

3. Definition of \( \int \phi(x) \, dx = \int \phi \, dm = \int \phi \) if \( \phi \) is a simple function such that the sets on which \( \phi \) assumes a non-zero value have finite measure. Or, as Royden puts it, \( \phi \) vanishes outside of a set of finite measure.

4. Lemma 1 on page 78.

5. Proposition 2 on page 78.

6. We now need a result that is part of Royden’s proof his Proposition 3 (sort of), but I want to have it as a separate result.

**Lemma 1** Let \( E \) be a measurable subset of \( \mathbb{R} \) and let \( f : E \to [0, \infty) \) be measurable. There exists a sequence \( \{ \varphi_n \} \) of non-negative simple functions such that:

(a) \( \varphi_1(x) \leq \varphi_2(x) \leq \cdots \) for all \( x \in E \).

(b) \( \lim_{n \to \infty} \varphi_n(x) = f(x) \) for all \( x \in E \). (The sequence converges pointwise to \( f \))

If \( f \) is bounded, then the sequence converges uniformly to \( f \) on \( E \).

**Proof.** For \( n = 1, 2, \ldots \) define \( \varphi_n \) as follows. Let

\[ E_{nk} = \{ x \in E : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \} \]

for \( k = 0, 1, 2, \ldots \). Actually, we only consider these sets for \( 0 \leq k \leq n2^n - 1 \).

Let \( x \in E \). If \( f(x) \geq n \), set \( \varphi_n(x) = n \). If \( 0 \leq f(x) < n \) there exists a unique \( k \) such that \( \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \), \( 0 \leq k \leq n2^n - 1 \). Set \( \varphi_n(x) = k/2^n \). An equivalent definition is

\[ \varphi_n = \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \chi_{E_{nk}} + n \chi_{F_n} \]

where \( F_n = E \setminus \bigcup_{k=0}^{2^n-1} E_{nk} \). The sets \( E_{nk}, F_n \) are measurable because \( f \) is measurable, thus \( \varphi_n \) is indeed a simple function for each \( n \); incidentally it vanishes outside of \( E \). Obviously \( \varphi_n(x) \geq 0 \) for all \( n \). It is also quite easy to see that \( \varphi_n(x) \leq \varphi_{n+1}(x) \leq f(x) \) for all \( x \in E \). In fact, in the first place the definition makes it quite clear that \( \varphi_n(x) \leq f(x) \) for all \( n \in \mathbb{N}, x \in E \). To see that \( \varphi_n(x) \leq \varphi_{n+1}(x) \) assume first that \( f(x) > n+1 \). Then \( \varphi_n(x) = n < n+1 = \varphi_{n+1}(x) < f(x) \). Assume \( n < f(x) \leq n+1 \). In this case \( \varphi_n(x) = n, \varphi_{n+1}(x) = k/2^{n+1} \), where \( k/2^{n+1} \leq f(x) < (k+1)/2^{n+1} \). Then \( (k+1)/2^{n+1} > n \), \( k + 1 > n2^{n+1} \), hence \( k \geq n2^{n+1} \) and \( \varphi_{n+1}(x) = k/2^{n+1} \geq n = \varphi_n(x) \). Finally, assume \( f(x) < n \).

Let \( k \) be such that \( k/2^n \leq f(x) < (k+1)/2^n \), so \( \varphi_n(x) = k/2^n \). Then either \( k/2^n \leq f(x) < (2k+1)/2^{n+1} \) or \( (2k+1)/2^{n+1} \leq f(x) < (k+1)/2^n \). (Notice that \( (2k+1)/2^{n+1} \) is the midpoint of the interval \( [k/2^n, (k+1)/2^n] \).) The former case can be written as \( (2k)/2^{n+1} \leq f(x) < (2k+1)/2^{n+1} \), hence \( \varphi_{n+1}(x) = 2k/2^{n+1} = k/2^{n} = \varphi_n(x) \); the latter case can be written as \( (2k+1)/2^{n+1} \leq f(x) < (2k+2)/2^{n+1} \), hence \( \varphi_{n+1}(x) = (2k+1)/2^{n+1} > k/2^n = \varphi_n(x) \). This completes the proof that the sequence \( \{ \varphi_n \} \) is increasing in the sense that \( \{ \varphi_n(x) \} \) is an increasing sequence of real numbers for each \( x \in E \).

Let \( x \in E \). If \( f(x) = \infty \) then \( \varphi_n(x) = n \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \varphi_n(x) = \infty = f(x) \). Assume \( f(x) \leq M \), some \( M \in \mathbb{R} \). Given \( \epsilon > 0 \) let \( N \in \mathbb{N} \) satisfy \( N > M \) and \( 2^{-N} < \epsilon \). If \( n \geq N \) then \( f(x) \leq n \) and there exists \( k \in \{ 0, 1, \ldots, n2^n - 1 \} \) such that \( k/2^n \leq f(x) < (k+1)/2^n \). Then \( \varphi_n(x) = k/2^n \) and

\[ 0 \leq f(x) - \varphi_n(x) < k/2^n - k/2^n = \frac{1}{2^n} < \epsilon. \]
We see that \( \lim_{n \to \infty} \varphi_n(x) = f(x) \) also in this case. This completes the proof that the sequence \( \{ \varphi_n \} \) converges pointwise to \( f \). If \( f \) is bounded, then there is \( M \in \mathbb{R} \) such that \( f(x) \leq M \) for all \( x \in E \) and the argument given above proves the sequence converges uniformly to \( f \).

7. Now come some major differences for a while. We extend the definition of the integral of a simple non-negative function to include the case in which the function assumes a positive value on a set of infinite measure. If \( \varphi = \sum_{j=1}^{m} c_j \chi_{E_j} \) where \( c_j \geq 0 \) for \( j = 1, \ldots, m \) and \( E_1, \ldots, E_m \) are measurable, then

\[
\int \varphi \, dm = \sum_{j=1}^{\infty} c_j m(E_j)
\]

with the convention that \( 0 \times \infty = 0 \). This is obviously the same definition as before if all the sets \( E_j \) have finite measure (or only a set \( E_j \) for which \( c_j = 0 \) has infinite measure) and extends the integral by setting it equal to \( \infty \) in case there \( \varphi \) assumes a positive value on a set of infinite measure.

8. We now **define, and this differs for a while from Royden**: If \( f : \mathbb{R} \to [0, \infty] \) is measurable, then

\[
\int f \, dm = \sup \{ \int \varphi \, dm : \varphi \text{ simple, } 0 \leq \varphi \leq f \}
\]

**Note:** If \( f, g : X \to [-\infty, \infty], X \) a set, then we define \( f \leq g \) to mean \( f(x) \leq g(x) \) for all \( x \in X \).

We also define, if \( E \) is a measurable subset of \( \mathbb{R} \), \( f : E \to [0, \infty] \) measurable,

\[
\int_{E} f \, dm = \int_{E} \tilde{f} \, dm
\]

where \( \tilde{f}(x) = f(x) \) if \( x \in E \), \( \tilde{f}(x) = 0 \) if \( x \in \mathbb{R} \setminus E \).

**Exercise 1** \( \tilde{f} : \mathbb{R} \to [0, \infty] \) is measurable.

**Exercise 2** An equivalent definition is

\[
\int_{E} f \, dm = \sup \{ \int \chi_{E} \varphi \, dm : \varphi \text{ simple, } 0 \leq \varphi(x) \leq f(x), x \in E \}
\]

It should be mentioned that these definitions make sense even if \( f \) is not measurable, but not much can be proved if we cannot invoke Lemma 1, which requires measurability.

While the theorems are the same as in the text, our proofs could be different. So take notes!

Task number one is to realize that if \( \varphi \) is simple, non-negative, we have two different definitions of \( \int_{E} \varphi \). The fact that they are equal, while very easy, is perhaps less obvious than one might think. My hope was that one of the exercises in Exam 2 would make you aware of the fact that certain “obvious” things do need proofs. Here it is very easy because we do have that if \( 0 \leq \psi \leq \varphi \) simple, then \( \int_{E} \psi \leq \int_{E} \varphi \) (old definition of integral of a simple function), thus the supremum of \( \int_{E} \psi \) as \( \psi \) ranges over all simple functions such that \( 0 \leq \psi \leq \varphi \) is \( \int_{E} \varphi \).

We need to supplement Royden’s Proposition 2 with the following simple result

**Lemma 2** Let \( \varphi \) be a non-negative simple function. The map

\[
m_{\varphi} : E \mapsto \int_{E} \varphi : \mathcal{M} \to [0, \infty]
\]

is a measure on \( \mathcal{M} \); that is, \( m_{\varphi}(0) = 0 \) and if \( \{ E_n \} \) is a family of pairwise disjoint measurable sets, then

\[
m_{\varphi}(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m_{\varphi}(E_n).
\]

Moreover, \( m_{\varphi}(E) = 0 \) for all Lebesgue null sets \( E \).
Proof. Write $\phi = \sum_{j=1}^{m} c_j \chi_{A_j}$ where $c_j \geq 0$, $A_j$ measurable for $j = 1, \ldots, m$. Then

$$m_\phi(E) = \sum_{j=1}^{m} c_j m(A_j \cap E)$$

for all $E \in \mathcal{M}$ and everything is fairly obvious. We just have to be a bit careful with the possible appearances of $\infty$. If $m(E) = 0$ it should be obvious that $m_\phi(E) = 0$. If $E = \bigcup_{n=1}^{\infty} E_n$ is a disjoint union, then we have

$$m(A_j \cap E) = m \left( \bigcup_{n=1}^{\infty} (A_j \cap E_n) \right) = \sum_{n=1}^{\infty} m(A_j \cap E_n)$$

from which the $\sigma$-additivity of $m_\phi$ follows at once.

Corollary 3 Let $\phi$ be a non-negative simple function. Then for every increasing sequence $\{E_n\}$ of measurable sets we have

$$m_\phi \left( \bigcup_{n=1}^{\infty} E_n \right) = \lim_{n \to \infty} m_\phi(E_n).$$

9. Parts i and iii of Proposition 8 of page 85, but part iii without the a.e. That is

**Proposition 4**

i. $\int_E c \phi = c \int_E \phi$ for all $c \in \mathbb{R}$, $c \geq 0$, $f : E \to [0, \infty]$ a non-negative measurable function.

iii. $\int_E f \leq \int_E g$ for all measurable $f, g : E \to [0, \infty]$ such that $f \leq g$.

Please notice that our definition of integral of a non-negative measurable function differs from that of Royden (that is, eventually we should be able to prove it is the same, but for now it is different) and we do NOT have his Proposition 5. Our proof is a consequence of Royden’s Proposition 2, which we do have, and of our Lemma 1. Other than that, the proof is identical.

Proof. i. If $c = 0$, and we stick to the convention $c \times \infty = 0$ (a convention that has to be used with care, only when justified*), the result is obvious. Assume, thus, $c > 0$. If $\phi$ is simple, $0 \leq \phi \leq f$, then $0 \leq c \phi \leq cf$. By the definition of $\int_E c \phi$ we have $\int_E c \phi \leq \int_E c f$; by Proposition 2 of Royden, this translates to $c \int_E \phi \leq \int_E cf$ and since $c > 0$ we proved

$$\int_E \phi \leq \frac{1}{c} \int_E cf$$

for all simple $\phi$, $0 \leq \phi \leq f$. By the definition of $\int_E f$, it follows that

$$\int_E f \leq \frac{1}{c} \int_E cf; \quad \text{i.e.,} \quad c \int_E f \leq \int_E cf$$

To get the inverse inequality, let $g = cf$. Then $g \geq 0$ is measurable, $1/c > 0$ and by what we proved (replacing $f$ by $g$ and $c$ by $1/c$)

$$\frac{1}{c} \int_E g \leq \frac{1}{c} \int_E g; \quad \text{i.e.,} \quad \frac{1}{c} \int_E cf \leq \int_E f.$$

iii. Assume now $0 \leq f \leq g$; $f, g$ measurable on $E$. If $\phi$ is simple, $0 \leq \phi \leq f$, then $0 \leq \phi \leq g$, thus $\int_E \phi \leq \int_E g$. Since $\phi$ was arbitrary subject to $0 \leq \phi \leq f$, we are done.

10. Lebesgue’s Monotone Convergence Theorem and its corollaries. (Points 10, 11, 12 in Royden, pages 87-88.)

*See note about infinity at the end of these notes.
Theorem 5 (Lebesgue's Monotone Convergence Theorem.) Assume $E$ is a measurable subset of $\mathbb{R}$ and for $n \in \mathbb{N}$ let $f_n : E \to [0, \infty]$ be measurable and satisfy $f_n \leq f_{n+1}$ for $n = 1, 2, \ldots$ (That is, $f_n(x) \leq f_{n+1}(x)$ for all $x \in E$, $n \in \mathbb{N}$). Then

$$\int_E (\lim_{n \to \infty} f_n) = \lim_{n \to \infty} \int_E f_n.$$ 

Proof. Let us notice first that both limits exist. $\lim_{n \to \infty} f_n$ exists because at every $x \in E$, the sequence of (extended) real numbers $\{f_n(x)\}$ is increasing, thus $(\lim_{n \to \infty} f_n)(x) = \lim_{n \to \infty} f_n(x)$ exists (possibly equal to $\infty$). Let us call it $f$; that is, let $f = \lim_{n \to \infty} f_n$. By results from Chapter 3, $f$ is measurable. By our limited version of Proposition 8 in Royden (our Proposition 4), the sequence of extended real numbers $\{\int_E f_n\}$ is increasing, thus $J = (\lim_{n \to \infty} \int_E f_n)$ exists (possibly equal to $\infty$). Moreover, because we must have $f_n \leq f$ (I hope this is clear), thus $\int_E f_n \leq \int_E f$ for all $n$ and hence also in the limit; that is $J \leq \int_E f$. All we need to prove is

(1) \[ \int_E f \leq J. \]

We may assume that $J < \infty$, otherwise we are done. Here is where things get nice and ingenious, sort of. To prove (1) we need to see (if we do it in the obvious way, and it is always good to begin with the obvious) $\int_E \varphi \leq J$

(2) \[ \int_E \varphi \leq J. \]

for all simple $\varphi$, $0 \leq \varphi \leq f$. So pick such a $\varphi$. It would be nice if one could say that $\varphi \leq f_n$ for some $n$, because then $\int_E \varphi \leq \int_E f_n \leq J$. But a moment’s reflection shows this is not necessarily true. But here is the trick. Let $a \in (0, 1)$. For $n = 1, 2, \ldots$ define

$$E_n = \{x \in E : a \varphi(x) \leq f_n(x)\}.$$ 

Notice that $E_1 \subset E_2 \subset E_3 \cdots$ and $E = \bigcup_{n=1}^\infty E_n$. In fact, if $x \in E_n$, then $a \varphi(x) \leq f_n(x) \leq f_{n+1}(x)$, hence $x \in E_{n+1}$. If $x \in E$ and $f(x) = 0$, then $f_n(x) = 0$ for all $n$ and $\varphi(x) = 0$, thus $0 \leq a \varphi(x) \leq f_n(x)$ and $x \in E_n$ for all $n$. If $f(x) > 0$, then (since $0 < a < 1$), $a \varphi(x) < f(x) = \lim_{n \to \infty} f_n(x)$ and there is $n$ such that $f_n(x) > a \varphi(x)$, hence $x \in E_n$. Now

$$J \int_E f_n \geq \int_{E_n} f_n \geq \int_{E_n} a \varphi = a m_{\varphi}(E_n).$$

By the corollary to our Lemma 2, (2) follows taking limits for $n \to \infty$. 

11. Now we can complete the proof of Proposition 8 of Royden:

Proposition 6 Let $E$ be a measurable subset of $\mathbb{R}$.

ii. $\int_E (f + g) = \int_E f + \int_E g$ for all non-negative measurable functions $f, g$ defined on $E$ (at least).

Proof. By Lemma 1 we can find increasing sequences $\{\varphi_n\}, \{\psi_n\}$ of non-negative simple functions converging at every point to $f, g$, respectively. Then $\{\varphi_n + \psi_n\}$ is an increasing sequence of non-negative simple functions converging to $f + g$ and since $\int_E (\varphi_n + \psi_n) = \int_E \varphi_n + \int_E \psi_n$ for all $n$, the result follows from Theorem 5

Corollary 7 Lebesgue’s monotone convergence theorem remains valid if $f_n \leq f_{n+1}$ and the existence of $\lim_{n \to \infty} f_n$ only hold a.e. That is:

Let $E$ be a measurable subset of $\mathbb{R}$ and for $n \in \mathbb{N}$ let $f_n : E \to [0, \infty]$ be measurable and satisfy $f_n \leq f_{n+1}$ for $n = 1, 2, \ldots$ a.e.; that is, $f_n(x) \leq f_{n+1}(x)$ for all $x \in E \setminus N_n$, $N_n$ a null set, $n \in \mathbb{N}$, and assume that $f = \lim_{n \to \infty} f_n$ a.e.; i.e., there is a null set $N$ such that $f(x) = \lim_{n \to \infty} f_n(x)$ for all $x \in E \setminus N$. Then

$$\int_E f = \lim_{n \to \infty} \int_E f_n.$$
Proof. Let \( C = N \cup \bigcup_{n=1}^{\infty} N_n \). Then \( C \) is a null set; by Theorem 5 we have

\[
\int_{E \setminus C} f = \lim_{n \to \infty} \int_{E \setminus C} f_n,
\]

and all integrals over \( C \) are zero\(^†\).

We also have the following corollary. Since it is one of the most common ways one applies Lebesgue’s Monotone Theorem we call it a theorem.

**Theorem 8** Let \( E \) be a measurable subset of \( \mathbb{R} \) and assume \( f_n : E \to [0, \infty] \) is measurable for \( n = 1, 2, \ldots \). Then

\[
\int_E \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} f_n.
\]

**Proof.** Notice that both limits exist. \( \sum_{n=1}^{\infty} f_n \) is the function which at every \( x \in E \) is defined by \( (\sum_{n=1}^{\infty} f_n)(x) = \sum_{n=1}^{\infty} f_n(x) \) and it exists (possibly equal to \( \infty \)) because we have a series of non-negative terms; for a similar reason the series on the right hand side above makes sense.

Let \( g_n = \sum_{k=1}^{n} f_k \). Then \( g_1 \leq g_2 \leq \cdots \) and \( \lim_{n \to \infty} g_n = \sum_{n=1}^{\infty} f_n \). By Theorem 5,

\[
\lim_{n \to \infty} \int_E g_n = \int_E \sum_{n=1}^{\infty} f_n.
\]

By Proposition 6,

\[
\lim_{n \to \infty} \int_E g_n = \sum_{n=1}^{\infty} \int_E f_n.
\]

The result follows.

This could be the point to throw in a very simple but actually quite frequently used result.

**Lemma 9** Let \( f : E \to [0, \infty] \) be measurable. If \( \int_E f < \infty \), then \( f \) is finite a.e.; i.e., the set \( \{x \in E : f(x) = \infty\} \) is a Lebesgue null set.

**Proof.** Let \( F = \{x \in E : f(x) = \infty\} \). Then \( F \) is measurable and \( c \chi_F \leq f \) for all \( c \in (0, \infty) \). Thus

\[
c \mu(F) = \int_E c \chi_F \leq \int f
\]

and since the right hand side of the last equality is finite, the only way this can hold for arbitrary \( c > 0 \) is if \( m(F) = 0 \).

12. Fatou’s Lemma, in a slightly more general version:

**Lemma 10** Let \( \{f_n\} \) be a sequence of non-negative measurable functions on a set \( E \). Then

\[
\int_E \left( \lim \inf_{n \to \infty} f_n \right) \, dm \leq \lim \inf_{n \to \infty} \int_E f_n \, dm.
\]

**Proof.** We use again the characterization of the \( \lim \inf \) of a sequence \( \{a_n\} \) of extended real numbers as a sup of an inf:

\[
\lim \inf_{n \to \infty} a_n = \sup_{n \in \mathbb{N}} \left( \inf_{k \geq n} a_k \right) = \lim_{n \to \infty} \left( \inf_{k \geq n} a_k \right).
\]

\(^†\)To be precise, we did not prove yet that this holds for a general non-negative measurable function, but it is an immediate consequence of the fact that it holds for non-negative simple functions.
For \( n \in \mathbb{N} \) set \( g_n = (\inf_{k \geq n} f_k) \) (that is, \( g_n \) is the function on the measurable set \( E \) defined by \( g_n(x) = (\inf_{k \geq n} f_k(x)) \) for each \( x \in E \)); then \( \{g_n\} \) is a sequence of non-negative measurable functions on \( E \) satisfying \( g_1 \leq g_2 \leq \cdots \) and \( \lim_{n \to \infty} g_n = \lim_{n \to \infty} f_n \). Theorem 5 (Lebesgue’s Monotone Convergence Theorem) implies

\[
\int_E \lim_{n \to \infty} f_n = \lim_{n \to \infty} \int_E g_n.
\]

Obviously (I hope) \( g_n \leq f_n \) for all \( n \), thus \( \int_E g_n \leq \int_E f_n \) and we must have

\[
\lim_{n \to \infty} \int_E g_n \leq \liminf_{n \to \infty} \int_E f_n.
\]

We are done.

13. **Definition.** A measurable function \( f : E \to [0, \infty] \) is said to be integrable (or Lebesgue integrable) iff

\[
\int_E f \, dm < \infty.
\]

(Same as in Royden)


**Proposition 11** (Royden, Chapter 4, Proposition 13) Let \( f, g : E \to [0, \infty] \) be measurable and assume \( 0 \leq f \leq g \). If \( g \) is integrable, so is \( f \).

**Proof.** By Proposition 4 of these notes, part (iii), we have

\[
\int f \leq \int g < \infty.
\]

The same result holds if we replace \( f \leq g \) by \( f \leq g \) a.e.

We can extend now Lemma 2 to the case of an arbitrary non-negative measurable function. But first a little exercise.

**Exercise 3** Let \( X \) be a set, let \( \mathcal{S} \) be a \( \sigma \)-algebra in \( X \). If \( Y \in \mathcal{S} \), define

\[
\mathcal{S}_Y = \{A : A \in \mathcal{S}, A \subseteq Y\}
\]

Prove that \( \mathcal{S}_Y \) is a \( \sigma \)-algebra in \( Y \) and that one has

\[
\mathcal{S}_Y = \{E \cap Y : E \in \mathcal{S}\}.
\]

**Proposition 12** Let \( E \) be a measurable subset of \( \mathbb{R} \) and let \( f : E \to [0, \infty] \) be measurable. Let \( \mathcal{M}_E \) be the family of all Lebesgue subsets of \( E \). Define \( m_f : \mathcal{M}_E \to [0, \infty] \) by

\[
m_f(A) = \int_{A \cap E} f \, dm = \int_A f \, dm = \int_E \chi_A f \, dm
\]

if \( A \in \mathcal{M}_E \). Then \( m_f \) is a measure on the \( \sigma \)-algebra \( \mathcal{M}_E \).
Proof. We already remarked that if \( A \) is a null set, then \( \int_A f \, dm = 0 \) for all measurable non-negative functions defined on \( A \). In particular, \( m_f(\emptyset) = 0 \). To see that \( m_f \) is \( \sigma \)-additive, let \( \{A_n\} \) be a sequence of pairwise disjoint measurable subsets of \( E \), and let \( A = \bigcup_{n=1}^{\infty} A_n \). Because of the pairwise disjointness, one sees that

\[
\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}.
\]

I hope this is evident and the proof seen as being trivial (“hope springs eternal!”). Then

\[
\chi_A f = \sum_{n=1}^{\infty} \chi_{A_n} f.
\]

and by the Theorem 8 (the corollary to LMCT),

\[
m_f(A) = \int_A f = \int_E \chi_A f = \sum_{n=1}^{\infty} \int_E \chi_{A_n} f = \sum_{n=1}^{\infty} m_f(A_n)
\]

For the next lemma we need the following observation. We don’t state it as a lemma or proposition because it is an obvious consequence of linearity; except that we don’t have linearity yet. Let \( f, g \) be measurable and nonnegative on \( E \). If \( 0 \leq f \leq g \), then \( g - f \geq 0 \) measurable; if \( f \) is integrable, then

\[
\int (g - f) \, dm = \int g \, dm - \int f \, dm
\]

Of course \( g - f \geq 0 \) measurable. Now \( g = f + (g - f) \) so that

\[
\int (g - f) \, dm + \int f \, dm = \int g \, dm.
\]

If \( f \) is integrable we can subtract \( \int f \, dm \) from both sides.

**Proposition 13** Let \( f : E \to [0, \infty] \) be integrable. Then for every \( \epsilon > 0 \) there is \( \delta > 0 \) such that if \( A \) is a measurable subset of \( E \) of measure \( < \delta \), then \( \int_A f \, dm < \epsilon \).

**Proof.** One can get this as a consequence of Proposition 12. But I'll do it more or less like Royden (though it may not seem so) and derive it from Lemma 1 in these notes. Let \( \epsilon > 0 \) be given. By the definition of \( \int_E f \, dm \) as a sup, there exists \( \varphi \) simple, \( 0 \leq \varphi \leq f \) such that

\[
\int f \, dm < \int_E \varphi \, dm + \frac{\epsilon}{2}.
\]

Then for every measurable subset \( A \) of \( E \), since \( \varphi \leq f \) and \( f \), hence also \( \varphi \), is integrable

\[
\int_A (f - \varphi) \, dm \leq \int_E (f - \varphi) \, dm = \int_E f \, dm - \int_E \varphi \, dm < \frac{\epsilon}{2},
\]

thus

\[
\int_A f < \int_A \varphi \, dm + \frac{\epsilon}{2}
\]

for all \( A \subset E \). Write \( \varphi = \sum_{i=1}^{n} c_i \chi_{E_i} \), where we will assume, as we may, \( c_i > 0 \), \( E_i \) measurable, for all \( i = 1, \ldots, n \). Let \( \delta = \epsilon/[2 \sum_{i=1}^{n} c_i] \). If \( m(A) < \delta \), then

\[
\int_A \varphi \, dm = \sum_{i=1}^{n} c_i m(A \cap E_i) \leq \sum_{i=1}^{n} c_i m(A) < \sum_{i=1}^{n} c_i \delta = \frac{\epsilon}{2}.
\]

The result follows.
15. We begin with the topics of Section 4. In general, given a function \( u : S \to [-\infty, \infty] \), \( S \) a set, one defines \( u^+, u^- \) by

\[
\begin{align*}
  u^+(x) &= \begin{cases} 
    u(x) & \text{if } x \in S, \ u(x) \geq 0, \\
    0 & \text{if } x \in S, \ u(x) < 0;
  \end{cases} \\
  u^-(x) &= \begin{cases} 
    -u(x) & \text{if } x \in S, \ u(x) < 0, \\
    0 & \text{if } x \in S, \ u(x) \geq 0.
  \end{cases}
\end{align*}
\]

Briefly: \( u^+ = \max(u, 0) \), \( u^- = -\min(u, 0) \).

**Exercise 4** Prove the following properties. Assume \( u : S \to \mathbb{R} \cup \{-\infty, \infty\} \), \( S \) some set. Just in case, assume \( S \) not empty (to avoid trivial nonsense).

(a) \( u^+ \geq 0, u^- \geq 0 \) (as usual, a function from a set to the extended reals is non-negative iff all its values are non-negative numbers or infinity).

(b) \( u = u^+ - u^- \), \( |u| = u^+ + u^- \).

(c) The previous property characterizes \( u^+, u^- \). That is, if \( v, w : S \to [-\infty, \infty] \), if \( u = v - w \) and \( |u| = v + w \), then \( v = u^+, w = u^- \).

Our first result is quite immediate; the proof is left as an exercise.

**Lemma 14** Let \( f : E \to [-\infty, \infty] \), where \( E \) is a measurable subset of \( \mathbb{R} \). Then \( f \) is measurable if and only if both \( f^+, f^- \) are measurable.

**Exercise 5** Prove Lemma 14.

Suppose now \( f : E \to [-\infty, \infty] \) is measurable. Then \( f^+, f^- \) are measurable nonnegative functions, and their integral is defined. If and only if at least one of \( f^+, f^- \) is integrable, we define

\[
\int_E f \, dm = \int_E f^+ \, dm - \int_E f^- \, dm.
\]

The requirement that at least one of \( f^\pm \) be integrable avoids the possibility of having \( \infty - \infty \). Obviously, if \( f \geq 0 \), this is the same integral as before. But we are going to be more interested in the finite valued cases and so we have a definition.

**Definition 1** Let \( f : E \to [-\infty, \infty] \) be measurable. We say \( f \) is (Lebesgue) integrable over \( E \) iff both \( f^+, f^- \) are integrable. In this case, \( \int_E f \, dm \in \mathbb{R} \).

**Lemma 15** Let \( f : E \to [-\infty, \infty] \) be measurable. Then \( f \) is integrable if and only if \( |f| \) is integrable. Moreover

\[
\left| \int_E f \, dm \right| \leq \int_E |f| \, dm.
\]

**Exercise 6** Prove Lemma 15. Should be very easy! The displayed inequality is actually true as long as \( \int_E f \) makes sense. The proof does not require results yet to come; it requires knowing that if a real number (or an extended real number) has been written in the form \( a = c - d \) where \( c, d \geq 0 \), then \( |a| \leq c + d \), and it does require knowing that for nonnegative functions the integral of the sum is the sum of the integrals.

One very common way of putting this is: Let \( f : E \to [-\infty, \infty] \) be measurable. Then \( f \) is integrable if and only if

\[
\int_E |f| \, dm < \infty.
\]

One of the really nice things about the Lebesgue integral is that, as long as the function is measurable, it is always defined for nonnegative functions.
Definition 2 Let $E$ be a measurable subset of $\mathbb{R}$. If $f : E \to [-\infty, \infty]$ we will say that $f \in L^1(E)$ if and only if $f$ is measurable and $\int_E |f| \, dm < \infty$. In other words, $L^1(E)$ is the set of all integrable functions of domain $E$.

We have reached Proposition 15 in Royden, which I will divide up into three propositions; Propositions 16, 17, 18. The proofs will be identical, so I omit them from these notes.

Proposition 16 Let $E$ be a measurable subset of $\mathbb{R}$. Then $L^1(E)$ is a real vector space and the map

$$f \mapsto \int_E f \, dm : L^1(E) \to \mathbb{R}$$

is a linear functional on $L^1(E)$.

(With a different phrasing, this is parts i and ii of Proposition 15 in Royden)

Proposition 17 Let $E$ be a measurable subset of $\mathbb{R}$ and let $f, g \in L^1(E)$. If $f \leq g$ a.e., then $\int_E f \, dm \leq \int_E g \, dm$.

Proposition 18 Let $E$ be a measurable subset of $\mathbb{R}$. Let $A, B$ be measurable subsets of $E$ such that $A \cap B = \emptyset$. Then

$$\int_{A \cup B} f \, dm = \int_A f \, dm + \int_B f \, dm$$

for all $f \in L^1(E)$.

Exercise 7 Let $f \in L^1(E)$.

(a) Assume $\{A_n\}$ is a sequence of pairwise disjoint measurable subsets of $E$. Let $A = \bigcup_{n=1}^{\infty} A_n$. Prove: The series

$$\sum_{n=1}^{\infty} \int_{A_n} f \, dm$$

converges absolutely and, in fact,

$$\sum_{n=1}^{\infty} \int_{A_n} f \, dm = \int_A f \, dm.$$

(b) Assume $\{A_n\}$ is an increasing sequence of measurable subsets of $E$. Let $A = \bigcup_{n=1}^{\infty} A_n$. Prove:

$$\lim_{n \to \infty} \int_{A_n} f \, dm = \int_A f \, dm.$$

(c) Assume $\{A_n\}$ is a decreasing sequence of measurable subsets of $E$. Let $A = \bigcap_{n=1}^{\infty} A_n$. Prove:

$$\lim_{n \to \infty} \int_{A_n} f \, dm = \int_A f \, dm.$$

16. We have reached what is probably the most important theorem of this Chapter, namely Lebesgue’s Dominated Convergence Theorem (LDCT). Our proof differs slightly from Royden’s proof. It doesn’t need two steps.
Theorem 19 Assume $E$ is a measurable subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of measurable function on $E$. If $\lim_{n \to \infty} f_n = f$ a.e., and there exists an integrable function $g$ such that $|f_n| \leq g$ a.e. for all $n \in \mathbb{N}$, then

$$\lim_{n \to \infty} \int_E f_n \, dm = \int_E f \, dm.$$ 

Proof. Clearly $|f| \leq g$ a.e. so that $|f - f_n| \leq 2g$ a.e. and $2g - |f - f_n|$ is a nonnegative measurable function. Moreover

$$\liminf_{n \to \infty} (2g - |f - f_n|) = \lim_{n \to \infty} (2g - |f - f_n|) = 2g.$$ 

By Fatou’s Lemma,

$$\int_E 2g \, dm = \int_E \left( \liminf_{n \to \infty} (2g - |f - f_n|) \right) \, dm \leq \liminf_{n \to \infty} \int_E (2g - |f - f_n|) \, dm \leq \liminf_{n \to \infty} \left( \int_E 2g \, dm - \int_E |f - f_n| \, dm \right) = \int_E 2g \, dm - \limsup_{n \to \infty} \int_E |f - f_n| \, dm,$$

where we used that $\liminf_n (a - b_n) = a - \limsup_n b_n$. All quantities involved here are finite, especially $\int_E 2g \, dm$. So we can subtract it to get

$$\limsup_{n \to \infty} \int_E |f - f_n| \, dm \leq 0.$$

But if the limsup of a sequence of nonnegative quantities is $\leq 0$, then the sequence converges to 0. \hfill \blacksquare

17. Section 2 of Chapter 4; with some big modifications of the proofs and without duplicate definitions.

First, a summary of how things were defined in these notes. And in class.

(a) The integral of a simple function that takes of all of its non-zero values on sets of finite measure was defined. Such simple functions form a real vector space and it was proved that the integral was a linear functional on such a space, and that it preserved order. That is, if $\phi \leq \psi$, meaning that $\phi(x) \leq \psi(x)$ for all $x$, then the integral of $\phi$ is less than or equal that of $\psi$. One conclusion obtained from all this is that if $\phi = \sum_{i=1}^{n} c_i \chi_{A_i}$, where each $A_i$ is measurable and $m(A_i) < \infty$ if $c_i \neq 0$, then

$$\int \phi = \sum_{i=1}^{n} c_i m(A_i).$$

(In all of this $0 \cdot \infty = 0$; to be used with care.)

Because there are many ways in which a simple function can be written in the form $\sum_{i=1}^{n} c_i \chi_{A_i}$, one could not use (3) as a definition, but had to prove it as a lemma or proposition. So far, all this is as in Royden.

(b) If $E$ is a measurable subset of $\mathbb{R}$, we define for a simple function $\phi$

$$\int_{E} \phi = \int_{\mathbb{R}} \chi_{E}\phi.$$ 

A simple function times the characteristic function of a simple function is still simple. It is a trivial exercise to verify that if $\phi = \sum_{i=1}^{n} c_i \chi_{A_i}$, where each $A_i$ measurable and $m(A_i) < \infty$ if $c_i \neq 0$, then $\chi_{E}\phi = \sum_{i=1}^{n} c_i \chi_{E \cap A_i}$

$$\int_{E} \phi = \sum_{i=1}^{n} c_i m(E \cap A_i).$$

It is also trivial to see that $\chi_{E}\phi$ is the simple function obtained from $\phi$ by setting $\phi = 0$ outside of $E$.

(c) The definition of $\int_{E} \phi$ was extended a bit to include the possibility of $\phi$ taking a non-zero value on a set of infinite measure, in case that value was positive. That is, we notice that the original definition of $\int \phi$ for a simple function makes sense if we just set, as one should $c \times \infty = \infty$ if $c > 0$, as long as there are no negative values to cause the cataclysmic mass-antimass effect one gets when subtracting infinity from infinity. One needs to verify that (3) is still valid, but that’s a simple exercise.
(d) The integral is defined next for all measurable \( f : E \to [0, \infty] \) by

\[
\int_E f = \sup \{ \int_E \phi : \phi \text{ simple}, \ 0 \leq \phi(x) \leq f(x) \forall x \in E \}.
\]

The integral can, of course, be equal to \( \infty \). At this point one has two definitions of \( \int_E \phi \), if \( \phi \) is simple. But because one knows that the integral is order preserving among simple functions, it is between very easy and trivial to prove that both definitions yield the same value for the integral.

(e) In our development, the approximation lemma Lemma 1 is a key element. Every non-negative measurable function is the pointwise limit of an increasing sequence of non-negative simple functions; the limit is uniform if the function is bounded.

(f) It is a trivial exercise to prove that if \( f : \mathbb{R} \to [0, \infty] \) is measurable, if \( E \) is a measurable subset of \( \mathbb{R} \), then

\[
\int_E f = \int_{E f}.
\]

(g) We now prove for non-negative valued measurable functions that \( \int_E cf = c \int_E f \) if \( c \in \mathbb{R}, c \geq 0 \), and that \( 0 \leq f \leq g \) on \( E \) implies \( \int_E f \leq \int_E g \). Proving that the integral of a sum is the sum of the integrals is postponed.

(h) A result we will need to prove the next important result, Lebesgue’s Monotone Convergence Theorem, is Lemma 2; more precisely its Corollary 3.

(i) Lebesgue’s Monotone Convergence Theorem (LMCT) can be proved now.

Because every non-negative measurable function is the pointwise limit of an increasing sequence of non-negative simple functions, and the integral is linear on simple functions, thanks to LMCT it is now easy to prove that the integral of a sum of non-negative measurable functions is the sum of the integrals.

(k) The next result is the series version of LMCT, Theorem 8. In our development, the approximation lemma Lemma 1 is a key element. Every non-negative measurable function is the pointwise limit of an increasing sequence of non-negative simple functions; the limit is uniform if the function is bounded.

We now move to general measurable functions. For a measurable \( f : E \to [-\infty, \infty] \) we define \( f^+, f^- \), and verify the obvious facts that \( f \) is measurable if and only if both \( f^+, f^- \) are measurable; \( f = f^+ - f^- \), \( |f| = f^+ + f^- \). We define \( L^1(E) \), the set of integrable functions over \( E \) as the set of all measurable \( f \) such that both \( f^+, f^- \) are integrable, and define for \( f \in L^1(E) \),

\[
\int_E f = \int_E f^+ - \int_E f^-.
\]

One should verify that if \( \phi \) is simple, then \( \phi \in L^1(E) \) if and only if any set on which \( \phi \) assumes a non-zero value has finite measure when intersected with \( E \), and that the integral as defined by (3) is the same as by (7). Another important but trivial thing to verify is that \( f \in L^1(E) \) if and only if \( f \) is measurable and \( \int_E |f| < \infty \). From the definition of the integral, it is also obvious that

\[
\left| \int_E f \right| \leq \int_E |f|
\]

for all \( f \in L^1(E) \).
(q) One proves now that $L^1(E)$ is a real vector space and the map $f \mapsto \int_E f$ is an order preserving functional on $L^1(E)$.

Also: If $f \in L^1(E)$ and we define $m_f : M \to \mathbb{R}$ by $m_f(A) = \int_A f = \int_E \chi_A f$ if $A \in M = \{A \in \mathcal{M} : A \subset E\}$, if $\{A_n\}$ is a sequence of pairwise disjoint sets and $A = \bigcup_{n=1}^{\infty} A_n$, then the series $\sum_n m_f(A_n)$ converges absolutely and its limit is $m_f(A)$. This is actually fairly immediate if one divides it into positive and negative parts.

(r) The big result: Lebesgue’s Dominated Convergence Theorem (LDCT), Theorem 19. This is proved as a consequence of Fatou’s Lemma. Our proof uses essentially the same idea as Royden.

And now we can get to Section 2. We do NOT redefine any integral. What we notice first (or should have done so) is the very obvious fact that if $f : E \to \mathbb{R}$ is bounded and $m(E) < \infty$, then $f \in L^1(E)$. In fact, if $|f(x)| \leq M$ for a.e. $x \in E$, then $|f| \leq M \chi_E$ a.e.; integrability of $f$ is a consequence of Proposition 11 and $\int M \chi_E = M m(E) < \infty$. Here is a fact mentioned in class:

**Proposition 20** Let $E$ be measurable, $f : E \to \mathbb{R}$ measurable and bounded. Then there exist sequences $\{\phi_n\}, \{\psi_n\}$ such that

$$\phi_1 \leq \phi_2 \leq \cdots \leq f \leq \cdots \leq \psi_2 \leq \psi_1 \quad \text{a.e. on } E,$$

and both sequences converge uniformly to $f$ on $E$.

**Proof.** Assume $|f(x)| \leq M$ for $x \in E$. Then $f + M$ (i.e., the function defined by $f(x) + M$ for all $x \in E$) is measurable and nonnegative; by Lemma 1 there exists a sequence $\{v_n\}$ of nonnegative simple functions increasing to $f$ and converging uniformly to $f$ on $E$. Let $\phi_n = v_n - M$. Since $f$ bounded, measurable, implies $-f$ is bounded measurable, by what we proved there is an increasing sequence $\{w_n\}$ of simple functions converging uniformly to $f$ on $E$; set $\psi_n = -w_n$. □

It is now quite trivial to PROVE

**Proposition 21** If $f : E \to \mathbb{R}$ is measurable and bounded; $m(E) < \infty$, then

$$\int_E f = \sup \left\{ \int_E \varphi : \varphi \text{ simple, } \varphi \leq f \right\} = \inf \left\{ \int_E \varphi : \varphi \text{ simple, } \varphi \geq f \right\}.$$

**Proof.** I’ll do the first equality in (??). The second one can be proved similarly. Or one gets it as a consequence of the first one, by replacing $f$ by $-f$.

In the first place, by Proposition 17 it is clear that

$$\int_E f \geq \sup \left\{ \int_E \varphi : \varphi \text{ simple, } \varphi \leq f \right\}.$$

To reverse the inequality, by Proposition 21 there is an increasing sequence of simple functions $\{\phi_n\}$ converging uniformly to $f$. We can now use either LDCT or LMCT to see that

$$\int_E \phi_n \to \int_E f.$$

To use LDCT, we use the fact that a uniformly convergent sequence whose limit is bounded and be uniformly bounded; there is thus $M$ such that $|\phi_n| \leq M \chi_E$, and because $m(E) < \infty$, $M \chi_E$ is integrable and (9) follows. If we prefer using LMCT, we can let $M$ be a bound for $f$; $|f(x)| \leq M$ for all (or almost all) $x \in E$. Then $f + M$ is non-negative. Letting $A_n = \{x \in E : \phi_n(x) < -M\}$, we can replace $\phi_n$ by $\chi_{E \setminus A_n} \phi_n$ and thus can assume that $\phi_n + M \geq 0$ for all $n$; LMCT now applies to prove that $\int_E (\phi_n + M) \to \int_E (f + M)$; subtracting $M m(E)$ we get (9). Having proved this we notice that if

$$s < \int_E f,$$

then there is $n$ such that

$$s < \int_E \phi_n \leq \sup \left\{ \int_E \varphi : \varphi \text{ simple, } \varphi \leq f \right\};$$

completing the proof of the first equality in (??). □

A result that could have been done earlier is:
**Proposition 22** Let \( f : E \to [0, \infty] \) be measurable. If \( \int_E f = 0 \), then \( f = 0 \) a.e. on \( E \).

**Proof.** Let \( A = \{ x \in E : f(x) > 0 \} \). Then \( A = \bigcup_{n=1}^\infty A_n \), where \( A_n = \{ x \in E : f(x) > 1/n \} \). Clearly \( 0 \leq \frac{1}{n} \chi_{A_n} \leq f \), thus

\[
0 \leq \frac{1}{n} m(A_n) = \int_E \frac{1}{n} \chi_{A_n} \leq \int_E f = 0.
\]

Thus \( m(A_n) = 0 \) for all \( n \), hence also \( m(A) = 0 \).

It is now quite easy to prove:

**Theorem 23** Let \( f : E \to \mathbb{R} \) be bounded, where \( E \) is a measurable set, \( m(E) < \infty \). Then \( f \) is measurable if and only if

\[(10) \sup \left\{ \int_E \varphi \mid \varphi \text{ simple}, \varphi \leq f \right\} = \inf \left\{ \int_E \varphi \mid \varphi \text{ simple}, \varphi \geq f \right\}.\]

**Proof.** Half of the theorem, that measurability implies (10) is the content of Proposition 21. For the converse, assume (10). For each \( n \in \mathbb{N} \) there will exist simple functions \( \phi_n, \psi_n \) such that \( \phi_n \leq f \leq \psi_n \) and \( \int_E \psi_n - \int_E \phi_n < 1/n \). (Given sets \( A, B \) of real numbers, or of extended real numbers, such that \( a \leq b \) for every \( a \in A, b \in B \), then \( supA = infB \) if and only if for every \( n \in \mathbb{N} \) there exist \( a_n \in A, b_n \in B \) such that \( b_n - a_n < 1/n \).) Let

\[
\phi^* = \sup_n \phi_n, \quad \psi^* = \inf_n \psi_n.
\]

Then \( \phi^*, \psi^* \) are measurable, \( \phi_n \leq \phi^* \leq f \leq \psi^* \leq \psi_m \) for all \( n, m \). All this should be evident (or not.). Now

\[
\int_E (\psi^* - \phi^*) = \int_E \psi^* - \int_E \phi^* \leq \int_E \psi_n - \int_E \phi_n < 1/n
\]

and it follows that \( \int_E (\psi^* - \phi^*) = 0 \). Since \( \psi^* - \phi^* \geq 0 \), Proposition 22 implies \( \psi^* = \phi^* \) a.e., forcing \( \phi^* = f = psi^* \) a.e.

Thus \( f \) is measurable being equal a.e. to a measurable function.

This takes care of most of Section 2. Lebesgue’s bounded convergence theorem is an immediate consequence of LDCT. Or one can use Royden’s proof, that also has some instructive value. The theorem about Riemann integrability at the end of the section will be left as an exercise; same as in Royden.

\[\blacksquare\]

**Note on infinity**

We should distinguish perhaps between a static infinity or 0 and a potential infinity or 0. Static zero times a static infinity is 0. Example: If \( A \) is a set of infinite measure, then \( 0 \cdot (A) = 0 \). Or, if \( \lim a_n = \infty \), then \( 0 \lim a_n = 0 \). But if we have \( \lim a_n = \infty, \lim b_n = 0 \), then \( \lim (a_n b_n) \) is not necessarily 0 because we are in the potential case here. That is, if we define in this course \( 0 \times \infty = 0 \), we have to be very careful to realize that \( \lim (a_n b_n) = (\lim a_n)(\lim b_n) \) is not always true.