Suppose we have a sequence \( \{a_n\} \) of real numbers. Here is a sort of practical way of determining the \( \lim \sup \) of the sequence.

**Step 1.** We ask, is there a largest term in the sequence?

If the answer is yes, let \( n_1 \) be the first index at which that largest term appears (it could appear more than once). If the answer is no, we can already declare that \( \lim \sup_{n \to \infty} a_n = \sup_{n \in \mathbb{N}} a_n \).

For example, suppose that \( \{a_n\} \) is the sequence \( \frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{5}{6}, \frac{1}{6}, \frac{7}{8}, \ldots \).

The odd terms decrease, the even terms increase; there is no largest term. We have

\[
\lim_{n \to \infty} \sup a_n = \sup_{n \in \mathbb{N}} a_n = 1.
\]

Now consider the sequence \( \{a_n\} \) defined recursively by

\[
a_1 = 1, a_2 = 3, \quad a_n = \frac{1}{2}(a_{n-1} + a_{n-2}) \text{ if } n \geq 3.
\]

The sequence looks like

\[
1, 3, 2, \frac{5}{2}, 9/4, 19/8, \ldots
\]

It is easy to see that the largest term is the second, thus in this case \( n_1 = 2, a_2 = 3 \).

For a third example, consider the sequence

\[
\sin 1, 2, \sin 3, 4, \sin 5, 6, \ldots
\]

Once again there is no largest term, so the \( \lim \sup \) of this sequence is the sup of its range, which is \( \infty \).

In sum, if the answer to the first question was NO, we are done. If it was YES, go to the next step.

**Step 2.** We now look at all the terms with index \( n > n_1 \) and repeat the question of Step 1: Is there a largest term among the terms \( a_n \) with \( n > n_1 \)? If the answer is yes, let \( n_2 \) be the index at which this term appears for the first time. If the answer is no, declare \( \lim \sup_{n \to \infty} a_n = \sup_{n > n_1} a_n \).

In sum, if the answer is NO, we are done. If yes go to the next step.

And so forth.

If we have gone as far as \( k \) steps, we’ll have selected terms \( a_{n_1}, \ldots, a_{n_k} \). Necessarily

\[
a_{n_1} \geq a_{n_2} \geq \cdots \geq a_{n_k}
\]

because each time we selected the largest of the remaining terms. The process will either terminate or go on forever. It terminates if after \( k \)-steps there is no largest term among the terms of index \( n > n_k \). In this case \( \lim \sup_{n \to \infty} a_n = \sup_{n > n_k} a_n \). If it goes on forever, then we have a decreasing subsequence \( \{a_{n_k}\} \) which will converge or diverge to \( -\infty \). Then \( \lim \sup_{n \to \infty} a_n = \lim_{k \to \infty} a_{n_k} \).

The precise definition (which is equivalent but I think preferable to the one in Rosenlicht) is: Let \( \{a_n\} \) be a sequence of real numbers. For each \( n \in \mathbb{N} \)
let \( t_n = \sup_{k \geq n} a_k \) (possibly \( t_n = \infty \)). Then \( \{t_n\} \) is a decreasing sequence of extended real numbers and we define

\[
\limsup_{n \to \infty} a_n = \lim_{n \to \infty} t_n = \inf_{n \in \mathbb{N}} t_n.
\]

You might notice that if \( \{a_n\} \) is increasing, then \( t_n = \sup n \in \mathbb{N} a_n \) for all \( n \in \mathbb{N} \). If \( \{a_n\} \) is decreasing, then \( t_n = a_n \) for all \( n \).

Similarly one defines the liminf. For each \( n \in \mathbb{N} \) let \( s_n = \inf_{k \geq n} a_k \) (possibly \( s_n = -\infty \)). Then \( \{s_n\} \) is an increasing sequence of extended real numbers and we define

\[
\liminf_{n \to \infty} a_n = \lim_{n \to \infty} s_n = \sup_{n \in \mathbb{N}} t_n.
\]

Exercise 1 Prove: If \( \{a_n\} \) is a sequence of real numbers, then \( \liminf_{n \to \infty} a_n = -\limsup_{n \to \infty} (-a_n) \).

The importance of the liminf, limsup is that they always exist (possibly equal to \( \infty \) or \( -\infty \)). Computing them by the definition is not always the best way. The following properties explain why these concepts are useful and suggest other ways of computing them.

Lemma 2 Let \( \{a_n\} \) be a sequence of real numbers. Let \( L = \limsup_{n \to \infty} a_n \in \mathbb{R} \cup \{\infty, -\infty\} \). Then \( L \) is the unique extended real number satisfying the following two properties:

1. If \( b \in \mathbb{R} \) and \( b > L \) then there exists \( N \in \mathbb{N} \) such that \( a_n < b \) for all \( n \geq N \).

2. If \( b \in \mathbb{R} \) and \( b < L \), then there exists an infinite number of indices \( n \) with \( a_n > b \). Equivalently, for every \( N \in \mathbb{N} \) there is \( n > N \) with \( a_n > b \).

Similarly, \( \ell = \liminf_{n \to \infty} a_n \in \mathbb{R} \cup \{-\infty, \infty\} \) is the unique extended real number satisfying the following two properties:

1. If \( b \in \mathbb{R} \) and \( b < \ell \) then there exists \( N \in \mathbb{N} \) such that \( a_n > b \) for all \( n \geq N \).

2. If \( b \in \mathbb{R} \) and \( b > \ell \), then there exists an infinite number of indices \( n \) with \( a_n < b \). Equivalently, for every \( N \in \mathbb{N} \) there is \( n > N \) with \( a_n < b \).

Proof. Let the sequences \( \{s_n\} \), \( \{t_n\} \) be defined as in the definition of liminf, limsup.

Assume \( L = \limsup_{n \to \infty} a_n \). We want to see properties 1 and 2 hold. Notice, incidentally, that if \( L = -\infty \), then property 1 is trivially (vacuously) satisfied: if \( L = \infty \) then property 2 holds trivially.

Assume \( b > L \). Since \( \limsup_{n \to \infty} t_n = L \), there exists \( N \) such that \( t_n < b \) for all \( n \geq N \); in particular for \( n = \) \( N \). But by the definition of \( t_n \) as the sup of the set of all terms with index \( \geq N \), that implies \( a_n < b \) for all \( n \geq N \) (also for all \( n > N \)).

Assume \( b < L \). Using again that \( \limsup_{n \to \infty} t_n = L \), there exists \( N \) such that \( t_n > b \) for all \( n \geq N \). By the definition of \( t_n \) as the sup of the set of all terms with index \( \geq N \), that implies that for each \( n \geq N \) there is \( k \geq n \) with \( a_k > b \). There is thus an infinite set of indices \( k \) for which \( a_k > b \); First there is \( k \geq N \) with this property (because \( t_N > b \). Next, since \( k + 1 > k \geq N \), thus \( t_{k+1} > b \), there is \( k^* \geq k + 1 > k \) such that \( a_{k^*} > b \). And so forth.

Conversely, assume \( L \in \mathbb{R} \cup \{-\infty, \infty\} \) satisfies the two properties. Let \( M = \limsup_{n \to \infty} a_n \). If \( L < M \), there is \( b \in \mathbb{R} \) such that \( L < b < M \). Because \( L \) satisfies property 1, there is \( N \) such that \( a_n < b \) for all \( n \geq N \). Because \( M \) satisfies property 2, there is an infinite number of indices \( n \) for which \( a_n > b \);
some of these indices (in fact, an infinity of them) must be \( \geq N \). If \( n \) is such an index, we have the contradiction \( a_n < b \) and \( a_n > b \). This proves \( L \geq M \).

To prove this we only used that \( L, M \) satisfy the two properties in question; we can thus reverse the roles of \( L \) and \( M \) and conclude \( M \geq L \); that is, \( L = M \).

The case of \( \liminf \) is proved similarly. Or one can use the exercise preceding this lemma.

**Theorem 3** Let \( \{a_n\} \) be a sequence of real numbers. Then \( \liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n \). One has \( \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n \) if and only if the sequence converges, or diverges to \( \infty \), or diverges to \( -\infty \). In any of these cases

\[
\liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = \lim_{n \to \infty} a_n.
\]

**Proof.** Assume first that the sequence \( \{a_n\} \) converges and let \( a = \lim_{n \to \infty} a_n \).

Then \( a \) satisfies the following two properties, which are easily seen to be equivalent to the definition of limit and unify the case of a finite limit with the case of the limit being \( \infty \) or \( -\infty \).

1. If \( b \in \mathbb{R} \) and \( b > a \) then there exists \( N \in \mathbb{N} \) such that \( a_n < b \) for all \( n \geq N \).

2. If \( b \in \mathbb{R} \) and \( b < a \), then there exists \( M \in \mathbb{N} \) such that \( a_n > b \) for all \( n \geq M \).

In fact, if \( a = \infty \), then 1 holds automatically, while 2 is the definition of \( \lim_{n \to \infty} a_n = \infty \); to say \( b \in \mathbb{R} \) and \( b < \infty \) is the same as saying \( b \in \mathbb{R} \).

Similarly if \( a = -\infty \). If \( a \in \mathbb{R} \), and the two conditions hold, given \( \epsilon > 0 \), one can take \( b = a + \epsilon \) to get \( N \) so \( a_n < a + \epsilon \) for all \( n \geq N \), \( b = a - \epsilon \) to get \( M \) so that \( a_n > a - \epsilon \) for all \( n \geq M \), thus \( |a_n - a| < \epsilon \) for all \( n \geq \max(N, M) \). Thus \( a \) is the limit of the sequence. Conversely, if \( a \) is the limit of the sequence, then we have theorems telling us that the two properties hold. But if \( a \) satisfies these two properties, it also satisfies the two properties of \( \limsup \) from the preceding lemma; in fact, the first property is identical and the second one much stronger.

The set \( \{n \geq M\} \) is certainly an infinite set. Similarly \( a \) satisfies the two defining properties of \( \liminf \).

Thus

\[
a = \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n.
\]

Conversely, if \( \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n \), then, by preceding lemma, the common value \( a = \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n \) satisfies the following four properties:

1. If \( b \in \mathbb{R} \) and \( b > a \) then there exists \( N \in \mathbb{N} \) such that \( a_n < b \) for all \( n \geq N \).

2. If \( b \in \mathbb{R} \) and \( b < a \), then there exists an infinite number of indices \( n \) with \( a_n > b \).

3. If \( b \in \mathbb{R} \) and \( b < a \) then there exists \( M \in \mathbb{N} \) such that \( a_n > b \) for all \( n \geq M \).

4. If \( b \in \mathbb{R} \) and \( b > a \), then there exists an infinite number of indices \( n \) with \( a_n < b \).

Of these properties, properties 2 and 4 are redundant, 2 is trivially implied by 3 and 4 by 1. Properties 1 and 3 are the properties we showed above define the
limit of the sequence, thus \( a = \lim_{n \to \infty} a_n \). 

Let \( \{a_n\} \) be a sequence of real numbers and let \( S \) be the set of all extended real numbers \( x \) such that there exists a subsequence of \( \{a_n\} \) with \( \lim_{n \to \infty} a_n = x \). \( S \) can never be empty. Since every bounded sequence has convergent subsequences, \( S \) is not empty if the sequence is bounded. If the sequence is unbounded, it either has a subsequence diverging to \( \infty \), or to \( -\infty \) (or has both). This set allows for another characterization of \( \limsup \) and \( \liminf \).

**Exercise 4**

1. Prove that the set \( S \) is closed in \([-\infty, \infty]\). Or, to be more specific, prove:

   - (a) If \( S \subseteq \mathbb{R} \), then \( S \) is closed.
   - (b) If \( S \) is not bounded above, then \( \infty \in S \).
   - (c) If \( S \) is not bounded below, then \( -\infty \in S \).

2. Prove that \( \limsup_{n \to \infty} a_n = \sup S \), \( \liminf_{n \to \infty} a_n = \inf S \).

**Hints:** For the “closed” part, suppose that \( \{s_k\} \) is a sequence of real numbers in \( S \) either converging to a real number \( s \), or diverging to \( \infty \) or to \( -\infty \). You should be able to argue that for each \( k \) there has to be \( n_k \) such that \( |s_k - a_{n_k}| < \) something going to 0 as \( k \to \infty \), for example \( |s_k - a_{n_k}| < 1/k \). With a bit of care you can get that \( n_k < n_{k+1} \) for all \( k \) and, voilà! there’s your subsequence converging to \( s \) or diverging to the same infinity \( \{s_n\} \) diverges to. To see that \( \limsup_{n \to \infty} a_n = \sup S \), you might want to see that \( \sup S \) has the two properties characterizing the \( \limsup \) mentioned in Lemma 2. If \( b > \sup S \), to see that there is \( N \) such that \( a_n < b \) for all \( n \geq N \), it might be best to proceed by contradiction. The contradiction should provide you with a subsequence \( \{a_{n_k}\} \) such that \( a_{n_k} > b \) for all \( k \). If this subsequence is bounded, it has in turn a convergent subsequence. If not bounded, it has a subsequence diverging to \( \infty \). A contradiction is sort of obvious from here. If \( b < \sup S \) and there is not an infinite number of terms \( > b \), then from a certain index \( N \) onward, \( a_n \leq b \) for all \( n \) and it should be easy to see from this that \( b \) is an upper bound of \( S \); a contradiction.

Notice that because \( S \) is closed, \( \limsup_{n \to \infty} a_n, \liminf_{n \to \infty} a_n \in S \); i.e., they are themselves limits of subsequences of \( \{a_n\} \).