The American educational system has treated mathematics as skill in numerical manipulations, and has used the term "quantitative reasoning" to describe the application of mathematics to other areas of study. This serious misconception has severely hampered the ability of our students to comprehend important developments in scientific and philosophic thought. Mathematics can be more properly regarded as a form of language, developed by humankind in order to converse about the abstract concepts of numbers and space. In addition to the intellectual appeal of its intricate linguistic structure, the language should be appreciated for the richness of the literature composed with it.

The immediate cause for the recent concern about mathematics education is the widespread difficulty students apparently experience with this subject. While some educators have expressed an interest in improving pedagogy, students have tended to opt for a more practical approach whenever it has been open to them: avoiding the subject altogether. Predictably, the reaction of the educational system has been an attempt to force students to learn mathematics anyway. The mandate to hold students' feet to the fire has been carried out with varying degrees of resolve, the debate over how much pain to inflict being dominated by two conflicting lines of reasoning. According to the first, there must be some minimum standards which can be applied to all students. In addition since it builds character, everyone deserves their fair share of suffering. Those determined to adopt a more merciful attitude, on the other hand, often argue that the dose of unpleasantness should be limited to what is absolutely necessary. By this logic, mathematics courses need be required only when they are listed as prerequisites for courses that are required for some other reason.

In the course of such arguments, the question of mathematics itself has been relegated to a position of secondary importance. Implicit in this view is the premise that mathematics is a highly esoteric subject which, despite having some practical applications, possesses only tenuous connections to other areas of scholarship. Any consideration of the intellectual and aesthetic appeal possessed by the discipline has been dismissed as irrelevant for all, save the mathematicians themselves. Within most curricula, mathematics has been so thoroughly dissociated from all other subjects that students generally encounter little evidence that these assumptions might be mistaken. Physical science, for example, is nearly always taught as if students know little or nothing of mathematics. Similarly, students are expected to learn biology without any reliance upon the concepts and formalism of the physical sciences. Many statisticians now insist that their subject is something quite apart from mathematics, so that statistics courses do not require any preparation in mathematics. Some computer scientists have gone so far as to suggest that if you understand the mathematics of their machines, it is not necessary to understand the mathematics of natural phenomena. To the students, nothing seems to be
amiss. They observe that the factors which determine whether a course is classified as hard or soft, general or specialized, are the complexity of the problem assignments and the thickness of the textbook. As the values of these parameters increase, the demand for proficiency with specific mathematical techniques also increases. Across the entire spectrum of course offerings, however, students are exposed only to the vaguest notions of what we mean by rigor.

A careful consideration of the nature of mathematics as an intellectual activity reveals the utter incongruity of this situation. Not only have our students failed to appreciate the beauty of mathematics, they have little grasp of the profound insights about the natural world which mathematics has made possible. Rather than a mere intellectual curiosity or useful skill, mathematics is an important facet of the most distinctive capability of the human species.

MATHEMATICS IS A LANGUAGE. As a general definition, mathematics could be called the study of numbers and space. In order to talk about such things, mathematicians first had to devise an appropriate vocabulary and alphabet. It is apparent that the objects defined by mathematicians are entirely abstract, and can never actually be observed in any way except by the human imagination. Although many students seem to be disturbed by this fact, mathematics is comprehensible precisely because it is abstract. We are not dependent upon observation to know that lines are completely straight and parallel lines never meet. It is possible to make these assertions with confidence because mathematics was not discovered, it was invented. It must also be remembered that imagination and abstraction are universal parts of the human experience, and not the exclusive domain of the mathematician. The idea of a line or a point is no more abstract than the idea of loyalty or freedom. Mathematics differs from other languages not because of its inclusion of abstraction, but because of its complete detachment from the complications of what we experience by direct observation.

If mathematics consisted only of new words and symbols, it could properly be considered as an extension of existing language. The reason mathematics is a new and separate language is that it also has its own syntax and grammar. Having devised both vocabulary and rules for the language, mathematicians seek to discover what things are possible or not possible to say about numbers and space. In this ideal, ordered world of the imagination, it is possible to apply relentless logic to any questions that arise. Euclid’s geometry should be appreciated for the beauty which can result from this process. His system of axioms, definitions, theorems, and constructions clearly lays out both the vocabulary and the rules for conversing about his chosen topic. By limiting his considerations to fanciful but specific abstractions of his imagination, he was able to produce an indisputably logical and thorough exposition. If one understands the language, one will not only know what a triangle is, but exactly what can be said about triangles. The lexicon and grammar of language are quite precise; the conclusions are quite inescapable.

FROM THE LINGUISTIC TO THE LITERARY. The natural sciences seek to explain the behavior of the things we encounter in nature. Experimental observation is the court of highest appeal for all such explanations. What then is the role of the abstract, detached language of mathematics in experimental science? The answer is not different than that for the roles of language in other forms of scholarship. The scientist uses the language of mathematics to construct metaphors which represent insights into the workings of nature. The use of metaphors is as commonplace in science as it is in poetry. We speak of radio “waves”, subatomic
"particles", and celestial "bodies". What distinguishes the mathematical metaphor is the extraordinary power of this language to uncover implications of the underlying idea. Calling the earth a sphere is an extremely powerful statement, for we know a great deal about the things we can say about spheres. The fact that most scientists prefer to call such constructions "models" or "laws" does not change their essential character. It is just as metaphorical to call the world a sphere as it is to call it a stage.

GETTING THE PICTURE. It is generally agreed that the object of studying science is to understand its fundamental concepts, but how does one come to understand a mathematical metaphor? It is widely believed that the first step is to find a way to remove the mathematical details so that the essential picture can be seen unobscured. Mathematics, then, need be introduced only if one is confronted with a specific problem that requires a numerical answer. Many educators are fond of using the term "quantitative reasoning" as a substitute for "mathematics". The idea that mathematical questions can be answered qualitatively without actually using any mathematics represents one of the most serious misconceptions about the nature of this discipline. Mathematics is not a way of handing numbers on things so that quantitative answers to ordinary questions can be obtained. It is a language that allows one to think about extraordinary questions. Saying that the earth has a round shape means only that it has no edges. This non-mathematical picture is not simply "qualitative", "verbal", or "intuitive"; it is primitive and empty. If we wish to construct a meaningful metaphor about the shape of the earth, we must use the language of shapes, which is mathematics. To those prepared to examine its implications, even the mathematically simple picture of the spherical shape has a great richness when used as a metaphorical device. Getting the picture does not mean writing the formula or crunching the numbers, it means grasping the metaphor. Without mathematics, one cannot even read the words.

READING CRITICALLY. Students of the physical sciences are often left with the impression that facility in solving mathematical problems is all that is necessary to understand scientific ideas. After all, the entire subject of classical physics can be reduced to seven compact, simple-looking equations, and most study time in this subject is spent working problems. This view, however, has the same basic flaw as the belief that mathematics can be replaced by less abstract language. Just as excluding numerical calculations from an argument does not eliminate the need for mathematics, one cannot bring mathematics to bear on a question by simply assigning numbers and symbols to its elements. Metaphors do not have answers, they have implications. When we struggle to find solutions to mathematical problems, we are exploring the specific implications of a writer's metaphors in particular physical circumstances.

In physics, as in all other intellectual disciplines, understanding derives from critical thought, not just hard work. The equations of electrodynamics, for example, are important not simply because they give rise to interesting and complicated mathematical problems. Maxwell's picture of flowing force fields which diverge and swirl according to precise mathematical relationships is an incredibly rich metaphor. In order to understand this literature, it must not only be read, it must be pondered carefully. By thoughtful consideration of wisely chosen examples, you can begin to discover the things about different physical situations that have important influences on the outcomes one observes. You begin to understand the
metaphor. As Paul Dirac put it: “I understand what an equation means if I have a way of figuring out the characteristics of its solution without actually solving it.” This is what we mean by physical insight; it is an intuition based upon critical analysis of key questions. Einstein’s surmisal that magnetism is fundamentally a relativistic effect of electricity, one of the truly remarkable insights of human history, came directly from his contemplation of Maxwell’s equations. Building such intuition may require considerable effort. Electricity and magnetism are more difficult to understand than the shape of the earth, and Maxwell’s metaphor is complex in language and construction. On the other hand, one could spend a lifetime working out solutions to very complex and difficult problems in electrodynamics and never discover the possibility of electromagnetic radiation, even though Maxwell’s equations directly imply its existence. A mason may indeed build great strength in his arm muscles, but his goal is to build the house. Likewise, the goal of the scholar is not technical dexterity, but insight.

**ANALYZING MATHEMATICAL LITERATURE.** While it is important to choose interesting examples to think about, as a practical matter we are still left with mathematical problems to solve. In asserting that mathematics is an idealized abstraction which lends itself to elegant, logical treatments, we have overstated our case in several particulars. First, not all mathematical problems have solutions. In fact, the only way a mathematical system can be made to give uniformly consistent answers is for some questions to be left completely unanswered. Even if solutions are known to exist, it is sometimes true that there can be no general method for finding them. (The most famous example is the problem of trisecting an angle.) Such philosophical difficulties with the positivist viewpoint pale in comparison to the practical difficulties in finding solutions, even when you know that they exist and that you should be able to find them. Specific mathematical problems, no matter how well posed, often lead one into an impenetrable thicket from which it can be exceedingly difficult to extract solutions by any means. In attempting to understand science, one should not let these kinds of mathematical difficulties take over. What the scientist wants to know is whether the original metaphor can reveal anything about what will be expected to happen in a particular situation. Often, a good approximate solution can lead out of the quagmire toward great insight. The concepts of field lines, resistance, and capacitance are examples of insightful approximations of Maxwell’s equations which allow one to picture and analyze the behavior of certain physical systems by avoiding purely mathematical entanglements. Hence, the use of these heuristic devices is another way to achieve the kind of understanding to which Dirac referred. One must never lose sight of the goal of this process. By itself, an individual solution or the particular technique used for finding it is important only insofar as it provides clues about the nature of the metaphor. It is in the original equations, not in a catalog of solutions, that the complete law, the entire metaphor, is to be found. The goal is to grasp the metaphor, not just to compile a catalog.

**THE NATURE OF KNOWLEDGE.** Finally, we have to consider that the metaphors themselves may have limited usefulness. Newton’s laws of motion have turned out to need corrections in order to be consistent with the theory of relativity. The problems with Maxwell’s laws are more serious: they are internally inconsistent at some points. Even the most imaginative metaphor cannot bring every facet of truth into plain view. In non-mathematical literature, we are used to the fact that meanings become distorted when ideas are stretched too far. (For example, the
cartoonist Johnny Hart observed some years ago that the trouble with a melting pot is that the bottom gets burnt and the scum comes to the top.) In interpreting mathematical language in the scientific literature, we tend to become confused by the Platonistic view that mathematical objects themselves actually exist. Regardless of whether this assertion about mathematics is philosophically correct, the use of this language does not imbue one's ideas about nature with an independent existence. The use of mathematical objects is still a metaphorical device. Mathematical metaphors are not different from other forms of human understanding. They may provide some significant insights, but are not a holy grail of ultimate truth. This should come as no surprise. We know that the earth is not exactly a sphere. Furthermore, the holy grail is a heretical idea, not just to Christian theology, but to Western philosophy as well. Despite our desires to the contrary, nature is not obliged to limit its complexity to the confines of the human imagination. Despair is not an appropriate reaction to this revelation. The importance of the insights provided by any single idea is not diminished because it fails to explain everything else in nature. In addition, our capabilities are not forever fixed. Mathematicians have continued to explore new territories in the abstract landscape they have created, and certainly do not believe that this is a task which will one day be completed. If all of human understanding is based on metaphors, then extending the reach of the languages we use to construct them can greatly expand the boundaries of our imagination. We have cited but a scant few of the important advances in the natural sciences which have been made possible by the development of mathematics. In a fuller accounting, we would need to discuss its influences in philosophy and in the social sciences, as well as the fresh insights provided by new mathematics into old conundrums. Mathematics, therefore, does not simply represent a narrow but important class of intellectual achievement. Its distinct and prominent place in the realm of human language and literature make it an essential foundational element of education.

HOW DOES ONE BECOME LITERATE? As educators, we seek to engage students in the exploration of literature, in the understanding of other people's ideas, and in the expression of their own views. If the starting place of this process is language, its goal must be literacy. This goal can be attained only through thoughtful reading and critical analysis. Having seen that all forms of language share the same basic literary device, we can draw rather extensive parallels between the meaning of literacy in the language of mathematics and that which applies to other languages. The following list of assertions about the nature of literacy hardly requires explanation or justification. When applied to mathematics education, however, they imply that our current approaches are grossly ill conceived, and require wholesale revision.

*Individuals can learn more than one language.* The assertion that mathematics requires a very special innate talent is often used as an excuse for not learning this subject and a reason for not trying to teach it. This sort of condescension must not be permitted in the educational community. Mathematics is not so different from other expressions of the human intellectual capacities for communication and imagination. The fact one can learn to say things in mathematical language that cannot be said in other languages is beside the point, as is the possibility that different regions of the brain may be used. Mathematics is not the exclusive provence of either the gifted or the deranged, it is for all who would seek to be truly educated.
One must learn to read before trying to study literature. This point seems so obvious that it is surprising that it is violated so systematically, especially in engineering and science curricula. Unsuspecting students are routinely confronted by passages of unintelligible mathematics. The accompanying explanation, if one is to be found, is typically cursory. Usually, it is unclear whether the mathematical interlude is given as an aside, whether it is a point of central importance, or whether it is something the students were supposed to have known already. Often, such narratives become more obscure as they progress. In reading any form of literature, one may occasionally find it necessary to consult a dictionary, and footnotes can be very helpful in clarifying points of grammar or usage. When one has to look up every other word and the footnotes begin to swallow up the text, however, a reader has very little chance of putting together the meaning of even a single complete sentence. If students are given the chance to study enough mathematics first, it can be startling how facilely they are able to grasp the central ideas of other disciplines.

Vocabulary is not enough. We all remember vocabulary tests from grade school: look up the word in the dictionary, memorize its definition, learn how to spell it, and go on to the next word. We compile the same sorts of lists for mathematics students: quadratic equations, logarithms, antiderivatives, Bessel functions, etc. Furthermore, we forbid our students to use a dictionary to look up the unfamiliar mathematics they may encounter in their studies. We insist that they commit the entire dictionary to memory. The difficulty with this approach is that knowing the formal definitions of a large number of words is no guarantee that one can actually say anything comprehensible. One must obtain practice in expressing complete thoughts with language. In mathematics, this means constructing examples which deal with specific objects and events; that is, applications. It does not matter so much whether the objects are observable or abstract, just as it does not matter whether the words are in the form of prose or verse. What matters is that words must be given a context if we are to be enriched by understanding their meanings.

Conversational fluency is of limited value. We spend a great deal of time telling our students “this is how you solve this kind of problem.” To divide by a fraction, for example, you just invert and multiply. This is just like saying “when you answer the telephone, pick up the receiver and say ‘hello’.” This might seem to be a step forward from the vocabulary lists, but most often it actually represents a retreat. You do not even have to know what the individual words mean in order to be able to fire back the correct response. Once you have learned the drill, you will never have to think about it again. In short, this is training rather than education. Whether or not any sort of training has a proper role in university curricula is a matter of longstanding controversy, but in this case, the point is moot. We have trained our students to become highly proficient in answering questions they will never be asked again. Outside the classroom, it is extremely unlikely that you will be asked to “solve for x”. The harm we inflict by insisting on this sort of raw skill development goes far beyond simply wasting our students’ time, however. Since they never really understood the meaning of the answers they were trained to give, they will be unable to answer the questions that do come up. Because they come to perceive mathematics as a collection of specialized skills, rather than a way of thinking and talking, our students are led to conclude that mathematics is either worthless or impossible to master. Imagine what we would think if they came to this conclusion about, say, Spanish. There was a line in one of our high school
dialogues that went: “they-always-serve-meatballs-in-the-cafeteria-on-Wednesday.” Since I have never had the opportunity to use Spanish to tell someone this, am I to conclude that Spanish is worthless for communicating with my fellow humans, or that it is too much to expect that I will ever be able to learn how to say anything really useful? By offering training rather than education in mathematics, we have produced a legion of mathematical sophomores, possessed of an extensive but superficial knowledge. Breaking this cycle is one of the most important pedagogical challenges in mathematics education.

Writing is not the same as penmanship. Whereas penmanship used to be considered a highly valuable skill, most of us would now refuse to accept a handwritten document, no matter how legible or graceful the hand. We would not say that a manuscript had been “written” by a machine just because it was typed. In the case of mathematics, our attitude is exactly the opposite. We insist on work which has been done by hand, and disallow all forms of mechanical or electronic assistance because it amounts to cheating. Whether the students are in college learning calculus or in the third grade learning multiplication, we insist that they concentrate on developing their skill at the mechanical manipulation of symbols required to arrive at the desired product. The time for us to abandon this antiquated and misguided attitude is long past. We are wasting our students’ time by teaching them how to write with a quill pen when they should be using a typewriter. The use of modern technology is beneficial to mathematical undertakings and literary tasks alike. One’s attention can be focused on the conceptual content of what is being written, rather than the manipulative processes required to produce the writing itself. The skills of using the typewriter, the calculator, and the computer are useful ones. The value of being able to add large columns of number in one’s head or remember the Fourier transforms of complicated functions is dubious at best. More to the point, in the outside world, it will be considered inappropriate to waste time performing such tasks by hand.

Complicated is not the same thing as sophisticated. Nearly everyone has had experience with long, tedious problems involving only elementary arithmetic. There is no trick to constructing similar problems at any level of mathematics. When a problem appears to be difficult, it is important to distinguish whether obtaining the solution requires new insights, a bit of cleverness, or just plain drudgery. The simple problems are not the only ones that are worth solving, but it is appropriate to offer some justification for tackling difficult ones.

The study of grammar can be overdone. While systematic consideration of mathematical grammar is generally ignored in lower level courses in mathematics, it often overwhelms everything else in advanced courses. For someone who is studying mathematics in order to do a bit of advanced reading, it is not always necessary to grasp all of the subtleties of mathematical arguments in order to appreciate the usefulness of their conclusions. Some such problems, like proving the existence of the derivative, turn out to be quite formidable. It is sometimes enough to understand the kinds of things mathematicians worry about, but to leave the actual worrying to them.

It is possible to be fluently ignorant. The ability to read the English language does not imply any knowledge of Shakespeare’s plays, nor is an understanding of the playwright’s ideas derived simply from hearing his words. Further, because a piece of writing is grammatically correct and draws from a large vocabulary is no
indication that it contains any worthwhile ideas. Similarly, the ability to perform mathematical gymnastics is not necessarily indicative of any underlying understanding of either the literature of mathematics or the nature of the language. Mathematical exercises should always be chosen in such a way as to provide the student with a keener insight into great concepts, not just practice in manipulations. Education is not just learning to read and write. The questions of what is read and what is written about are central to any practical definition of literacy.

*Literature must be studied from original sources.* The application of this principle to science and mathematics is often misunderstood. We are not suggesting that the only proper way to learn Newtonian physics is to study the *Principia*. Ideas and arguments presented in the language of mathematics can be reproduced very concisely. When that part of the exposition written in other languages is paraphrased or even condensed, little or no loss of meaning need occur as long as the mathematics is preserved. On the other hand, major distortions of perception will inevitably result if mathematics is replaced by non-mathematical pictures or lists of specific examples. It is essential to present the original formulation, the equations, which define the author’s metaphor. This is true even if the students’ command of the language is limited, and they are able to grasp only some of the simpler implications of the metaphor. Having seen the original idea in its entirety, they will understand the underlying basis of however much they are ready to learn. Furthermore, they can approach the subject again in the future without having anything to unlearn.

*The accomplishments of the past cannot be dismissed.* We have become accustomed to considering pre-modern science as so much primitive nonsense: domed sky, flat earth, flies erupting spontaneously from dead meat, et cetera. In the humanities, such an attitude would appear silly. The works of Shakespeare did not relegate those of Sophocles to the dust bin. Because mathematics is generally, and quite incorrectly, categorized with science rather than with philosophy, most people, including our students, tend to think that the mathematics being taught in the present day represents modern developments. In fact, most college students are exposed to very few mathematical ideas which originated since the fifteenth century. The accomplishments of the early Greek, Hindu, and Islamic scholars are of more than historical interest. They were not made obsolete by subsequent developments, but rather formed the basis for those very advances. What is distressing about the present state of education is that so few students have even an inkling of the progress made in mathematics during the last half millennium.

*There is no substitute for learning how to read.* The reason mathematics was devised in the first place was the inability of existing language to deal with this subject matter. Newton, for example, found it necessary to invent the calculus in order to develop and express his ideas. Trying to understand Newton without calculus is not like trying to understand Sophocles without Greek, it is like trying to understand Sophocles without words. One could represent *Antigone* as a series of pictures. If done with sufficient skill, the major points of the story line could be made evident; but the play is not just a story about some people who end up dead under tragic circumstances. To pretend that one can understand Newton without using his mathematics would be equally delusional. At best, one might hope to be trained in solving certain types of problems involving throwing things up in the air or sliding them down inclined planes. With few exceptions, such training will prove
useless in the long run. Visual images, no matter how important they may be to human communication, do not constitute a form of language. Creating a picture book is not an act of translation; and, despite the aphorism to the contrary, no number of pictures can match the power of the written or spoken word. Mathematics, on the other hand, is precisely a form of language. If we truly wish to understand the things that have been said with this language, we must eventually take up some books with words in them.

**Illiteracy has unfortunate consequences.** In demanding picture books, one chooses illiteracy and is cut off from some of the most important ideas in the history of western civilization. This is a decision that should not be taken lightly by anyone who truly desires an education. That Americans seem unaware of the consequences of such a decision is hardly surprising, since they do not make it for themselves. If it makes us feel better about our own ignorance to say that we know that the earth is spherical, it is only because we were never told that the educated people of the world had known that for at least 1800 years prior to the time Columbus set sail. Americans may not think that the earth is flat, but most do believe that Michael Jordan can hang in the air and change direction in mid-flight. Only the ignorant could be convinced of this. The educated have known better for three hundred years. Our own educational system has given out nothing but picture books, and as a result, our society has been left illiterate and vulnerable to manipulation. We have told the public that they should leave the technical stuff to the experts, that they are not smart enough to understand anyway. They have been convinced that they are stupid. They are not stupid, they have been cheated.

**Literacy is the goal of education.** The study of mathematics not only allows one to read and understand the work of others, it also increases one’s own powers of thought, imagination, and expression. That so many regard this subject as frightening, boring, or otherwise unpleasant represents a disappointing failure of our educational efforts. Literacy in mathematics is not simply a question of how much or for whom. To improve our present condition will require substantive reform across the spectrum of curricula. It will not be enough for our departments of mathematics to teach our students how to read. All the rest of us must see to it that they do read, that what they read is worthwhile and important, and that they are able to comprehend what they read. Becoming literate requires a real education, the kind that does not become out of date, but prepares one for a lifetime of learning. Just as there is no aspect of the human experience unworthy of serious study, there is no student unworthy of a genuine education. For once, a popular buzz word has captured the true essence of a national problem. Our difficulty is not that education has failed to keep pace with rapidly changing knowledge, it is that we have misplaced the goal of all true education: literacy.

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