1. Let $X$ be a metric space and consider the set $\{0, 1\}$ as a metric space with the discrete metric.

(In other words, $d$ is defined in $\{0, 1\}$ by $d(0, 0) = d(1, 1) = 0$; $d(0, 1) = d(1, 0) = 1$.)

Proof: $X$ is disconnected if and only if there exists a continuous $f$ from $X$ onto $\{0, 1\}$ (onto=surjective).

Hint: For one direction consider the sets $f^{-1}(\{0\}), f^{-1}(\{1\})$. They might be open. Or perhaps closed. Or something.

Proof. Suppose first that $X$ is disconnected. Then $X = U \cup V$ for some open subsets $U, V$ of $X$, $U \neq \emptyset \neq V$, $U \cap V = \emptyset$. Since $U, V$ are disjoint, we can define $f : X \rightarrow \{0, 1\}$ by

$$f(p) = \begin{cases} 0, & \text{if } p \in U, \\ 1, & \text{if } p \in V. \end{cases}$$

Since $U, V$ are not empty, $f$ assumes both values of 0 and 1; that is $f$ is onto. Moreover, $f|_U$ is clearly continuous, being the constant function 0. Since $U$ is open in $X$, $f$ as a function on $X$ is continuous at all points of $U$. Similarly, $f$ is continuous at all points of $V$ since $f|_V$ is continuous and $V$ is open in $X$. Thus $f$ is continuous.

Conversely, assume there exists a continuous $f$ from $X$ onto $\{0, 1\}$. Then defining $U, V$ by

$$U = f^{-1}(0) = \{p \in X : f(p) = 0\}, \quad U = f^{-1}(1) = \{p \in X : f(p) = 1\}$$

we see that $U, V$ are not empty because $f$ is onto, $X = U \cup V$ because $f$ maps all of $X$ to $\{0, 1\}$ and $U \cap V = \emptyset$ because of course 1 $\neq 0$. Finally, because $f$ is continuous and $\{0\}, \{1\}$ are closed subsets of $\{0, 1\}$, $U, V$ are closed in $X$. (Alternatively, because $\{0\}, \{1\}$ are open subsets of $\{0, 1\}$, $U, V$ are open in $X$.) Thus $U, V$ disconnect $X$.

\[
\square
\]

2. Let $X$ be a metric space, $d$ its distance function. If $A \subset X$, $A \neq \emptyset$, if $x \in X$, define the distance from $x$ to $A$ by

$$d(x, A) = \text{g.l.b.} \{d(x, a) : a \in A\}.$$

Let $\emptyset \neq A \subset X$. Define $f : X \rightarrow \mathbb{R}$ by $f(x) = d(x, A)$. Prove: $f$ is uniformly continuous; in fact Lipschitz.

(The objective should be to prove $|f(x) - f(y)| \leq d(x, y)$ for all $x, y \in X$.)

Proof. Let $x, y \in X$. If $a \in A$ then $d(x, a) \leq d(x, y) + d(y, a)$. Now by the definition of $d(x, A)$, $d(x, A) \leq d(x, a)$ for all $a \in A$, thus $d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a)$ for all $a \in A$. Thus $d(x, A) - d(x, y) \leq d(y, a)$ for all $a \in A$ and $d(x, A) - d(x, y)$ is a lower bound for $\{d(y, a) : a \in A\}$. Thus

$$d(x, A) - d(x, y) \leq \text{g.l.b.} \{d(y, a) : a \in A\} = d(y, A).$$

Thus $d(x, A) - d(y, A) \leq d(x, y)$. (There is a perhaps subtle point here; it is the first time we actually use the hypothesis that $A \neq \emptyset$. If $A = \emptyset$ one could argue that $d(x, A)$ is meaningless, but one can also argue
that then \(d(x, A) = \infty\) for all \(x \in X\). Then the inequality \(d(x, A) - d(x, y) \leq d(y, A)\) is perfectly O.K.; it states that \(\infty - d(x, y) \leq \infty\). But now subtracting \(d(y, A)\) from both sides results in the nonsense \(\infty - \infty \leq d(x, y)\). Exchanging the roles of \(x, y\) we get \(d(y, A) - d(x, A) \leq d(y, x) = d(x, y)\), proving that
\[
|f(x) - f(a)| = |d(x, A) - d(y, A)| \leq d(x, y).
\]

\[\blacksquare\]

3. Let us continue in the vein of the previous exercise. Let \(X\) be a metric space, \(d\) its distance function, and assume \(\emptyset \neq A \subset X\). Prove:

(a) \(\{p \in X : d(p, A) = 0\} = \bar{A}\).

(b) Let \(A, B\) be closed non-empty subsets of \(X\) and assume \(A \cap B = \emptyset\). Then \(d(p, A) + d(p, B) > 0\) for all \(p \in X\) (this requires a proof) and it makes sense to define \(f : A \rightarrow B\) by
\[
f(x) = \frac{d(x, B)}{d(x, A) + d(x, B)}.
\]

Then \(f : X \rightarrow [0, 1]\) is continuous and \(f(x) = 1\) for all \(x \in A\), \(f(x) = 0\) for all \(x \in B\).

Proof.

(a) Let \(B = \{p \in X : d(p, A) = 0\}\). Then \(B\) is the inverse image under the continuous function \(p \mapsto d(p, A)\) (the function \(f\) of the previous exercise) of the closed set \(\{0\} \subset \mathbb{R}\), thus \(B\) is closed. It being obvious that \(A \subset B\), we see that \(A \subset B\). Assume now \(p \in B\). To see \(p \in A\), assume \(p \notin \bar{A}\). Then (\(\bar{A}\) being closed) there exists \(r > 0\) such that \(B(p, r) \cap A = \emptyset\), hence also \(B(p, r) \cap A = \emptyset\). But then it is obvious that \(d(p, a) > r\) for all \(a \in A\), hence \(d(p, A) \geq r > 0\), hence \(p \notin B\).

We conclude \(p \in \bar{A}\).

An alternative argument: \(p \in \bar{A}\) if and only if there exists a sequence \(\{p_n\}\) of points of \(A\) converging to \(p\). If such a sequence exists, then by continuity of the map \(p \mapsto d(p, A)\),
\[
d(p, A) = \lim_{n \to \infty} d(p_n, A) = 0
\]

since \(d(p_n, A) = 0\) for all \(n\) since \(p_n \in A\). Conversely, assume \(d(p, A) = 0\). We may recall that the greatest lower bound of a set is in the closure of the set, thus \(d(p, A)\) is in the closure of \(\{d(p, a) : a \in A\}\), hence the limit of a sequence in that set. There is thus (always) a sequence of the form \(\{d(p, a_n)\}\) converging to \(d(p, A)\), where \(a_n \in A\) for all \(n\). In our case \(d(p, A) = 0\) implies \(d(p, a_n) \rightarrow 0\), thus \(p = \lim_{n \to \infty} a_n \in \bar{A}\).

(b) Obviously \(d(p, A) + d(p, B) = 0\) implies \(d(p, A) = d(p, B) = 0\). Because \(A, B\) are closed, this implies \(p \in A \cap B\), contradicting \(A \cap B = \emptyset\). Thus \(d(p, A) + d(p, B) > 0\) for all \(p \in X\) and \(f\) as defined makes sense. The continuity of \(f\) is immediate from the previous exercise. Finally,
\[
f(p) = \begin{cases} \frac{d(p, B)}{0 + d(p, B)} = 1 & \text{if } p \in A, \\ 0 & \text{if } p \in B. \end{cases}
\]
4. Rosenlicht, Chapter IV, Exercise 16.

Proof. Let us verify that, in fact, \((f^{-1})^{-1}(A) = f(A)\) in this case. The notation is a bit confusing, so let us set \(g = f^{-1} : E' \to E\); here \(f^{-1}\) is the inverse function which exists because \(f\) is one-to-one and onto. Let us see that \(g^{-1}(A) = f(A)\) for all \(A \subset E\). Here \(g^{-1}(A)\) is the inverse image of \(A\); thus \(q \in g^{-1}(A)\) if and only if \(g(q) \in A\) Assume \(q \in g^{-1}(A)\). Then \(g(q) \in A\). Applying \(f\) we see that \(q = f(g(q)) \in f(A)\). Conversely, if \(q \in f(A)\), then \(q = f(p)\) for some \(p \in A\); then \(g(q) = g(f(p)) = p \in A\), thus \(q \in g^{-1}(A)\).

With this out of the way, to prove \(g\) continuous it suffices to prove that \(g^{-1}(F)\) is closed whenever \(F\) is closed in \(E\). But if \(E\) is closed in \(E\), then \(F\) is compact because \(E\) is compact, hence by continuity of \(f\), \(g^{-1}(F) = f(F)\) is compact, hence closed.

5. Rosenlicht, Chapter IV, Exercise 21.

Proof. We are given \(f : S \to E'\) uniformly continuous, where \(S\) is a dense subset of a metric space \(E\) and \(E'\) is a complete metric space. We have to show that \(f\) has a unique continuous extension to a continuous \(f : E \to E'\) and that this extension is uniformly continuous. This exercise has a bit more to it than may be at first visible. Once \(f\) is extended one may have to verify that it still is continuous on \(S\); the continuity on \(S\) could have been destroyed. Anyway, let’s go ahead with the proof.

We establish first a useful property of uniformly continuous functions: Let \(X, Y\) be metric spaces, let \(f : X \to Y\) be uniformly continuous. If \(\{x_n\}\) is a Cauchy sequence in \(X\), then \(\{f(x_n)\}\) is a Cauchy sequence in \(Y\). In fact, assume \(\{x_n\}\) is a Cauchy sequence in \(X\). Let \(\epsilon > 0\). By uniform continuity, there is \(\delta > 0\) such that \(x, y \in X\) and \(d(x, y) < \delta\) implies \(d(f(x), f(y)) < \epsilon\). Because \(\delta > 0\) and \(\{x_n\}\) is Cauchy, there is \(N\) such that \(d(x_n, x_m) < \delta\) whenever \(n, m \geq N\). Thus \(n, m \geq N\) implies \(d(f(x_n), f(x_m)) < \epsilon\).

We need to define \(f(p)\) if \(p \in E \setminus S\). Since such a \(p\) is in \(S\) there is an obvious way to proceed:

Let \(p \in E \setminus S\). Claim: \(\lim_{q \to p} f(q)\) exists. This is equivalent to proving that there is some \(z \in E'\) such that defining \(f(p) = z\), \(f : S \cup \{p\} \to E'\) is continuous. Since \(p \in E'\) there is a sequence \(\{p_n\}\) in \(S\) converging to \(p\); select and fix any such sequence. because \(\{p_n\}\) converges, it is Cauchy, hence \(\{f(p_n)\}\) is Cauchy in \(E'\) because \(f\) is uniformly continuous. Since \(E'\) is complete, \(\{f(p_n)\}\) converges in \(E\), say to an element \(z \in E'\). We now prove that \(z = \lim_{q \to p} f(q)\). We proceed by contradiction, assume it isn’t. Then there exists \(\epsilon > 0\) and for every \(\delta > 0\) an element \(q_0 \in S\) such that \(d(q_0, p) < \delta\) but \(d(f(q_0), z) \geq \epsilon\). Specialize to \(\delta = 1/n, n \in \mathbb{N}\) and write \(q_n\) for \(q_{1/n}\) so that \(q_n \in S\), \(d(q_n, p) < 1/n\) and \(d(f(q_n), z) \geq \epsilon\). Now consider the sequence \(p_1, q_1, p_2, q_2, p_3, q_3, \ldots\). This sequence converges to \(p\), thus is Cauchy, thus the sequence \(f(p_1), f(q_1), f(p_2), f(q_2), \ldots\) is Cauchy in \(E'\), thus it converges. Since the sequence of odd terms of this last sequence converges to \(z\), the whole sequence must converge to \(z\), hence also \(\lim_{n \to \infty} f(q_n) = z\), contradicting \(d(f(q_n), z) \geq \epsilon\) for all \(n\).
The claim is established.

We now can define \( f(p) \) for \( p \in E \setminus S \) by \( f(p) = \lim_{q \to p} f(q) \). We still have to prove continuity. We go directly for uniform continuity. Let \( \epsilon > 0 \) be given. By uniform continuity of \( f \) on \( S \), there exists \( \delta > 0 \) such that if \( p, q \in S \), \( d(p, q) < \delta \), then \( d(f(p), f(q)) < \epsilon/3 \). We have to see this keeps holding also if one or both of \( p, q \) is not in \( S \). Or something like it.

Let \( p, q \in E \), \( d(p, q) < \delta/3 \). We always have, whether \( p \in S \) or not,

\[
f(p) = \lim_{x \to p, x \in S} f(x);
\]

similarly for \( f(q) \). There is thus \( \delta_1 > 0 \) such that if \( p' \in S \), \( d(p', p) < \delta_1 \), then \( d(f(p'), f(p)) < \epsilon/3 \). Similarly, there is \( \delta_2 > 0 \) such that if \( q' \in S \), \( d(q', q) < \delta_2 \), then \( d(f(q'), f(q)) < \epsilon/3 \). Because \( p \in \bar{S} \), there is \( p' \in S \) such that \( d(p', p) < \min(\delta/3, \delta_1) \). Similarly, because \( q \in \bar{S} \), there is \( q' \in S \) such that \( d(q', q) < \min(\delta/3, \delta_2) \). Then \( d(q', p') \leq d(q', q) + d(q, p) + d(p, p') < \delta/3 + \delta/3 + \delta/3 = \delta \), hence \( d(f(q'), f(p')) < \epsilon/3 \). Since \( d(p', p) < \delta_1 \), \( d(q', q) < \delta_2 \), we also have \( d(f(p'), f(p)) < \epsilon/3 \) and \( d(f(q'), f(q)) < \epsilon/3 \), thus \( d(f(p), f(q)) \leq d(f(p), f(p')) + d(f(p'), f(q')) + d(f(q'), f(q)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \).

This proves \( f \) is uniformly continuous. Finally, to see that \( f \) as defined is the unique continuous extension, assume \( g : E \to E' \) is continuous and \( g|_S = f \). If \( p \in E \), since \( E = \bar{S} \), there is a sequence \( \{p_n\} \) of points of \( S \) converging to \( p \). Then, by continuity,

\[
g(p) = \lim_{n \to \infty} g(p_n) = \lim_{n \to \infty} f(p_n) = f(p).
\]

\[\square\]

6.* Let \( S \) be a closed subset of \( \mathbb{R} \) and let \( f : S \to \mathbb{R}^n \) be continuous. Show that \( f \) has a continuous extension to all of \( \mathbb{R} \); that is, show there exists a continuous \( g : \mathbb{R} \to \mathbb{R}^n \) such that \( g(x) = f(x) \) if \( x \in S \).

Hints: This probably has very little to do with exercise 21 of Rosenlicht, but a lot to do with exercise 5 of Chapter III. What you need to use is that if \( U \) is the complement of \( S \), then \( U = \bigcup_{n=1}^{\infty} (a_n, b_n) \), where \((a_n, b_n) \cap (a_m, b_m) = \emptyset \) if \( n \neq m \). Don’t make too many further assumptions about these open intervals; their distribution can be extremely weird. You must argue that in each case \( a_n, b_n \in S \) (obvious) thus \( f(a_n), f(b_n) \) are defined and since the range is \( \mathbb{R}^n \) it makes sense to define \( g(x) \) if \( a_n < x < b_n \) linearly; say

\[
g(x) = f(a_n) + \frac{x - a_n}{b_n - a_n} f(b_n).
\]

If \( x \in S \), then set \( g(x) = f(x) \). Now you have to prove \( g|_S = f \) (obvious) and \( g \) is continuous on \( \mathbb{R} \); not totally absolutely trivial.

7.** Rosenlicht, Chapter IV, Exercise 31. Substitute the previous exercise for Prob. 5 of Chapter III.