1. Let $\sum_{n=0}^{\infty} a_n$ be a convergent series of complex numbers (or real numbers; that doesn’t matter). Prove: The power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $r \geq 1$.

**Solution.** This is a totally silly exercise. Since the series converges at 1, the radius $r$ satisfies $r \geq |1| = 1$.

2. **The space $\ell^1$.** The space $\ell^1$ is defined as the set of all sequences $a = \{a_1, a_2, \ldots \}$ of real numbers such that $\sum_{n=1}^{\infty} |a_n|$ converges. If $a = \{a_n\}$, $b = \{b_n\}$ are in $\ell^1$, one defines $a + b = \{a_n + b_n\}$; if $a = \{a_n\} \in \ell^1$, $c \in \mathbb{R}$, one defines $ca = \{ca_n\}$.

In other words, addition and scalar multiplication are defined in the obvious way.

(a) Prove: With the operations as defined, $\ell^1$ is a real vector space. This being a senior level/graduate course, please do not waste time with trivialities like proving the existence of a zero element, associativity of the sum, etc. All you need to do is show that $\ell^1$ is closed under the operations. That is also close to trivial, but at least not incredibly trivial.

(b) Define (as one does) a norm in $\ell^1$ by

$$\|a\|_1 = \sum_{n=1}^{\infty} |a_n|, \quad \text{if } a = \{a_n\}.$$ 

Prove: $\| \cdot \|_1$ is indeed a norm in $\ell^1$.

(c) Prove: With the norm as defined, $\ell^1$ is a Banach space.

**Solution.** Everything is (or should be) quite obvious except, perhaps the completeness. To prove $\ell^1$ is complete with the metric as defined, let us suppose we have a Cauchy sequence. Since this is a sequence of sequences, we need to have some notation involving either subscripts and superscripts, or double subscripts. I’ll opt for the double subscripts.

$$\|a_m - a_n\|_1 = \sum_{k=1}^{\infty} |a_{mk} - a_{nk}| < \epsilon$$

if $m,n \geq N$. For every $j \in \mathbb{N}$ we have that $m,n \geq N$ implies

$$|a_{mj} - a_{nj}| \leq \sum_{k=1}^{\infty} |a_{mk} - a_{nk}| < \epsilon$$

so that $\{a_{nj}\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers for each $j \in \mathbb{N}$, hence converges. Define $a_j = \lim_{n \to \infty} a_{nj}$ for $j \in \mathbb{N}$. Let $a = \{a_j\}_{j=1}^{\infty}$.

**A small digression.** In proving completeness of a space one usually runs into three difficulties: One is finding something to which the Cauchy sequence converges. Assuming one overcomes the first one, the second difficulty is showing that what one found as supposed limit of the sequence is in the space. The third difficulty is proving the sequence actually converges to the assumed limit. The second and third difficulty are frequently interchangeable. We just overcame the first difficulty; if the sequence converges to anything it has to be to $a$. **Back to the proof.**

The sequence $\{a_n\}$ is a Cauchy sequence, thus is bounded. (recall that Cauchy sequences are bounded!). In other words, there is some real number $C > 0$ such that $\|a_n\|_1 \leq C$ for $n = 1, 2, 3, \ldots$ For every $K \in \mathbb{N}$ we have

$$\sum_{k=1}^{K} |a_{nk}| \leq \sum_{k=1}^{\infty} |a_{nk}| = \|a_n\|_1 \leq C$$

for $n = 1, 2, 3, \ldots$. Since (it being a finite sum!)

$$\lim_{n \to \infty} \sum_{k=1}^{K} |a_{nk}| = \sum_{k=1}^{K} \lim_{n \to \infty} a_{nk} = \sum_{k=1}^{K} |a_k|$$
we see that
\[ \sum_{k=1}^{K} |a_k| \leq C \]
for all \( K \), hence also
\[ \|a\|_1 = \sum_{k=1}^{\infty} |a_k| \leq C, \]
proving \( a \in \ell^1 \). Finally, we need to show that \( \|a_n - a\|_1 \to 0 \) as \( n \to \infty \). The problem here is to control well the tails of the sequences.

Let \( \epsilon > 0 \) be given. Because \( a \in \ell^1 \), there is \( K \) such that
\[ \sum_{k=K+1}^{\infty} |a_k| < \frac{\epsilon}{2}. \]

By the Cauchy condition, there is \( N \) such that \( \|a_n - a_m\|_1 < \frac{\epsilon}{2} \) whenever \( n, m \geq N \). Thus, for every \( p \geq 1 \), \( n, m \geq N \),
\[ \sum_{k=K+1}^{K+p} |a_{nk} - a_{mk}| \leq \|a_n - a_m\|_1 < \frac{\epsilon}{2}. \]

Keeping \( p \) fixed for a moment we let \( m \to \infty \) (the sum being finite, the limit can be interchanged with the sum) to conclude that

\[ \sum_{k=K+1}^{K+p} |a_{nk} - a_k| < \frac{\epsilon}{2} \]

for all \( p \in \mathbb{N}, n \in \mathbb{N}, n \geq N \). This being true for all \( p \geq 1 \) it is also true for the limit as \( p \to \infty \); in other words,

\[ \sum_{k=K+1}^{\infty} |a_{nk} - a_k| < \frac{\epsilon}{2} \]

if \( n \geq N \). Now (again because the sum is finite)
\[ \lim_{n \to \infty} \sum_{k=1}^{K} |a_{nk} - a_k| = 0, \]
thus there is \( N_1 \) such that
\[ \sum_{k=1}^{K} |a_{nk} - a_k| < \frac{\epsilon}{2} \]
if \( n \geq N_1 \). Let \( n \geq \max(N, N_1) \). Then
\[ \|a_n - a\|_1 = \sum_{k=1}^{\infty} |a_{nk} - a_k| = \sum_{k=1}^{K} |a_{nk} - a_k| + \sum_{k=K+1}^{\infty} |a_{nk} - a_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \]
proving that \( \{a_n\} \) converges to \( a \) in \( \ell^1 \).

NOTES:

1. If you got only 5 points for this exercise it is because I assumed that everybody would have no difficulty proving everything but the completeness, thus I emphasized that part of the problem. A 5 means that, in my opinion, your proof of completeness has profound flaws. But see also Note 3.

2. Several of you used (essentially) the following result without justification:
   If \( \lim_{n \to \infty} a_{nk} = a_k \), then
   \[ \sum_{k=1}^{\infty} |a_k| = \lim_{n \to \infty} \sum_{k=1}^{\infty} |a_{nk}|. \]
Exchanging infinite processes is a delicate operation; it has to be done with care. For example, suppose we take

\[ a_{nk} = \begin{cases} 
\frac{1}{n2^k} & \text{if } k \neq n, \\
 k & \text{if } k = n
\end{cases} \]

For a fixed \( k \), once \( n > k \) the sequence is the same as \( 1/(2^k n) \to 0 \) as \( n \to \infty \), thus \( a_k = \lim_{n \to \infty} a_{nk} = 0 \) for all \( k \). However, it should be clear that

\[ \sum_{k=1}^{\infty} a_{nk} > a_{kk} = k \]

for all \( n \). So we have here a situation where each series \( \sum_{k=1}^{\infty} a_{nk} \) converges (absolutely since all terms are positive), the series coming from the limit sequence \( \{a_k\} \) converges even more so since all terms are 0, but (1) is quite false.

3. Several of you got 5 for this exercise for using a correct proof incorrectly. It gave me the impression that you were not quite sure of what you were doing. Before using that proof, a lot of things have to be proved. Here is a sketch of how it works.

Let \( X \) be a normed vector space. Because it is a vector space, \( \sum_{k=1}^{n} a_k \) makes sense for any finite number \( a_1, \ldots, a_k \in X \). If \( a_n \in X \) for \( n = 1, 2, 3, \ldots \), one can talk now of the series \( \sum_{n=1}^{\infty} a_n \) and say that it converges if and only if the sequence \( \{\sum_{n=1}^{\infty} a_k\}_{n=1}^{\infty} \) converges, and write \( a = \sum_{n=1}^{\infty} a_n \) if \( \lim_{n \to \infty} \sum_{k=1}^{n} a_k = a \); i.e., if \( \lim_{n \to \infty} \|\sum_{k=1}^{n} a_k - a\| = 0 \). We say the series converges absolutely if the series of nonnegative real terms \( \sum_{n=1}^{\infty} \|a_n\| \) converges. Then one has:

**Theorem 1** If \( X \) is a complete normed space; i.e., a Banach space, absolute convergence implies convergence.

The proof is similar (or identical) to the proof in the complex case, but it requires a proof.

Another result needed for this other proof of completeness, a result that cannot be used so freely, that requires a proof, is one that is valid in all metric spaces, namely:

**Lemma 2** Let \( \{p_n\} \) be a Cauchy sequence in a metric space \( M \). Then there is a subsequence \( \{p_{n_k}\} \) such that \( d(p_{n_{k-1}}, p_{n_k}) < 2^{-k} \).

The proof is easy, but at our level it is needed.

The main result involved is the converse of Theorem 1, namely:

**Theorem 3** If \( X \) is a normed vector space in which every absolutely convergent sequence converges, then \( X \) is a Banach space.

People who used this approach provided what amounts to a moderately incorrect proof of this result.

All in all, I felt there were too many gaps, and several errors, in the proofs I saw, to warrant more points than I gave.

3. Let \( \{f_n\} \) be a sequence of continuously differentiable functions on \([0, 1]\) with \( f_n(0) = f_n'(0) \) and \( |f_n'(x)| \leq 1 \) for all \( x \in [0, 1] \) and \( n \in \mathbb{N} \). Show that if \( \lim_{n \to \infty} f_n(x) = f(x) \) for all \( x \in [0, 1] \), then \( f \) is continuous on \([0, 1]\). Must the sequence converge? Must there be a convergent subsequence?

(This exercise appeared in the last analysis qualifier. You have the advantage over the people who took the exam of knowing that it has to do with something we covered recently.)

**Solution.** Claim that the sequence \( \{f_n\} \) is equicontinuous and bounded. In fact, given \( x, y \in [0, 1] \) we have that \( |f_n(x) - f_n(y)| \leq |x - y| \) by the Mean Value Theorem, thus given \( \epsilon > 0 \) we can take \( \delta = \epsilon \) to conclude that \( |f_n(x) - f_n(y)| < \epsilon \) for all \( x, y \in [0, 1] \) with \( |x - y| < \delta, n = 1, 2, 3, \ldots \). This establishes equicontinuity. For boundedness, we can use that

\[ f_n(x) = f_n(0) + \int_0^x f_n'(t) \, dt, \]
hence
\[ |f_n(x)| \leq |f_n(0)| + \int_0^x |f_n'(t)| \, dt \leq 1 + \int_0^x x \, dt = 1 + x \leq 2 \]
for all \( n \in \mathbb{N}, \, x \in [0, 1] \). The claim is established.

By the Theorem of Arzela-Ascoli, it follows that \( \{f_n\} \) has a uniformly convergent subsequence \( \{f_{n_k}\} \). Since this sequence also converges pointwise, the limit must be \( f \). As the uniform limit of a sequence of continuous functions, \( f \) must be continuous.

The rest of the exercise is a bit ambiguous. I think I copied it from a non final version of the qualifier without checking too carefully. “Must the sequence converge?” can be interpreted in at least two ways:

(a) Given all the other hypothesis, is the hypothesis \( \lim_{n \to \infty} f_n(x) = f(x) \) for all \( x \in [0, 1] \) redundant; is it needed? Interpreted in this way the answer is no; the sequence does not have to converge; counterexamples provided upon request.

(b) Must the full sequence converge \textit{uniformly}? The answer in this case is yes, and I made this an exercise Homework 5.

In either case, the answer to the last question is yes.

I will only grade your proof of the continuity of \( f \) (or lack thereof).

4. Let \([a, b]\) be a closed and bounded interval and let \( \mathcal{E} \subset C([a, b]) \). Assume that \( \mathcal{E} \) is equicontinuous and bounded. Define \( f : [a, b] \to \mathbb{R} \) by: if \( a \leq x \leq b \), then
\[
    f(x) = \sup\{g(x) : g \in \mathcal{E}\}.
\]

Prove: \( f \) is continuous.

\textbf{Solution.} Because \( \mathcal{E} \) is bounded, \( \sup\{g(x) : g \in \mathcal{E}\} < \infty \) for all \( x \in [a, b] \); \( f \) is a well defined, real valued, function on \([a, b]\). By equicontinuity, given \( \epsilon > 0 \) there is \( \delta > 0 \) such that if \( x, y \in [a, b] \) satisfy \( |x - y| < \delta \), then \( |g(x) - g(y)| < \epsilon/2 \) for all for all \( g \in \mathcal{E} \). Now let \( x, y \in [a, b] \) and assume \( |x - y| < \delta \). By definition of \( f(x) \) as the sup of the set of all values \( g(x), \, g \in \mathcal{E} \), there is \( g \in \mathcal{E} \) such that
\[
    f(x) - \frac{\epsilon}{2} < g(x), \quad \text{thus} \quad f(x) < g(x) + \frac{\epsilon}{2}.
\]
By the choice of \( \delta \) and because \( g \in \mathcal{E}, \, |x - y| < \delta \),
\[
    |g(x) - g(y)| < \frac{\epsilon}{2}, \quad \text{thus} \quad g(x) < g(y) + \frac{\epsilon}{2}.
\]
By the definition of \( f(y) \) as a sup,
\[
    g(y) \leq f(y).
\]
Putting it all together, we get
\[
    f(x) < g(x) + \frac{\epsilon}{2} < g(y) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq f(y) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = f(y) + \epsilon;
\]
that is \( f(x) - f(y) < \epsilon \). Changing the roles of \( x \) and \( y \), we similarly get that \( f(y) - f(x) < \epsilon \), thus proving that \( |f(x) - f(y)| < \epsilon \) if \( |x - y| < \delta \). Continuity follows.