
**Solution.** We have to show that isometries preserve volumes. By definition, a parameterized manifold of class \( C^r \), dimension \( k \), is the pair \((\alpha, Y_\alpha)\) where \( \alpha : A \to \mathbb{R}^n \) is of class \( C^r \), \( A \) is open in \( \mathbb{R}^k \) and \( Y_\alpha = \alpha(A) \).

Given such a parameterized manifold, if \( h : U \to \mathbb{R}^n \) is of class \( C^r \), where \( U \) is an open subset of \( \mathbb{R}^n \) such that \( Y_\alpha \subset U \), then \((h \circ \alpha, h(Y_\alpha))\) is also a \( k \)-dimensional parameterized manifold of dimension \( k \), class \( C^r \). It follows that

\[
V(h(Y_\alpha)) = \int_A |\det((Dh \circ \alpha)^T Dh \circ \alpha)| = \int_A |\det((D\alpha)^T (Dh)^T Dh D\alpha)|.
\]

In general, this might be a bit difficult to resolve. But since \( h \) is an isometry we’ll use Theorem 20.6 stating that \( h(y) = Ay + p \), where \( A \) is an orthogonal linear transformation and \( p \in \mathbb{R}^n \). The \( Dh \equiv A \) so that \( (Dh)^T Dh = A^T A = I \) and

\[
V(h(Y_\alpha)) = \int_A |\det((Dh \circ \alpha)^T Dh \circ \alpha)| = \int_A |\det((D\alpha)^T (Dh)^T Dh D\alpha)| = \int_A |\det((D\alpha)^T D\alpha)| = v(Y_\alpha).
\]


**Solution.** We are supposed to express \( v(Y) \) as an integral, where \( Y = \alpha([r^{k+1}], \alpha : \mathbb{R}^k \to \mathbb{R}^{k+1}) \) defined by \( \alpha(x) = (x, f(x)) \); \( f : A \to \mathbb{R} \) of class \( C^r \), \( A \) open in \( \mathbb{R}^k \). We have

\[
D\alpha(x) = \begin{bmatrix} I & Df(x) \end{bmatrix}
\]

where \( I \) denotes the \( k \times k \) identity matrix. Thus

\[
(D\alpha(x))^T D\alpha(x) = I + Df(x)^T Df(x) = \begin{bmatrix}
1 + D_1 f(x)^2 & D_1 f(x) D_2 f(x) & \cdots & D_1 f(x) D_k f(x) \\
D_2 f(x) D_1 f(x) & 1 + D_2 f(x)^2 & \cdots & D_2 f(x) D_k f(x) \\
\vdots & \vdots & \ddots & \vdots \\
D_k f(x) D_1 f(x) & D_k f(x) D_2 f(x) & \cdots & 1 + D_k f(x)^2
\end{bmatrix}
\]

But since we are graduate math students we cannot stop here!!! A bit of experimenting suggests the following result: Let \( a_1, \ldots, a_k \in \mathbb{R} \) and let \( A \) be the \( k \times k \) matrix whose \((i, j)\)-th entry is \( a_i a_j \). Then \( \det(I + A) = 1 + \sum_{j=1}^k a_j^2 \).

I suggested to several people who came to my office how to prove this result. But here is a much simpler proof. It uses some facts about eigenvalues and eigenvectors of matrices. Specifically it uses the following results:

**Lemma 1** Let \( B \) be a \( k \times k \) matrix with a basis of eigenvectors. That is assume there exists a basis \( \{v_1, \ldots, v_k\} \) of \( \mathbb{R}^k \), numbers \( \lambda_1, \ldots, \lambda_k \) (not necessarily distinct) such that \( Bv_j = \lambda_j v_j \) for \( j = 1, \ldots, k \). Then the matrix \( I + B \) has the same basis of eigenvectors, with eigenvalues \( 1 + \lambda_1, \ldots, 1 + \lambda_k \).

**Proof.** We have \((I + B)v_j = v_j + \lambda_j v_j = (1 + \lambda_j)v_j \) and since a basis is a basis is a basis, we are done. \( \blacksquare \)

Suppose now \( B \) has a basis of eigenvalues \( \{v_1, \ldots, v_k\} \) with eigenvalues (not necessarily distinct) \( \lambda_1, \ldots, \lambda_k \). Setting \( U = [v_1, \ldots, v_k] \) (the \( k \times k \) matrix whose columns are the basis vectors \( v_j \)) we have that \( U \) is invertible and \( U^{-1} B U \) is the diagonal matrix \( D \) with entries \( \lambda_1, \ldots, \lambda_k \) in the diagonal. Then

\[
\det B = \det(U DU^{-1}) = \det(U) \det(D) \det(U)^{-1} = \det(D) = \prod_{j=1}^k \lambda_j.
\]
All of this is absolutely standard linear algebra fare. Let us apply it to the matrix $A = [a_{ij}]$ where $a_1, \ldots, a_n \in \mathbb{R}$. We want to prove that $\det(I + A) = 1 + \sum_{j=1}^{k} a_{jj}^2$. If all the $a_j$’s are zero, then the result is trivial, so assume at least one $a_j \neq 0$. Let $a = (a_1, \ldots, a_k)$; by assumption $a \in \mathbb{R}^k$, $a \neq 0$. We can thus find a basis of $\mathbb{R}^k$ consisting of vectors $v_1, \ldots, v_k$ where $v_1 = a$ and $v_j \cdot a = 0$ for all $j \geq 2$ (we can even assume that $v_i \cdot v_j = 0$ if $i \neq j$) but we don’t need this. Now, for $j = 1, \ldots, k$, writing $v_j = v_{j1}, \ldots, v_{jk}$

$$(Av_j)_\ell = \sum_{i=1}^{k} a_{\ell i} v_{ji} = a_{\ell} a \cdot v_j.$$  

It follows that $Av_j = (a \cdot v_j)a$ for $j = 1, \ldots, k$. This gives $Av_j = 0$ if $j \geq 2$, $Av_1 = ||a||v_1$. In brief, $A$ has $k-1$ repeated eigenvalues equal to $0$, one eigenvalue equal to $||a||^2$. By Lemma 1, $I + A$ has eigenvalues $1 + ||a||^2, 1, \ldots, 1$. By Lemma 1, it has a basis of eigenvectors (namely $v_1, \ldots, v_k$) so that its determinant is the product of these eigenvalues, namely $1 + ||a||^2$ as asserted.

Applying all this to our case, we see that

$$\det(D\alpha(x)^T D\alpha(x)) = 1 + \sum_{j=1}^{k} D_k f(x)^2.$$  

We thus get the formula we learned (and perhaps forgot) in Calculus:

$$v(Y) = \int_A \left[ 1 + \sum_{j=1}^{k} (D_k f)^2 \right].$$

**Second Proof of** $\det(I + A) = 1 + \sum_{j=1}^{k} a_{jj}^2$ **if** $A = [a_{ij}]_{1 \leq i, j \leq k}$. This is the complete proof I suggested to some people who came to see me in my office. It is a bit more complicated, but since nobody who used it seemed to have done it right, here it is.

For $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ define

$$g_k(\lambda) = \det(\lambda I + A).$$

We prove by induction on $k$ that $f_k(\lambda) = \lambda^k + \lambda^{k-1} \sum_{j=1}^{k} a_{jj}^2$. We proceed by induction on $k$. The case $k = 1$ is $f_1(\lambda) = \lambda + a_1^2$; obvious since $\lambda I + A = [\lambda + a_1^2]$ if $k = 1$. It isn’t really necessary, but the case $k = 2$ is also easy:

$$f_2(\lambda) = \det \left[ \begin{array}{cc} \lambda + a_1^2 & a_1 a_2 \\ a_2 a_1 & \lambda + a_2^2 \end{array} \right] = (\lambda + a_1^2)(\lambda + a_2^2) - (a_1 a_2)^2 = \lambda^2 + (a_1^2 + a_2^2)\lambda + a_1^2 a_2^2 - (a_1 a_2)^2 = \lambda^2 + (a_1^2 + a_2^2)\lambda$$

proving the formula for $k = 2$. I now need to use the formula for derivative of a determinant of variable entries. Briefly, the derivative with respect to $\lambda$ of an $n \times n$ determinant $D$ whose entries depend on $\lambda$, is the sum of $n$ determinants $D_1, \ldots, D_n$, where $D_i$ is obtained from $D$ by replacing all entries of the $i$-th row by their derivatives with respect to $\lambda$, and leaving all other entries unchanged. Assume now the formula is proved for some $n \geq 1$, then

$$\frac{df_{n+1}}{\lambda} = \sum_{i=1}^{n+1} D_i,$$

where $D_i$ has all rows the same as $f_{n+1}(\lambda)$ except the $i$-th row, which has 0’s everywhere but in the diagonal where the entry is one. This is because the only entry in the $i$-th row of $f_{n+1}(\lambda)$ that depends on $\lambda$ is the diagonal entry, the $i$-th one, which is $\lambda + a_i^2$ and differentiates to 1. Here is how it looks, assuming $n$
is the same as in the previous exercise, and

Thus, by the induction hypothesis,

\[ D_i = \text{det} \begin{bmatrix} \lambda + a_1^2 & a_1a_2 & \cdots & a_1a_i & a_1a_{i+1} & \cdots & a_1a_{n+1} \\ a_2a_1 & \lambda + a_2^2 & \cdots & a_2a_i & a_2a_{i+1} & \cdots & a_2a_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1}a_1 & a_{i-1}a_2 & \cdots & 1 + a_{i-1}^2 & a_{i-1}a_{i+1} & \cdots & a_{i-1}a_{n+1} \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ a_{i+1}a_1 & a_{i+1}a_2 & \cdots & a_{i+1}a_i & \lambda + a_{i+1}^2 & \cdots & a_{i+1}a_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n+1}a_1 & a_{n+1}a_2 & \cdots & a_{n+1}a_i & a_{n+1}a_{i+1} & \cdots & 1 + a_{n+1}^2 \end{bmatrix}. \]

Developing this determinant by the \( i \)-th row we see that

\[ D_i = f_n(\lambda, a_1, \ldots, a_i, \ldots, a_n); \]

in other words, \( D_i \) equals \( f_n \) applied to \( \lambda \) and the \( n \)-vector obtained from \( a \) by omitting the \( n \)-th component. Thus, by the induction hypothesis,

\[ \frac{df_{n+1}}{\lambda} = \sum_{i=1}^{n+1} f_n(\lambda, a_1, \ldots, \hat{a}_i, \ldots, a_n) = \sum_{i=1}^{n+1} \left( \lambda^n + \lambda^{n-1} \sum_{1 \leq j \leq n, j \neq i} a_j^2 \right) \]

If we expand this sum, \( \lambda^n \) appears \( n + 1 \) times, while each \( a_j^2 \) appears \( n \) times, so that

\[ \frac{df_{n+1}}{\lambda} = (n + 1)\lambda^n + n\lambda^{n-1} \sum_{j=1}^{n+1} a_j^2. \]

Integrating from 0 to \( \lambda \) we see that

\[ f_{n+1}(\lambda) = \lambda^{n+1} + \lambda^n \sum_{j=1}^{n+1} a_j^2 + f(0). \]

The proof is complete once one shows that \( f(0) = 0 \). This is easily done, \( f(0) = [a_i a_j] \); we simply need to notice (for example) that the first row times \( a_2 \) equals the second row times \( a_1 \).


**Solution.** We have to show that if \( M = \alpha(\mathbb{R}) \), where \( \alpha : \mathbb{R} \to \mathbb{R}^2 \) is the map \( \alpha(x) = (x, x^2) \), then \( M \) is a 1-manifold in \( \mathbb{R}^2 \) covered by the single coordinate patch \( (\mathbb{R}, \alpha) \).

This exercise is a straightforward verification of the definition. We need to answer positively the following questions.

(a) Is \( M \) open in \( M \)? Of course it is!

(b) Is \( \mathbb{R} \) open in \( \mathbb{R} \)? Absolutely!

(c) Is \( \alpha \) of class \( C^r \) for some \( r \geq 1 \)? Yes, for ALL \( r \geq 1 \).

(d) Is \( \alpha \) one-to-one? Obviously!

(e) Is \( \alpha(x) \) of rank 1 for all \( x \in \mathbb{R} \)? Since \( D\alpha(x) = \begin{bmatrix} 1 \\ 2x \end{bmatrix} \), this is due to \( 1 \neq 0 \).

(f) Is \( \alpha^{-1} : M \to \mathbb{R} \) continuous? Since this is the map \( (x, x^2) \to x \), the answer is obviously yes.

And we are done!


**Solution.** This exercise is almost as simple as the previous one. In this case we have \( \beta = \alpha|_{H^1} \), where \( \alpha \) is the same as in the previous exercise, and \( N = \beta(H^1) \). We are asked to show \( N \) is a 1-manifold in \( \mathbb{R}^2 \). This time the questions (and answers) are:
(a) Is $N$ open in $N$? But, of course.
(b) Is $H^1$ open in $H^1$? Totally!
(c) Is $\beta$ of class $C^r$ for some $r \geq 1$? Since $\alpha$ is of class $C^\infty$ and extends $\beta$, the answer is most certainly yes.
(d) Is $\beta$ one-to-one? Clearly so!
(e) Is $\beta(x)$ of rank 1 for all $x \in \mathbb{R}$? Since $D\beta(x) = D\alpha(x)$ for all $x$, the reason $\beta$ has rank 1 is the same as that for $\alpha$.
(f) Is $\beta^{-1} : N \to \mathbb{R}$ continuous? Since this is again the map $(x, x^2) \to x$, the answer is a resounding yes.

Solution. We are supposed to show a) that $S^1$ is a 1-manifold in $\mathbb{R}^2$ and b) that the map
\[ t \mapsto (\cos 2\pi t, \sin 2\pi t) : [0, 1) \to S^1 \]
does not define a coordinate patch. Concerning part a), there are many ways of doing it. Here is a very direct one. Define $L, R, T, B$ by
\[ L = \{(x, y) \in S^1 : x < 0\}, \quad R = \{(x, y) \in S^1 : x > 0\}, \quad T = \{(x, y) \in S^1 : y > 0\}, \quad B = \{(x, y) \in S^1 : y < 0\}. \]
It is clear that all these sets are open in $S^1$ (being obvious intersections of open subsets of $\mathbb{R}^2$ with $S^1$) and that
\[ S^1 = L \cup R \cup T \cup B. \]
Let $I = (-1, 1)$, which is an open subset of $\mathbb{R}$. We define maps $\alpha, \beta, \gamma, \delta$ from $I$ to $\mathbb{R}^2$ by
\[ \alpha(t) = (-\sqrt{1-t^2}, t), \quad \beta(t) = (\sqrt{1-t^2}, t), \quad \gamma(t) = (t, \sqrt{1-t^2}), \quad \delta(t) = (t, -\sqrt{1-t^2}). \]
It is clear that all these maps are $C^\infty$ and that they satisfy
\[ \alpha(I) = L, \quad \beta(I) = R, \quad \gamma(I) = T, \quad \delta(I) = B. \]
Moreover, in each case the inverse is the restriction of the projection of $\mathbb{R} \times \mathbb{R}$ onto the first or second component, thus continuous. Moreover, the Jacobian matrix of $\alpha$ is
\[ D\alpha(t) = \begin{bmatrix} -2\pi \sin 2\pi t \\ 2\pi \cos 2\pi t \end{bmatrix} \]
and, since sine and cosine can never be simultaneously 0, $D\alpha$ is always of rank 1. This takes care of part a)
For part b) we notice that $[0, 1)$ is open in $H^1$, $\alpha$ is $C^\infty$ and one-to-one from $[0, 1)$ to $\mathbb{R}^2$, so we would have a patch except for one detail: The continuity of $\alpha^{-1}$. But $\alpha^{-1}$ is not continuous. In complex analysis we would express this by saying there is no continuous determination of the logarithm defined on the whole unit circle. A quick proof is: let
\[ a_n = \begin{cases} \cos 2\pi/n, \sin 2\pi/n & \text{if } n \text{ is even,} \\ \cos(2n\pi - 1)/n, \sin(2n\pi - 1)/n & \text{if } n \text{ is odd.} \end{cases} \]
Then $\lim_{n \to \infty} a_n = (1, 0)$, but $\{\alpha^{-1}(a_n)\}$ oscillates between 0 and 1.

This really a more general version of exercise 1. Let $M$ be the graph of $f$, so $M = \{(x, f(x)) : x \in A\}$. Define $\alpha : A \to \mathbb{R}^{k+1}$ by $\alpha(x) = (x, f(x))$. Now we ask and answer the same questions. The answer is obvious in every case.
(a) Is $M$ open in $M$? Yes.
(b) Is $A$ open in $\mathbb{R}^k$? Yes; it is a given.
(c) Is $\alpha$ of class $C^r$ for some $r \geq 1$? Yes, because $f$ is of class $C^r$.
(d) Is $\alpha$ one-to-one? Yes.
(e) Is $\alpha(x)$ of rank $k$ for all $x \in \mathbb{R}$? Since $D\alpha(x) = \begin{bmatrix} I \\ Df(x) \end{bmatrix}$, where $I$ is the $k \times k$ identity matrix, this is due to $\det I = 1 \neq 0$.

(f) Is $\alpha^{-1}: M \to \mathbb{R}$ continuous? Since this is the restriction of the map $(x_1, \ldots, x_{k+1}) \mapsto (x_1, \ldots, x_k)$ to the set $M$, the answer is obviously yes.

And we are done!

7. Textbook, §23, #5 (p. 202). We have to show that if $M$ is a $k$-manifold in $\mathbb{R}^m$ and $N$ is an $\ell$-manifold in $\mathbb{R}^n$, $M$ without boundary, then $M \times N$ is a $k + \ell$-manifold in $\mathbb{R}^{m+n}$.

It will be convenient to use the following notation. We identify, of course, $\mathbb{R}^{m+n}$ with $\mathbb{R}^m \times \mathbb{R}^n$ in the obvious way, so that $M \times N \subset \mathbb{R}^{m+n}$.

Let $(p, q) \in M \times N$. Since $p \in M$, there exists $V$ open in $M$, $U$ open in $\mathbb{R}^k$, $\alpha: U \to \mathbb{R}^m$ one-to-one and of class $C^r$ ($r \geq 1$) such that $\alpha(U) = V$ and $\alpha^{-1}: V \to U$ is continuous. Similarly, since $q \in N$, there exists $Y$ open in $N$, $X$ open in $\mathbb{H}^\ell$, $\beta: X \to \mathbb{R}^n$ one-to-one and of class $C^r$ ($r \geq 1$) such that $\beta(X) = Y$ and $\beta^{-1}: Y \to X$ is continuous.

It is then obvious that the map $\alpha \otimes \beta: (x, y) \mapsto (\alpha(x), \beta(y))$ is one-to-one and $C^r$ from $U \times X$ into $\mathbb{R}^m \times \mathbb{R}^n$ and that it maps $U \times X$ onto $V \times Y$, an open subset of $M \times N$. There are two more things that need to be verified; one is obvious, the other one easy but slightly subtle. We have to see the inverse function is continuous. This is obvious because

$$(\alpha \otimes \beta)^{-1}(p, q) = (\alpha^{-1}(p), \beta^{-1}(q)).$$

The slightly subtle part is to verify the one missing link: $U \times X$ is open in $H^{k+\ell}$. This is obvious if $X$ is open in $\mathbb{R}^\ell$; then $U \times X$ is open in $\mathbb{R}^{k+\ell}$. In general, if $X$ is open in $H^\ell$ there exists $W$ open in $\mathbb{R}^\ell$ such that $W \cap H^\ell = X$. Then

$$(U \times W) \cap H^{k+\ell} = U \times X,$$

and we see $U \times X$ is open in $H^{k+\ell}$. Finally

$$D(\alpha \otimes \beta)(x, y) = \begin{bmatrix} D\alpha(x) & 0 \\ 0 & D\beta(y) \end{bmatrix}$$

is of rank $k + \ell$, as is immediately proved.

We are done.

Notice that $H^k \times H^\ell$ is NOT open in $H^{k+\ell}$, so that if both $M, N$ have boundaries, in general $M \times N$ will not be a manifold.