbf{In other words:} To a \textit{series} \( \sum_{n=1}^{\infty} a_n \) one associates the \textit{sequence} of partial sums \( \{s_n\} \), where \( s_1 = a_1 \), \( s_2 = a_1 + a_2 \), in general

\[ s_n = \sum_{k=1}^{n} a_k \]

for \( n = 1, 2, \ldots \); most properties of the series are defined in terms of this sequence.

Here are some examples.

**Example 1.** Let \( a_1 = -1, \ a_2 = 3, \ a_3 = 5 \) and \( a_n = 0 \) if \( n > 3 \). Then

\[ \sum_{n=1}^{\infty} a_n = 7. \]

Why? Because if we form the sequence of partial sums we get

\[ -1, 2, 7, 7, 7, 7, 7, 7, \ldots, \]

a sequence which clearly converges to 7.

**Example 2.** Consider the series

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots \]
If we start forming partial sums, a pattern might emerge.

\[
s_1 = \frac{1}{2}, \\
s_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, \\
s_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}, \\
s_4 = s_3 + \frac{1}{20} = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}
\]

At this stage, we should make a conjecture and try to prove it. Could it be that \( s_n = n/(n+1) \)? Proving it means to show that the pattern will continue (also known as mathematical induction), notice that if \( s_n \) has the conjectured form, then

\[
s_{n+1} = s_n + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}
\]

showing the pattern will go on repeating once it started, so we now can say

\[
s_n = \frac{n}{n+1}
\]

for \( n = 1, 2, \ldots \). Because \( \lim_{n \to \infty} s_n = 1 \), we have shown:

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.
\]

**Example 3.** Let \( x \) be a real number. We consider the series

\[
1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n.
\]

It is more convenient here to start counting at 0, so the first term is \( x^0 = 1 \). We also start the partial sums at 0, so that we have \( s_0 = 1, s_1 = 1 + x, \) etc. The general expression of the partial sum is

\[
s_n = \sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n.
\]

There are several ways to get an expression for \( s_n \). For example, one can notice that the following equation is true:

\[
x s_n = s_n + x^{n+1} - 1,
\]

and solve for \( s_n \). One gets \( s_n = (1 - x^{n+1})/(1 - x) \), at least if \( x \neq 1 \). If \( x = 1 \), then the series is \( 1 + 1 + 1 + \cdots \), with \( s_0 = 1, s_1 = 2, s_2 = 3, \) in general, \( s_n = n + 1 \). Suppose first \( x \neq 1 \). In the expression

\[
s_n = \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x} - \frac{x^{n+1}}{1 - x}
\]
only $x^{n+1}$ depends on $n$. We will use that
\[ \lim_{n \to \infty} x^{n+1} = 0 \text{ if } |x| < 1, \]
and does not exist (or is infinite) when $x \leq -1$ or $x > 1$. So for $x > 1$ or
$x < -1$, the series diverges, for $|x| < 1$ it converges:
\[ \sum_{n=0}^{\infty} x^n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( \frac{1}{1-x} - \frac{x^{n+1}}{1-x} \right) = \frac{1}{1-x} - \frac{0}{1-x}. \]
Since $\lim_{n \to \infty} n+1 = \infty$ we also have divergence for $x = 1$. We can collect
our results in the form: The geometric series of ratio $x$ converges if and
only if $|x| < 1$; we have
\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ if } |x| < 1. \]

**Exercise.** Analyze for convergence. In case of convergence find the sum.

1. $\sum_{n=0}^{\infty} 2^{-n}$.

2. $\sum_{n=0}^{\infty} (-1)^n (3/4)^n$.

3. $\sum_{n=0}^{\infty} (4/3)^n$.

4. $\sum_{n=2}^{\infty} (1/3)^n$.

**Answers** All are geometric series.

1. The ratio is $1/2$, hence
\[ \sum_{n=0}^{\infty} 2^{-n} = \frac{1}{1 - \frac{1}{2}} = 2. \]

2. The ratio is $-3/4$, since $|-3/4| = 3/4 < 1$,
\[ \sum_{n=0}^{\infty} (-1)^n \left( \frac{3}{4} \right)^n = \frac{1/4}{1 + 3/4} = \frac{4}{7}. \]

3. The ratio is $4/3$ is larger than 1; the series diverges.
4. The ratio is $1/3$, but there is a slight difference, the series starts with the ratio to the power 2. Here is how we proceed:

$$\sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^{2} + \left(\frac{1}{3}\right)^{3} + \left(\frac{1}{3}\right)^{3} + \cdots$$

$$= \left(\frac{1}{3}\right)^{2} (1 + \frac{1}{3} + \left(\frac{1}{3}\right)^{2} + \cdots)$$

$$= \left(\frac{1}{3}\right)^{2} \left(\frac{1}{1 - \frac{1}{3}}\right) = \frac{1}{6}.$$

We used a rule here: If $\sum_{n=1}^{\infty} a_n$ converges and $k$ is a real number, then $\sum_{n=1}^{\infty} ka_n$ also converges and, in fact,

$$\sum_{n=1}^{\infty} ka_n = k \sum_{n=1}^{\infty} a_n.$$

**Exercise.** Evaluate

$$\sum_{n=0}^{\infty} \frac{2^n + 5 \cdot 3^n}{7^n}.$$

Here we use another simple rule which says that IF the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then the series $\sum_{n=1}^{\infty} (a_n + b_n)$ also converges and, in fact,

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

Now we solve the exercise by

$$\sum_{n=0}^{\infty} \frac{2^n + 5 \cdot 3^n}{7^n} = \sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n + 5 \sum_{n=0}^{\infty} \left(\frac{3}{7}\right)^n = \frac{203}{20}.$$

Most series are quite hard to add up. For most series it is hard (and frequently impossible) to find a nice pattern for the sequence of partial sums. Also, given that only very few numbers have names associated with them, what right do we have to expect that a series always ends by being such a number? For example, why couldn’t a series work out to a number like

$$1.23456789101112131415161718192021222324252627282930...$$

or worse, maybe a number with no discernible pattern whatsoever in its decimal expansion? Only a few such numbers have names. It becomes of some importance to be able to say whether a series converges or diverges without having to find its limit (sum), an impossibility in most cases. A number of tests or criteria are available to decide convergence (or divergence). Here is a first one.
1. **Divergence criterion.** (See textbook, [6] or [7], Chapter 11.2, p. 708) If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

**Example.** Analyze the following series for convergence or divergence

1. $\sum_{n=1}^{\infty} (-1)^{n+1}$.
2. $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$.
3. $\sum_{k=1}^{\infty} \frac{k - 5}{k + 2}$.

**Analysis.**

1. Since $\lim_{n \to \infty} (-1)^n$ does not exist (much less can be 0), the series diverges.
2. Since $\lim_{n \to \infty} n \sin(1/n) = 1$ (and $1 \neq 0$), the series diverges.
3. Since $\lim_{k \to \infty} (k - 5)/(k + 2) = 1$ (and $1 \neq 0$), the series diverges.

**SOME NOTES ON COMMON MISTAKES.** Finding sums of series is hard and can be so hard as to baffle even very experienced mathematicians. Only a genius of the first magnitude can look at a series he or she has never seen before and take only a second (or a few minutes) to know what its sum (also called its limit) is. So, if you can do this, if you can say what the sum of a series is just by a quick inspection, you are either a great genius or totally confused and wrong. One common mistake is to confuse

$$\lim_{n \to \infty} a_n$$

with

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k.$$
Since \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \), we have \( \sum_{n=1}^{\infty} \frac{1}{n^2} = 0 \).

Wouldn’t it be nice if things were so easy? Because both series satisfy \( \lim_{n \to \infty} a_n = 0 \), both could converge. However, **NOT TO 0**. That’s quite impossible, the terms of the series are positive so with each additional term you add, you get further away from 0. As it turns out, the first series diverges, the second one converges, as the following criterion, known as the **integral test** shows.

**Integral Test.** (Textbook, page 715). Suppose all the terms \( a_n \) of the series \( \sum_{n=1}^{\infty} a_n \) are positive and \( a_1 \geq a_2 \geq \cdots \). Suppose we can find a continuous, decreasing function \( f(x) \) such that \( f(n) = a_n \) for all positive integers \( n \). Then: The series \( \sum_{n=1}^{\infty} a_n \) converges if and only if the improper integral

\[
\int_1^{\infty} f(x) \, dx
\]

converges. (Please, look at the pictures on page 717 of the textbook.)

With this test, we can repeat the previous exercise, at least the testing for convergence part.

(a) The series

\[
\sum_{n=1}^{\infty} \frac{1}{n}
\]

diverges because

\[
\int_1^{\infty} \frac{dx}{x} = \lim_{b \to \infty} \int_1^{b} \frac{dx}{x} = \lim_{b \to \infty} \ln b = \infty.
\]

(b) The series

\[
\sum_{n=1}^{\infty} \frac{1}{n^2}
\]

converges because

\[
\int_1^{\infty} \frac{dx}{x^2} = \lim_{b \to \infty} \int_1^{b} \frac{dx}{x^2} = \lim_{b \to \infty} \left( 1 - \frac{1}{b} \right) = 1.
\]

Is the series equal to the integral, in case of convergence? One would wish, but the answer is “yes, when pigs fly.” That is, no, not a chance. The mathematician Leonhard Euler (a genius of the first magnitude) was the first to figure out that

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},
\]

not an obvious result at all (I think). Series are usually MUCH harder to sum than integrals. The integral test tests for convergence. It can actually tell you a little about the sum, but it does not give the actual value of the sum.
Exercise. Let \( p \) be a real number. Show that the series

\[
\sum_{n=1}^{\infty} \frac{1}{n^p}
\]

converges if and only if \( p > 1 \).

Once one has a number of series classified as convergent or divergent, one can get others by several comparison tests. While every term plays a role in what the sum of the series is, only the terms with very large index play a role in convergence.

Limit Comparison Test. (see Textbook, page 723) Suppose we want to decide if a series of non-negative terms \( \sum_{n=1}^{\infty} a_n \) converges. If we can find a series \( \sum_{n=1}^{\infty} b_n \), also of non-negative terms, of which we know whether it converges or diverges and are able to compute

\[
L = \lim_{n \to \infty} \frac{a_n}{b_n}
\]

Then:

1. If \( 0 < L < \infty \) (best case, case to strive for) both series have the same convergence behavior. That is, if \( \sum_{n=1}^{\infty} b_n \) converges, so does \( \sum_{n=1}^{\infty} a_n \); if \( \sum_{n=1}^{\infty} b_n \) diverges, so does \( \sum_{n=1}^{\infty} a_n \).

2. If \( L = 0 \), then the terms \( a_n \) get much smaller than the \( b_n \) terms, so we can only say: If \( L = 0 \) and if \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.

3. If \( L = \infty \), then it is the \( a_n \)’s who grow most; in this case, if \( \sum_{n=1}^{\infty} b_n \) diverges, then \( \sum_{n=1}^{\infty} a_n \) diverges.

All this is not too hard to remember. First of all, it refers to series of non-negative terms. Second, what the limit \( L \) measures is how the terms of the first series compare with those of the second one.

Exercises. Analyze for convergence:

1. \[
\sum_{n=2}^{\infty} \frac{1}{\ln n}
\]

2. \[
\sum_{n=2}^{\infty} \frac{1}{n \ln n}
\]

Solution:

1. We can compare with \( \sum_{n} 1/n \). Since

\[
\lim_{n \to \infty} \frac{\frac{1}{n}}{1/n} = \lim_{n \to \infty} \frac{n}{\ln n} = \infty,
\]

this series behaves worse than the series \( \sum_{n} 1/n \), hence diverges.
2. Suppose we compare again with $\sum_n 1/n$. We get

$$\lim_{n \to \infty} \frac{1}{n \ln n} = \lim_{n \to \infty} \frac{1}{\ln n} = 0.$$ 

What does this tell us? Not much, merely that the series is better than $\sum_n 1/n$ but, since this last series diverges, to be better may not be good enough. In fact, the series in question diverges, as one can see using the integral test (Do this?).

Summary. In the limit comparison test we compare the series of terms $a_n$ with one of terms $b_n$; of the last one we know whether it converges or diverges. We consider the ratio $a_n/b_n$. If this ratio stays away from zero and from $\infty$, both series exhibit the same convergence behavior. If the ratio goes to 0, the $a_n$ series is “better” and will converge if the $b_n$ series does. If the ratio goes to $\infty$, the $a_n$ series is “worse” than the $b_n$ series; if the latter one diverges, so will the former one. The test should be applied ONLY to series of non-negative terms.

**Alternating Series.** These are series in which positive terms alternate with negative ones. Here are a few examples.

1. \(\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 \pm \cdots\),
2. \(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \cdots\),
3. \(\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} = \frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} \pm \cdots\),
4. \(\sum_{n=0}^{\infty} (-3)^n = 1 - 3 + 9 - 27 + \cdots\).

The general form of an alternating series is

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n,$$

where $a_n > 0$ for all positive integers $n$. For example a series that starts like

$$1 - \frac{1}{2} \frac{1}{3} \pm \cdots$$

already missed the alternating boat; alternating means that if a term is positive, the next one is negative; and vice-versa.

There is a very simple and reasonable criterion for convergence of these series, it almost says alternating series converge if they have any chance to do so. Specifically:
Alternating Series Convergence Criterion. Assume $a_1 \geq a_2 \geq a_3 \geq \cdots$ and $\lim_{n \to \infty} a_n = 0$. Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n,$$

converges.

Of the alternating series given in the examples, two converge and two diverge. Can you tell which?

It may be instructive to understand why the alternating series criterion works. We’ll say something in class; here we remark that if $s_n$ denotes the $n$-th partial sum of the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, if the conditions of the criterion hold and if the sum of the series is $s$, then $|s - s_n| \leq a_n$. This estimate is actually fairly accurate.

Exercise. It is known that the alternating series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$ converges to $\frac{\pi}{4}$. Suppose we want to use the series to evaluate $\frac{\pi}{4}$ with an error $\leq 10^{-4}$. How many terms of the series are needed?

Solution. We find $k$ so that $\frac{1}{2k+1} \leq 10^{-4}$. We get (since $k$ has to be an integer) $k = 5,000$, we need five thousand terms for such a precision.

Absolute and Conditional Convergence. Not all convergences are equally good. For example, consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. It converges, but not very well. It converges because there is some compensation; the positive terms are compensated by the negative terms. If we take the series of absolute values (i.e., replace each term by its absolute value), we get the series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. We say that the original series converges conditionally. On the other hand, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely, because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

The formal definitions are: A series $\sum_{n=1}^{\infty} a_n$ converges absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Notice that the definition does NOT require that the series $\sum_{n=1}^{\infty} a_n$ converge, only that the series of absolute values should converge. It is, however, an important fact that if a series converges absolutely, it converges. Given a series $\sum_{n=1}^{\infty} a_n$ which converges absolutely, let $S$ be the sum of the series of absolute values:

$$\sum_{n=1}^{\infty} |a_n| = S.$$
What can one say about the sum of $\sum_{n=1}^{\infty} a_n$? Not much; it exists but could almost be anything as long as it isn’t larger than $S$. About all one can say is that one has

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|$$

and if you can find a better relation which holds in a general sense, please let me know. Of course, if all terms of the series are positive (or negative), then absolute convergence is the same as convergence.

A series which converges but which does not converge absolutely is said to converge conditionally: The series $\sum_{n=1}^{\infty} a_n$ converges conditionally if it converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Conditionally convergent series are treacherous objects, they do weird stuff. We may see some of it later on, for now let’s keep in mind that absolute is better. There is also the fact that series of positive terms are easier to handle than those having both positive and negative terms, so there are zillions of convergence tests for them, which can be also seen as tests for absolute convergence. Here is an important fact about series of positive terms, which lies behind all the comparison tests:

**Important Fact:** Assume $a_1, a_2, a_3, \ldots$ are all non-negative; that is, $a_n \geq 0$ for all positive integers $n$. Then the series $\sum_{n=1}^{\infty} a_n$ either converges or it diverges to $\infty$. Convergence happens if and only if all the partial sums are bounded; that is, if there exists a number $M$ such that

$$s_n = \sum_{k=1}^{n} a_k \leq M$$

for all $n$.

The reason for this fact is actually simple. If all terms are non-negative, then the sequence of partial sums $\{s_n\}$ is monotone increasing and will converge if and only if it is bounded above.

**Example.** We’ll use this fact to show

(a) The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \cdots$$

converges;

(b) give a new argument showing the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges.

(a) If $n \geq 2$, then $n! = n \cdot (n - 1) \cdots 2$ is a product of $n - 1$ factors each of which is $\geq 2$, so that $n! \geq 2^{n-1}$ for $n \geq 2$. Since $1! = 1 = 2^0$, the inequality also
holds (as an equality) for \( n = 1 \). We thus get

\[
s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + \left( \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right)
\]

\[
\leq 1 + \left( \frac{1}{2^0} + \frac{1}{2^1} + \cdots + \frac{1}{2^{n-1}} \right)
\]

\[
= 1 + \frac{1 - 2^{-n}}{1 - 2^{-1}}
\]

where we used the formula for sum of a geometric progression (of ratio \( \frac{1}{2} = 2^{-1} \)). Since

\[
1 + \frac{1 - 2^{-n}}{1 - 2^{-1}} = 1 + (2 - 2^{1-n}) < 3,
\]

we have shown that \( s_n \leq 3 \) for all \( n \), so the sequence of partial sums is bounded above; the series converges. Do you know what the limit is? Can you provide an alternative proof of convergence using the limit comparison test? (Please, answer yes to the second question. And justify your answer)

(b) Suppose we write out \( s_n \) for a very large \( n \), suppose also this \( n \) is a power of 2, say \( n = 2^m \). We can write

\[
s_n = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \left( \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \right) - \cdots + \left( \frac{1}{2^{m-1} + 1} + \frac{1}{2^{m-2} + 2} + \cdots + \frac{1}{2^m} \right).
\]

The parentheses were placed there for a reason. In every case, the terms grouped between parentheses add up to more than \( \frac{1}{2} \): Since \( \frac{1}{3} \geq \frac{1}{4}, \frac{1}{3} + \frac{1}{4} \geq 1/4 + 1/4 = 1/2 \), of the four next terms in parentheses, namely \( \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8} \), the smallest one is \( 1/8 \) so that

\[
\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq \frac{1}{8} = \frac{1}{2};
\]

similarly,

\[
\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \geq \frac{1}{6} = \frac{1}{2}
\]

and so forth. We start with \( 1 + (1/2) \) followed by (am I right?) \( m - 2 \) such groupings so that

\[
s_n \geq 1 + \frac{1}{2} + (m - 2)\frac{1}{2} = \frac{m + 1}{2}.
\]

Here are two questions you should be able to answer fast, without need of a calculator, if you understood this.
1. Explain why
\[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{1000000} \geq 11.5 \]

(Hint: \(2^{19} < 1000000 < 2^{20}\))

2. Find \(n\) such that
\[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \geq 10000 \]

To conclude with the example, since we can make \(\frac{m+1}{2}\) very large (the limit as \(m \to \infty\) is \(\infty\)), it is impossible to find an upper bound for the sequence of partial sums.

Before getting to power series, Maclaurin series, Taylor series and all that nice stuff, one final convergence test; the ratio test. It applies to series of positive terms (or can be used to test for absolute convergence).

Consider a series \(\sum_{n=1}^{\infty} a_n\) where \(a_n > 0\) for all \(n\). Such a series will converge if the terms get small fast enough, otherwise it will diverge. How fast is fast enough? There is no exact cut-off point. The ratio test considers the ratio
\[ \frac{a_{n+1}}{a_n} \]

Notice the way it goes, if this ratio is always less than 1 then the terms are getting smaller, though perhaps not small enough. The ratio test guarantees convergence if this ratio is a bit better than merely \(< 1\):

**Ratio test for series of positive terms.** If
\[ L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \]
exists and \(L < 1\), then the series \(\sum_{n=1}^{\infty} a_n\) converges. If \(L > 1\), the series diverges (and it should be quite obvious that it diverges even without using this test). If \(L = 1\), the test is inconclusive and one should go back to the drawing board.

**Comments.**

i. Consider the series \(\sum_{n=1}^{\infty} (1/n)\). We know it diverges. Notice that here
\[ \frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} < 1, \]
so the test does **NOT** say that if the ratio is \(< 1\) then the series converges. It says that if the limit of the ratios is \(< 1\), then the series converges. Not the same thing.

ii. One can show (can you?) that if the limit \(L\) of the ratios exists and \(L > 1\), then \(\lim_{n \to \infty} a_n = \infty\).
iii. The main use we will make of this test is to decide on convergence of power series.

**Power Series**

A power series is a series of the form

\[ \sum_{n=0}^{\infty} a_n (x - a)^n \]

where the coefficients \( a_0, a_1, a_2, \ldots \) are given, the center \( a \) is given, and \( x \) denotes a variable free to range over the whole real axis. The new feature is that for every value of \( x \), we get a series. Here are some examples of power series

1. The geometric series

\[ \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots \]

In this case \( a_0 = a_1 = a_2 = \ldots = 1, \ a = 0 \). We recall that this series converges for \(-1 < x < 1\), diverges for all other values of \( x \).

2. All polynomials can be written as power series. For example, consider \( 3x^4 - 2x^3 + 1 \). We can write

\[ 3x^4 - 2x^3 + 1 = \sum_{n=0}^{\infty} a_n x^n \]

by taking \( a_0 = 1, \ a_1 = 0, \ a_2 = 0, \ a_3 = -2, \ a_4 = 3, \) all the other \( a_n \)'s equal to 0. We took \( a = 0 \). This series converges for all values of \( x \).

3. \[ \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]

As we shall see, this series converges for all values of \( x \).

Let us recapitulate a bit. A power series is one in which a variable appears, raised to an integer power, the \( n \)-th term is the term where the power is \( n \). The first term is usually the 0 term. The variable is just that, a variable. It doesn’t have to be called \( x \). For example,

\[ \sum_{n=0}^{\infty} x^n, \sum_{n=0}^{\infty} t^n, \sum_{n=0}^{\infty} z^n, \]

denote the same series. But one should be consistent. Here is a minor exercise:

**Exercise.** One of the following two statements is correct, the other one is wrong. Decide which and explain why the correct one is correct, the wrong one wrong.
1. $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n t^n$.

2. $\sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_k x^k$.

The first question about a power series in which we called the variable $x$ is: For what values of $x$ does the series converge? Every time we give $x$ a concrete value, we get a concrete series; for what concrete $x$-values is this series convergent? The following monster theorem tells us what we can expect, it also tries to list all the relevant properties of power series.

**Monster Theorem.** Suppose given a power series $\sum_{n=0}^{\infty} a_n (x - a)^n$.

1. The series always converges for $x = a$. In fact, setting $x = a$ produces a series in which every term but the first one is 0, so

   $$\sum_{n=0}^{\infty} a_n (a - a)^n = a_0$$

2. The set of points $x$ for which it converges **ALWAYS** is an interval centered at $a$. This interval is known as the interval of convergence. The possible cases are (no less, no more):

   (a) The interval is $[a, a]$ (worst case, the series only converges for $x = a$).

   (b) The interval is a regular interval and is of one of the four possible forms $(c, d)$, $(c, d]$, $[c, d]$, or $[c, d]$. Because it must be centered at $a$ one introduces its half-length $r$, $r = a - c = d - a$. One has $r > 0$ (otherwise we are in the previous case) and then we can restate what happens in this case as follows: There exists a number $r > 0$ such that the interval of convergence has the form

   $$(a - r, a + r), [a - r, a + r), (a - r, a + r]$$

   (c) The interval of convergence is $(-\infty, \infty)$; the series converges for every $x$.

   The number $r$ of is usually called the radius of convergence. It is also defined in case a and in case c. In case a, we say the series has radius of convergence 0, in case c, that its radius is $\infty$.

3. Convergence is **absolute** in the interior of the interval of convergence. If the interval of convergence is $(-\infty, \infty)$, then all points are interior; if it has the form $(a - r, a + r)$, or $[a - r, a + r)$, or $(a - r, a + r]$, or $[a - r, a + r]$, then all points but the endpoints are interior (the end points being $a \pm r$).
4. The interval of convergence can usually be determined by the ratio test or by the properties still to come.

5. The **differentiated series** is the series obtained from the given one differentiating term by term. Specifically, (the 0-th term is constant and differentiates to 0), it is the series

\[
\sum_{n=1}^{\infty} na_n(x-a)^{n-1} = a_1+2a_2(x-a)+3a_3(x-a)^2+\cdots = \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-a)^n.
\]

The radius of convergence of the differentiated series is the same as the radius of the original series. So if the original series had radius \(r\), so does the differentiated one. However, if the original series converges for \(x = a + r\) or for \(x = a - r\), it is possible that the differentiated series may diverge at one or both of these points.

6. The **integrated series** is the series obtained from the given one integrating term by term. It looks like

\[
C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1} = C + a_0(x-a) + \frac{a_1}{2}(x-a)^2 + \frac{a_2}{3}(x-a)^3 + \cdots
\]

(we added an arbitrary constant \(C\) as 0-term). The radius of convergence of the integrated series is the same as the radius of the original series. However, the integrated series may also converge at \(a - r\) or at \(a + r\) (assuming \(0 < r < \infty\)) even if the original series diverges at those points.

7. Suppose now the radius of convergence \(r\) is positive or \(\infty\) and let \(I\) be the interval of convergence. (If \(r = \infty\), then \(I = (-\infty, \infty)\); if \(0 < r < \infty\), then \(I = (a - r, a + r)\), \(I = (a - r, a + r)\), \(I = (a - r, a + r)\), or \(I = [a - r, a + r]\).)

We can then define a function which we’ll call \(f(x)\) by saying that \(f(x)\) is the sum of the series at that point \(x\); in symbols

\[
f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = \lim_{n \to \infty} (a_0 + a_1(x-a) + \cdots + a_n(x-a)^n).
\]

We have

(a) The function \(f(x)\) can be differentiated as many times as one wishes in \((a - r, a + r)\) (or \((-\infty, \infty)\) if \(r = \infty\)), each successive derivative being the sum of the corresponding differentiated series; i.e.,

\[
\begin{align*}
f'(x) &= \sum_{n=1}^{\infty} na_n(x-a)^{n-1}, \\
f''(x) &= \sum_{n=2}^{\infty} n(n-1)a_n(x-a)^{n-2}, \\
f'''(x) &= \sum_{n=3}^{\infty} n(n-1)(n-2)a_n(x-a)^{n-3},
\end{align*}
\]
etc.

(b) \( a_n = \frac{1}{n!} f^{(n)}(a) \) for \( n = 0, 1, 2, \ldots \)

(c) As we know, the integrated series also has radius \( r \) and, in fact, converges in all of \( I \) (at least). Let \( F(x) \) be its sum; that is

\[
F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1}.
\]

Then \( F'(x) = f(x) \) for all points in \((a-r, a+r)\).

**End of the monster theorem** Take a deep breath.

We will try to illustrate the theorem with examples and exercises. But we need a definition first.

**Definition.** Let \( f(x) \) be a function defined (at least) in an interval centered at the point \( a \). We say \( f(x) \) can be expanded in a Taylor’s series at \( a \) if there exists a power series \( \sum_{n=0}^{\infty} a_n (x-a)^n \) of positive or infinite radius of convergence such that

\[
(\ast) \quad f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n
\]

for all \( x \) in the interval of convergence of the power series. In case \( a = 0 \), one frequently calls the Taylor series a **Maclaurin series**.

**Notice:** The \( n \)-th partial sum of the Taylor series is precisely the \( n \)-th Taylor polynomial. In fact, by Property .... of the Monster Theorem, an equation like \((\ast)\) is only possible if \( a_n = (1/n!) f^{(n)}(a) \) for \( n = 0, 1, 2, \ldots \). We might as well have written

\[
f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a) (x-a)^n.
\]

**Exercise. Finding intervals of convergence by the ratio test.** Find the intervals and radii of convergence of the following series.

1. \( \sum_{n=1}^{\infty} \frac{1}{n} x^n \).
2. \( \sum_{n=1}^{\infty} \frac{1}{n^2} x^n \).
3. \( \sum_{n=1}^{\infty} (-2)^n (x + 3)^n \).
4. \( \sum_{n=0}^{\infty} \frac{1}{n!} x^n \).

16
5. \( \sum_{n=0}^{\infty} n! x^n \).

6. \( \sum_{n=0}^{\infty} (-3)^n (x - 4)^{2n} \).

Solutions.

1. \( \sum_{n=1}^{\infty} \frac{1}{n} x^n \). We use the ratio test. Because the ratio test assumes positive terms, we have to take absolute values. We have

\[
\left| \frac{\frac{1}{n+1} x^{n+1}}{\frac{1}{n} x^n} \right| = \frac{n|x|^{n+1}}{(n+1)|x|^n} = \frac{n}{n+1} |x|;
\]

so that

\[
\lim_{n \to \infty} \left| \frac{\frac{1}{n+1} x^{n+1}}{\frac{1}{n} x^n} \right| = \lim_{n \to \infty} \frac{n}{n+1} |x| = |x|.
\]

The ratio test says that we have convergence for \(|x| < 1\), divergence for \(|x| > 1\); it doesn’t tell us anything about what happens when \(x = 1\) or \(x = -1\). From what it tells us, we see that the radius is 1, interval of convergence is one of \((-1, 1], [-1, 1), (-1, 1], [-1, 1);\) to decide which we have to test the end-points directly. When \(x = -1\), the series is

\[
\sum_{n=1}^{\infty} \frac{1}{n} x^n = -1 + \frac{1}{2} - \frac{1}{3} + \cdots,
\]

which converges by the alternating series test. When \(x = 1\) we get the harmonic series, which diverges. \textbf{Answer:} The interval of convergence is \([-1, 1)\).

2. \( \sum_{n=1}^{\infty} \frac{1}{n^2} x^n \). We apply the ratio test.

\[
\lim_{n \to \infty} \left| \frac{\frac{1}{(n+1)^2} x^{n+1}}{\frac{1}{n^2} x^n} \right| = \lim_{n \to \infty} \frac{n^2|x|^{n+1}}{(n+1)^2|x|^n} = \lim_{n \to \infty} \frac{n^2}{(n+1)^2} |x| = |x|;
\]

as before, we have convergence for \(|x| < 1\), divergence for \(|x| > 1\), so the radius of convergence is 1. At 1 we have the series

\[
\sum_{n=1}^{\infty} \frac{1}{n^2}.
\]
which converges; because this series converges, so does the series
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}. \]
(The latter series also converges by the alternating series test; however absolute convergence is a more powerful form of convergence). Answer: The radius is 1, the interval of convergence is \([-1, 1]\).

3. \[ \sum_{n=1}^{\infty} (-2)^n(x + 3)^n. \]

\[ \lim_{n \to \infty} \frac{|(-2)^{n+1}(x + 3)^{n+1}|}{(-2)^n(x + 3)^n} = \lim_{n \to \infty} (2|x + 3|) = 2|x + 3|. \]
The radius of convergence is 1/2. Why? Because the ratio test gives convergence for \(2|x+3| < 1\); i.e., \(|x+3| < 1/2\), divergence for \(|x+3| > 1/2\).
The endpoints of the interval of convergence (an interval of half-length 1/2 centered at -3) are \(-7/2, -5/2\). Replacing \(x\) by the end-points we get the series
\[ \sum_{n=1}^{\infty} (-2)^n(-\frac{1}{2})^n = \sum_{n=1}^{\infty} 1; \sum_{n=1}^{\infty} (-2)^n(\frac{1}{2})^n = \sum_{n=1}^{\infty} (-1)^n, \]
both highly divergent. Answer: The radius of convergence is 1/2, the interval of convergence \((-7/2, -5/2)\).

4. \[ \sum_{n=0}^{\infty} \frac{1}{n!}x^n. \]

\[ \lim_{n \to \infty} \frac{\frac{1}{(n+1)!}x^{n+1}}{\frac{1}{n!}x^n} = \lim_{n \to \infty} \frac{1}{n+1} |x| = 0. \]
The radius of convergence is infinite; in fact, convergence happens for all \(x\) for which \(0 < 1\); since this happily always happens, we have convergence everywhere. Answer: The radius of convergence is \(\infty\), the interval of convergence \((-\infty, \infty)\).

5. \[ \sum_{n=0}^{\infty} n!x^n. \]

\[ \lim_{n \to \infty} \frac{|(n+1)!x^{n+1}|}{n!x^n} = \lim_{n \to \infty} (n + 1)|x| \]
\[ = \begin{cases} \infty & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \]
The conclusion is that the series converges only for \(x = 0\). Answer: The radius of convergence is 0, the interval is \([0, 0]\).
6. \[ \sum_{n=0}^{\infty} (-3)^n (x - 4)^{2n}. \]

\[
\lim_{n \to \infty} \frac{|(-3)^{n+1} (x - 4)^{2(n+1)}|}{|(-3)^n (x - 4)^{2n}|} = 3|x - 4|^2.
\]

We have convergence when \(3|x - 4|^2 < 1\); i.e., \(|x - 4| < 1/\sqrt{3}\), convergence for \(|x - 4| > 1/\sqrt{3}\). The radius is \(1/\sqrt{3}\). The endpoints of the interval of convergence are the points \(4 \pm 1/\sqrt{3}\) giving the series

\[
\sum_{n=0}^{\infty} (-3)^n \left( \frac{x - 4}{\sqrt{3}} \right)^{2n} = \sum_{n=0}^{\infty} 1, \sum_{n=0}^{\infty} (-3)^n \left( \frac{1}{\sqrt{3}} \right)^{2n} = \sum_{n=0}^{\infty} (-1)^n,
\]

both divergent. **Answer:** The radius is \(1/\sqrt{3}\), the interval of convergence \((4 - 1/\sqrt{3}, 4 + 1/\sqrt{3})\).

The ratio test works most of the time (not always), but it can get very tedious. Usually, one can avoid using it, specially if one has a few intervals of convergence already figured out.

**Examples.**

1. Find the radius and interval of convergence of the series \(\sum_{n=0}^{\infty} x^n\).

This is just our geometric series, we know it converges precisely for \(|x| < 1\).

**Answer:** Radius 1, interval \((-1, 1)\).

2. Find the radius and interval of convergence of the series \(\sum_{n=0}^{\infty} (-3)^n (x - 4)^{2n}\)

(Same series as in exercise 6 above.) This is a geometric series; in fact,

\[
\sum_{n=0}^{\infty} (-3)^n (x - 4)^{2n} = \sum_{n=0}^{\infty} \left((-3)(x - 4)^2\right)^n,
\]

so the ratio is \((-3)(x - 4)^2\). It converges (precisely) for \(|(-3)(x - 4)^2| < 1\); i.e., for \(3|x - 4|^2 < 1\), which works out to \(|x - 4| < 1/\sqrt{3}\). We get the same answer we got before.

**Exercise.** Find the Taylor series for \(\ln x\) centered at 1.

**Solution #1.** If \(f(x) = \ln x\), the only possible Taylor series at 1 will have coefficients \(a_n = (1/n!) f^{(n)}(1)\) (See Monster Theorem, Property ....). We compute the derivatives of \(\ln x\), hoping that a pattern develops. We are in luck, a pattern does develop; it is

\[
f^{(n)}(x) = (-1)^{n-1} (n-1)! x^{-n}
\]
for \( n = 1, 2, \ldots \). Since \( f^{(0)}(1) = f(1) = \ln 1 = 0 \) and \( f^{(n)}(1) = (-1)^{n-1} \) for \( n \geq 0 \), the series works out to

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} (x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = (x-1)^{-1} - \frac{1}{2} (x-1)^{2} + \frac{1}{3} (x-1)^{3} \pm \cdots.
\]

You should be able to see that this series has radius of convergence 1, interval of convergence \((0, 2]\). The equality

\[
\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n
\]

can ONLY be true for \( 0 < x \leq 2 \), outside of that interval the series diverges. Is it true there? We will state here that it is, leaving an attempt at proof for later. Notice that for \( x = 2 \) we get

\[
\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.
\]

**Some important Maclaurin series, and how to get them.** NOTE: Not knowing the material appearing between stars could have very serious consequences during the final.

1. **The series for** \( e^x \). Since

\[
\frac{d^n}{dx^n} e^x \bigg|_{x=0} = 1
\]

for all integers \( n \geq 0 \), the only possible Maclaurin series for \( e^x \) is

\[
\sum_{n=0}^{\infty} \frac{1}{n!} x^n.
\]

As we saw before, the series in question has radius of convergence \( \infty \). We also know from Taylor polynomials that

\[
e^x = \sum_{k=0}^{n} \frac{1}{k!} x^k + R_n(x)
\]

so that the series will converge to \( e^x \) for a given \( x \) if and only if \( \lim_{n \to \infty} R_n(x) = 0 \). Well, by Taylor’s theorem with remainder,

\[
R_n(x) = \frac{e^c}{(n+1)!} |x|^{n+1} \leq \frac{e^{|x|}}{(n+1)!} |x|^{n+1}.
\]

(Because \( c \) is between \( x \) and 0, we have

\[
e^c \leq e^x = e^{|x|} \text{ if } x \geq 0; \quad e^c \leq e^0 = 1 < e^{|x|} \text{ if } x < 0.
\]
Since
\[ \lim_{n \to \infty} \frac{|x|^{n+1}}{(n + 1)!} = 0 \]
no matter what value \( x \) has, we conclude \( \lim_{n \to \infty} R_n(x) = 0 \) holds for all \( x \) and
*The equality
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]
holds for all real numbers \( x \).*

2. **The series for** \( \sin x \). We have
\[
\frac{d^n}{dx^n} \sin x \bigg|_{x=0} = \begin{cases} 
0 & \text{if } n = 2k \text{ is even}, \\
(-1)^k & \text{if } n = 2k + 1 \text{ is odd}.
\end{cases}
\]
One can see the remainder goes to 0 for all \( x \) so that we end with
*The equality
\[ \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1} \]
holds for all real numbers \( x \).*

3. **The series for** \( \cos x \). We could proceed as for \( \sin x \), but there is now an easier way. Differentiate both sides of
\[ \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1} \]
to get
*The equality
\[ \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \]
holds for all real numbers \( x \).*

4. **The series for** \( \arctan x \). We could again find derivatives. But an easier (much easier) way is:
\[
\frac{d}{dx} \arctan x = \frac{1}{1 + x^2} ;
\]
if in the formula \( \sum_{n=0}^{\infty} x^n = 1/(1 - x) \), valid for \( |x| < 1 \), we replace \( x \) by \(-x^2\), we get
\[
\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-x^2)^n
\]
for $|−x^2| < 1$. However, to say $|−x^2| < 1$ is the same as saying $|x| < 1$ and $(-x^2)^n = (-1)^n x^{2n}$, so that we have established that

$$\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for (precisely) $|x| < 1$. We now integrate term by term to get

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C,$$

for $|x| < 1$, where $C$ is a constant to be determined. Setting $x = 0$, we get the equation $\arctan 0 = 0 + C$, so that $C = 0$. With a bit of care one can show that the equality is also valid for $x = \pm 1$ (as happens, one can gain one or both end-points of the interval on integrating) so we have

*The equality

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1},$$

holds for all real numbers $x$ such that $−1 \leq x \leq 1$*

In particular, for $x = 1$ we get the so called series of Gregory,

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

5. **The series for $\ln(1-x)$.** We have

$$\frac{d}{dx} \ln(1-x) = -\frac{1}{1-x} = -\sum_{n=0}^{\infty} x^n,$$

the last inequality being true for $|x| < 1$. Integrating (and evaluating at 0 to get rid of the constant of integration) we get

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n},$$

for $|x| < 1$. For what it may be worth, we prove that the equality also holds for $x = -1$. To this effect, we notice that for any integer $N > 0$, $x \neq 1$,

$$\left| \frac{1}{1-x} - \sum_{n=0}^{N} x^n \right| = \left| \frac{1}{1-x} \frac{1-x^{N+1}}{1-x} \right| = \frac{|x|^{N+1}}{1-x},$$

22
we can integrate from $-1$ to $0$ to get

$$\left| \int_{-1}^{0} \left( \frac{1}{1-x} - \sum_{n=0}^{N} x^n \right) dx \right| \leq \int_{-1}^{0} \left| \frac{1}{1-x} - \sum_{n=0}^{N} x^n \right| dx = \int_{-1}^{0} \frac{|x|^{N+1}}{1-x} dx.$$  

In the interval $[-1, 0]$ we have $1 - x \geq 1$ so that the last integral can be bounded by

$$\int_{-1}^{0} |x|^{N+1} dx = \frac{1}{N+2}.$$  

We have therefore shown that

$$\left| \int_{-1}^{0} \left( \frac{1}{1-x} - \sum_{n=0}^{N} x^n \right) dx \right| \leq \frac{1}{N+2};$$  

i.e., that

$$\left| \ln 2 - \sum_{n=0}^{N} \frac{(-1)^{n+1}}{n+1} \right| \leq 1N + 2.$$  

**Exercise.** Explain why the last inequality proves whatever still had to be proved of the following fact:

*The equality

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n},$$

holds for all real numbers $x$ such that $-1 \leq x < 1^*$

Two things to notice are:

1) If we replace $x$ by $1 - x$ we get a previous result, namely that

$$\ln x = -\sum_{n=1}^{\infty} \frac{(1-x)^n}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x-1)^n}{n}$$

is valid for $0 < x \leq 2$ (since $-1 \leq 1 - x < 1$ is equivalent to $0 < x \leq 2$).

2) We proved

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$
### Addendum.

Writing a function as a Taylor series.

**Objective.** Given $f(x)$ and a point $a$, find $a_0, a_1, a_2, \ldots$ so that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

in some interval centered at $a$.

**Minimum Requirements on $f$.** The function $f$ must have derivatives of all orders in an interval around $a$.

**Doing it the hard way.** The only possibility for the coefficients $a_n$ is:

$$a_n = \frac{1}{n!} f^{(n)}(a),$$

for $n = 0, 1, 2, \ldots$. Take derivatives of $f$, evaluate at $a$, see if a pattern emerges. The fact that this is the only possibility, does not yet establish that the formula is valid.

**Establishing validity.** For what $x$ is the formula

$$(*) \quad f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

true? The rule of thumb is: If $f(x)$ is a function which can be expressed in a single formula in terms of algebraic, trigonometric and exponential functions, then $(\ast)$ is valid in the whole interval of convergence of the power series. For a rigorous proof one has to realize that the partial sums are the Taylor polynomials, so that convergence is equivalent to the remainder going to 0; then one has to prove the remainder goes to 0.

**An example.** Find a power series development centered at 0 for $e^{-x^2}$.

The hard way: Let $f(x) = e^{-x^2}$.

$$f(0) = 1, \quad f'(x) = -2xe^{-x^2}, \quad f'(0) = 0,$$

$$f''(x) = (4x^2 - 2)e^{-x^2}, \quad f''(0) = -2,$$

$$f'''(x) = (-8x^3 + 12x)e^{-x^2}, \quad f'''(0) = 0,$$

$$f^{(4)}(x) = (16x^4 - 48x^2 + 12)e^{-x^2}, \quad f^{(4)}(0) = 12,$$

A pattern is emerging. For example, it is becoming sort of clear that all odd derivatives evaluate to 0; it may take more computations to begin to see that the even derivatives also follow a simple pattern. And how can one be sure?

The easy way: We know by now that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

24
for all real numbers $x$. Well, if $x$ is real so is $-x^2$, thus

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} \pm \cdots$$

is valid for all real numbers $x$.

Sad comment: The easy way is not always available. For example, there is no easy way for the function $\tan x$. 