1. **(10 points)** We saw in class that a subset $A$ of $\mathbb{R}$ is Lebesgue measurable if and only if for every $\epsilon > 0$ there exists an open set $U$ such that $A \subset U$ and $m(U \setminus A) < \epsilon$, also if and only if there exists a closed set $F$ such that $F \subset A$ and $m(A \setminus F) < \epsilon$. Use this to prove: A subset $A$ of $\mathbb{R}$ is Lebesgue measurable if and only if for every $\epsilon > 0$ there exists an open set $U$ and a closed set $F$ such that $F \subset A \subset U$ and $m(U \setminus F) < \epsilon$. (Yes, this is very easy, so do it right.)

2. **(20 points)** Let $X$ be a set, $\mathcal{M}$ a $\sigma$-algebra in $X$ and let $\mu : \mathcal{M} \to X$ satisfy:

(a) $\mu(\emptyset) = 0$.

(b) If $A, B \in \mathcal{M}$ and $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Prove: If $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n)$ whenever $\{A_n\}$ is a sequence of sets in $\mathcal{M}$ such that $A_1 \subset A_2 \subset A_3 \cdots$, then $\mu$ is $\sigma$-additive; that is, $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ whenever $\{A_n\}$ is a sequence of sets in $\mathcal{M}$ such that $A_n \cap A_m = \emptyset$ for $n \neq m$.

3. Let $\tilde{\ell}$ be a function from open subintervals of $\mathbb{R}$ to $[0, \infty]$ so that if $I = (a, b)$, $-\infty \leq a \leq b$, then $\tilde{\ell}(I) \in [0, \infty)$. Assume it sends the empty interval to 0. If $A \subset \mathbb{R}$ define

$$
\mu^*(A) = \inf \{ \sum_{n=1}^{\infty} \tilde{\ell}(I_n) : I_1, I_2, \ldots \text{open intervals}, A \subset \bigcup_{n=1}^{\infty} I_n \}.
$$

(a) **(20 points)** Prove that $\mu^*$ is an outer measure; in other words prove that $\mu^*(\emptyset) = 0$, $A \subset B$ implies $\mu^*(A) \leq \mu^*(B)$, $\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

(b)* Give an example to show that $\mu^*(I) = \tilde{\ell}(I)$ for open intervals $I$ may fail to hold.

(c)* Let $f : \mathbb{R} \to \mathbb{R}$ be increasing and continuous. If $I = (a, b)$ is an open interval, define $\tilde{\ell}(I) = f(b) - f(a)$. Prove: Defining $\mu^*$ as above, all Borel sets are measurable. Is it necessary for $f$ to be continuous?

4. **(10 points)** Let $A \subset \mathbb{R}$. Prove that $A$ is measurable if and only if the characteristic function $\chi_A$ of $A$ is measurable. Here $\chi_A(x) = 1$ if $x \in A$, 0 if $x \notin A$.

5. **(20 points)** Let $f_n : \mathbb{R} \to \mathbb{R}$ be measurable for $n = 1, 2, \ldots$ and let

$$
D = \{ x : \lim_{n \to \infty} f_n(x) \text{ exists} \}.
$$

Prove: $D$ is measurable.

6. **(20 points)** Let $E$ be a null set; that is, $E \subset \mathbb{R}$ and $m(E) = 0$, where $m$ is Lebesgue measure. Prove that its complement is dense; that is, the closure of $\mathbb{R} \setminus E$ is $\mathbb{R}$.