General Comments.

- Most everybody did most everything right; sort of. You might think I am being nasty or pedantic in taking away points for what could be minor faults. But, to put it in mountaineering terms, we have to learn to watch our step while we are still in relatively flat territory; these small missteps, once we are higher up, can have us falling off cliffs into the bottomless mathematical abyss, where darkness reigns forever.

- Here are some simply proved facts. I will assume you know how to prove them, thus in the future you can use them without further comments:

1. Let \( \{a_n\} \) be a sequence of real numbers and assume \( a_n \leq b \) for all \( n \in \mathbb{N} \). Then \( \limsup_{n \to \infty} a_n \leq b \). In particular, in case of convergence, \( \lim_{n \to \infty} a_n \leq b \). I mention this because there is no need to bring in the constant sequence \( \{b, b, b \ldots\} \) into the picture so as to say, in case of convergence, \( \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b = b \). The result you are using to get \( \lim_{n \to \infty} a_n \leq b \), namely that if \( \{a_n\}, \{b_n\} \) are sequences with \( a_n \leq b_n \) for all \( n \), then (if both converge) \( \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n \) is not simpler in any way than the first result I stated, namely that if \( a_n \leq b \) for all \( n \), then \( \limsup_{n \to \infty} a_n \leq b \).

2. Similarly, if \( a_n \geq b \) for all \( n \in \mathbb{N} \). Then \( \liminf_{n \to \infty} a_n \geq b \). This can be used without proof, except if it comes up in exam 1 as an exercise.

Exercise 1  Given two convergent sequences \( \{s_n\}, \{t_n\} \), if we know that \( s_n < t_n \) for all \( n \), we can only conclude that \( \lim_{n \to \infty} s_n \leq \lim_{n \to \infty} t_n \); the limits could be equal. However, if \( \sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n \) are series of non-negative terms and we have \( 0 \leq a_n < b_n \) for all \( n \in \mathbb{N}_0 \), prove: If both series converge, then

\[
\sum_{n=0}^{\infty} a_n < \sum_{n=0}^{\infty} b_n.
\]

Solution. A much stronger version of this exercise holds: If \( 0 \leq a_n \leq b_n \) for all \( n \) and there is \( k \) such that \( a_k < b_k \), that is enough for

\[
\sum_{n=0}^{\infty} a_n < \sum_{n=0}^{\infty} b_n,
\]

assuming that \( \sum_{n=0}^{\infty} a_n \) converges. In fact, then from elementary College Algebra (or grade school) arithmetic we have

\[
\sum_{n=0}^{k} a_n = a_0 + a_1 + \cdots + a_k < b_0 + b_1 + \cdots + b_k = \sum_{n=0}^{k} b_n.
\]

On the other hand, because \( a_n \leq b_n \) for all \( n \), hence also for \( n \geq k + 1 \), again by grade school arithmetic (or, at most Calculus 1 arithmetic)

\[
\sum_{n=k+1}^{N} a_n \leq \sum_{n=k+1}^{N} b_n \quad \text{for all} \ N \geq k + 1;
\]

\[
\sum_{n=k+1}^{N} a_n < \sum_{n=k+1}^{N} b_n.
\]

\[
\sum_{n=0}^{\infty} a_n < \sum_{n=0}^{\infty} b_n
\]
Taking limits for $N \to \infty$ we get
\[
\sum_{n=k+1}^{\infty} a_n \leq \sum_{n=k+1}^{\infty} b_n.
\]
If $\sum_{n=k+1}^{\infty} a_n < \infty$, then we conclude that
\[
\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{k} a_n + \sum_{n=k+1}^{\infty} a_n < \sum_{n=0}^{k} b_n + \sum_{n=k+1}^{\infty} b_n = \sum_{n=0}^{\infty} b_n.
\]

Exercise 2 Prove: If $\sum_{n=0}^{\infty} a_n$ converges, then
\[
\left| \sum_{n=0}^{\infty} a_n \right| \leq \sum_{n=0}^{\infty} |a_n|.
\]

Solution. Let $s_n = \sum_{k=1}^{n} a_k$. By the triangle inequality
\[
|s_n| \leq \sum_{k=1}^{n} |a_k|
\]
for all $n \in \mathbb{N}$. The result follows taking limits for $n \to \infty$.

Exercise 3 Let $a_n \geq 0$ for $n \in \mathbb{N}_0$. Prove
\[
\sum_{n=1}^{\infty} a_n = \sup \{ \sum_{n \in F} a_n : F \subset \mathbb{N}_0, F \text{ finite} \}.
\]

Proving this includes seeing that the sup on the right hand side is finite if and only if the series on the right converges; equivalently, divergence of the series (which means divergence to $\infty$) is equivalent to the set of finite sums being unbounded, hence the sup being $\infty$.

Solution. Let $\alpha = \sup \{ \sum_{n \in F} a_n : F \subset \mathbb{N}_0, F \text{ finite} \}$ and $\beta = \sum_{n=1}^{\infty} a_n$; from general theory both are well defined extended real numbers (i.e., one or both could be $\infty$).

Let $F$ be a finite subset of $\mathbb{N}$. If $N = \max F$, then $F \subset \{1, \ldots, N\}$ and since all terms of the series are non-negative it is clear that $\sum_{n \in F} a_n \leq \sum_{n=1}^{N} a_n \leq \beta$. Thus $\beta$ is an upper bound of the set of which $\alpha$ is the sup, hence $\alpha \leq \beta$. On the other hand, for every $N \in \mathbb{N}$, $F = \{1, \ldots, N\}$ is a finite subset of $\mathbb{N}$, thus
\[
\sum_{n=1}^{N} a_n = \sum_{n \in F} a_n \leq \alpha.
\]
This being true for all $N \in \mathbb{N}$, we can let $N \to \infty$ to get $\beta \leq \alpha$.

Exercise 4 Consider the series $\sum_{n=0}^{\infty} a_n$ where
\[
a_n = \begin{cases} 
2^{-n} & \text{if } n \text{ is even}, \\
3^{-n} & \text{if } n \text{ is odd}.
\end{cases}
\]

That is, the series
\[
1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{27} + \frac{1}{16} + \cdots.
\]
Show that the ratio test is inconclusive, but the root test applies, proving convergence. Is there any case in which the root test can be inconclusive, but the ratio test is conclusive? Conclude that the root test is superior to the ratio test.
Solution. First, the ratio test. There is only one sequence of ratios. There is not an even and an odd case to consider; however, one can look at what happens to the subsequence of odd terms and of even terms. One converges to 0, the other one diverges to $\infty$. Since all ratios are positive, no subsequence can converge to a value $< 0$, nor is there anything larger than $\infty$, this proves that
\[
\liminf_{n \to \infty} \frac{a_{n+1}}{a_n} = 0, \quad \limsup_{n \to \infty} \frac{a_{n+1}}{a_n} = \infty,
\]
hence the ratio test is inconclusive. For the root test, it is quite easy to see that the set of limit points of subsequences $\{n^{\sqrt{n}}\}$ is the set $\{1/3, 1/2\}$, thus
\[
\limsup_{n \to \infty} n^{\sqrt{n}} = 1/2 < 1 \text{ and we have convergence.}
\]
The root test is inconclusive only if $\limsup_{n \to \infty} n^{\sqrt{n}} |a_{n+1}|/|a_n| = 1$, in which case $\limsup_{n \to \infty} |a_{n+1}|/|a_n| \leq 1$ and the ratio test is also inconclusive.

Exercise 5 (Rosenlicht, Chapter VII, #11) Show the convergence of the series
\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+x} \right)
\]
of real valued functions on $\mathbb{R} \setminus \{-1, -2, -3, \ldots\}$.

Solution. I will try to present a proof which can then be easily modified to prove the stronger version I proposed as an optional exercise, namely the uniform convergence on any compact subset of $\mathbb{R} \setminus \{-1, -2, -3, \ldots\}$. Suppose $x \in \mathbb{R} \setminus \{-1, -2, -3, \ldots\}$. Due to this I won’t divide into cases ($x \geq 0$, $x < 0$, not that there is anything wrong with that) and simply select $N \in \mathbb{N}$ such that $N > |x|$. If $n > N$, then $n + x \geq n - |x| \geq n - N$ and
\[
\left| \frac{1}{n} - \frac{1}{n + x} \right| = \frac{|x|}{n(n + x)} \leq \frac{N}{n(n - N)}.
\]
It follows that the series
\[
\sum_{n=N+1}^{\infty} \left( \frac{1}{n} - \frac{1}{n + x} \right)
\]
converges, by comparison with the convergent (and even telescoping) series
\[
\sum_{n=N+1}^{\infty} \frac{N}{n(n - N)} = N \sum_{n=1}^{\infty} \frac{1}{(n + N)n}.
\]
Thus
\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n + x} \right)
\]
also converges. This argument is easily modified to get the desired uniform convergence. Assume now that $C$ is a compact subset of $\mathbb{R} \setminus \{-1, -2, -3, \ldots\}$. Being compact, there is $N \in \mathbb{N}$ such that $|x| < N$ for all $x \in C$ and the previous estimates show that the series
\[
\sum_{n=N+1}^{\infty} \left( \frac{1}{n} - \frac{1}{n + x} \right)
\]
converges uniformly for $x \in C$ by the Weierstrass $M$-test. So does the full series.

Exercise 6 (Rosenlicht, Chapter VII, #12.) Show that if $a_1 + a_2 + a_3 + \cdots$ is an absolutely convergent series of real numbers, then $a_1^2 + a_2^2 + a_3^2 + \cdots$ converges.
Solution. Here are two perfectly valid solutions.

I. Since $\sum_{n=1}^{\infty} |a_n|$ converges, one has $\lim_{n \to \infty} |a_n| = 0$, hence there is $N \in \mathbb{N}$ such that $|a_n| < 1$ if $n \geq N$. Then $a_n^2 < |a_n|$ for $n \geq N$, thus $\sum_{n=N}^{\infty} a_n^2$ converges by comparison with $\sum_{n=N}^{\infty} |a_n|$, and so does $\sum_{n=1}^{\infty} a_n^2$.

II. Let $\alpha = \sum_{n=1}^{\infty} |a_n|$. For every $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} a_k^2 \leq \left( \sum_{k=1}^{n} |a_k| \right)^2 \leq \alpha^2.$$ 

Thus all partial sums of $\sum_{n=1}^{\infty} a_n^2$ are bounded; since the terms are all non-negative, the series converges.