I'm only adding solutions in cases where I felt most proofs that I saw where not quite satisfactory. There is a typo in Exercise 1, part f. The corrected version has been made part of Homework 7.

1. Assume \( A \) is an algebra of subsets of the set \( X \). Prove: If \( A, B \in A \), then so are \( A \setminus B = \{ x \in A : x \notin B \} \), \( A \triangle B = (A \cup B) \setminus (A \cap B) \).

Show also that \( A \triangle B = (A \setminus B) \cup (B \setminus A) \). It is called the symmetric difference between \( A \) and \( B \).

2. Classify the following families \( C \) of subsets of a set \( X \) according to:
   (a) ABNS (Algebra but not \( \sigma \)-algebra)
   (b) SA (\( \sigma \)-algebra)
   (c) N (Neither)

No proofs are required, but it is my fervent hope that you would all know how to prove it if so required.

(a) \( X \) any set, \( A \subset X \), \( \emptyset \neq A \neq X \), \( C = \{ \emptyset, X, A, X \setminus A \} \).

Solution. SA

(b) \( X = \mathbb{R} \) and \( C \) consists of all sets that are either finite or have a finite complement.

Solution. ABNS

(c) \( X = \mathbb{R} \) and \( C \) consists of all sets that are either countable or have a countable complement. (Finite sets count as countable!)

Solution. SA

(d) \( X = \mathbb{R} \) and \( C \) is the family of all open subsets of \( \mathbb{R} \).

Solution. N

(e) \( X = \mathbb{R} \) and \( C \) is the family of all sets that are either open or closed.

Solution. N

(f) \( X = \mathbb{R} \) and \( C \) is the family consisting of \( \mathbb{R} \), and all sets of one of the following forms:

i. \( \bigcup_{k=1}^{n} (a_k, b_k] \),

ii. \( (a_0, \infty] \cup \bigcup_{k=1}^{n} (a_k, b_k] \),

iii. \( (-\infty, b_0] \cup \bigcup_{k=1}^{n} (a_k, b_k] \).

(The number of sets \( n \) in the unions above varies, all \( n \in \mathbb{N} \) are involved.)

Solution. This was supposed to be ABNS, but is actually N. See Homework 7 for a corrected version.

3. The smallest \( \sigma \)-algebra containing all open subsets of a metric space \( X \) is known as the Borel \( \sigma \)-algebra of \( X \), or the family of Borel subsets of \( X \). We will write \( \mathcal{B}(X) \) to denote the \( \sigma \)-algebra of Borel subsets of \( X \). To repeat, \( \mathcal{B}(X) \) is the \( \sigma \)-algebra generated by the family of open subsets of \( X \). Prove that \( \mathcal{B}(X) \) is also the \( \sigma \)-algebra generated by all closed sets of \( X \).

Solution. Let \( \mathcal{O} \) be the family of all open subsets of \( X \), \( C \) the family of all closed subsets of \( X \). Let \( A \in \mathcal{O} \). Then \( A^c \) is closed, hence \( A^c \in \sigma(C) \), hence also \( A \in \sigma(C) \). This proves that \( \sigma(C) \) contains all open sets, being a \( \sigma \)-algebra containing all open sets, it must contain \( \sigma(\mathcal{O}) = \mathcal{B}(X) \). Similarly one sees that \( \mathcal{B}(X) = \sigma(\mathcal{O}) \supset \sigma(\mathcal{O}) \).

4. Let \( C \) be the family of all singleton subsets of \( \mathbb{R} \). That is, \( A \in C \) if and only if \( A \) is a set consisting of a single element. Describe explicitly:
(a) The algebra generated by $\mathcal{C}$.
(b) The $\sigma$-algebra generated by $\mathcal{C}$.

5. Let $X$ be a set. An outer measure in $X$ is any map $\mu : \mathcal{P}(X) \to [0, \infty]$ with the following properties.

(a) $\mu(\emptyset) = 0$.
(b) If $A \subset B \subset X$, then $\mu(A) \leq \mu(B)$.
(c) If $\{A_n\}$ is a sequence of subsets of $X$, then

$$\mu \left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Suppose we define for $A \subset \mathbb{R}$, $\mu(A) = \max(\sup A, 0)$. Is $\mu$ an outer measure in $\mathbb{R}$?

You should use here, incidentally, that $\sup \emptyset = -\infty$ and that a set is bounded above if and only if its sup is $< \infty$; in other words, the sup of sets that are not bounded above is $\infty$.

**Solution.**

We prove all the properties hold.

(a) $\mu(\emptyset) = \max(\sup \emptyset, 0) = \max(-\infty, 0) = 0$.
(b) If $A \subset B \subset \mathbb{R}$, then every upper bound of $B$ is also an upper bound of $A$; in particular $\sup B$ is an upper bound of $A$, hence $\sup B \geq \sup A$. Thus also

$$\mu(B) = \max(\sup B, 0) \geq \max(\sup A, 0) = \mu(A).$$

(c) Let $\{A_n\}$ be a sequence of subsets of $\mathbb{R}$. Let $x \in \bigcup_{n=1}^{\infty} A_n$. Then there is $m \in \mathbb{N}$ such that $x \in A_m$, hence

$$x \leq \sup A_m \leq \mu(A_m) \leq \sup_{n \in \mathbb{N}} \mu(A_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Thus $\sum_{n=1}^{\infty} \mu(A_n)$ is an upper bound of $\bigcup_{n=1}^{\infty} A_n$, hence $\sup \bigcup_{n=1}^{\infty} A_n \leq \sum_{n=1}^{\infty} \mu(A_n)$. Moreover, $\sum_{n=1}^{\infty} \mu(A_n) \geq 0$, thus

$$\mu \left( \bigcup_{n=1}^{\infty} A_n \right) = \max \left( \sup_{n=1}^{\infty} A_n, 0 \right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Simple, isn’t it?