Note: This homework consists of a lot of very simple exercises, things you should do on your own. A minimum part of it will be due Monday, October 5, 2009.

1. Let $X$ be a set and let $d_1, d_2$ be two distance functions for $X$. We say they are equivalent if they define the same family of open sets. That is, $d_1 \sim d_2$ if and only if whenever a subset of $X$ is open with respect to $d_1$ it is open with respect to $d_2$ and whenever a subset of $X$ is open with respect to $d_2$ it is open with respect to $d_1$.

Prove:

(a) Two distance functions for a set $X$ are equivalent if and only if they both have the same convergent sequences. That is, if the functions are $d_1, d_2$, then a sequence converges with respect to $d_1$ if and only if it converges with respect to $d_2$.

(b) A metric space is said to be discrete if and only if its metric is equivalent to the discrete metric. Prove that a metric space is discrete if and only if it has no accumulation points (i.e., all points are isolated).

(c) Prove that $\mathbb{Z}$, as a subspace of $\mathbb{R}$, is discrete.

2. Let $X$ be a metric space and let $Y \subset X$; consider $Y$ as a metric space with the distance function of $X$ (restricted to $Y$). Prove:

(a) A subset $A$ of $Y$ is open in $Y$ (i.e., is an open subset of the metric space $Y$) if and only if $A = U \cap Y$ for some open subset $U$ of $X$.

(b) A subset $A$ of $Y$ is closed in $Y$ (i.e., is a closed subset of the metric space $Y$) if and only if $A = F \cap Y$ for some closed subset $F$ of $X$.

(c) Let $A$ be a subset of $Y$. The closure of $A$ in $Y$ is the intersection of the closure of $A$ in $X$ with $Y$.

3. Let $X_1, \ldots, X_n$ be metric spaces with distance functions $d_1, \ldots, d_n$ respectively. Let

$$X = X_1 \times \cdots \times X_n = \{p = (p_1, \ldots, p_n) : p_i \in X_i, i = 1, \ldots, n\}.$$ 

Consider the following three functions from $X \times X$ to $\mathbb{R}$. In the definition, $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n) \in X$.

$$\delta_1(p, q) = \sum_{i=1}^{n} d_i(p_i, q_i).$$

$$\delta_2(p, q) = \left( \sum_{i=1}^{n} d_i(p_i, q_i)^2 \right)^{1/2}.$$

$$\delta_\infty(p, q) = \max\{d_i(p_i, q_i) : i = 1, \ldots, n\}.$$

(a) Prove all three metrics for $X$ are equivalent and a sequence $\{p_m\}_{m=1}^{\infty}$, where $p_m = (p_{m1}, \ldots, p_{mn})$ converges to $p = (p_1, \ldots, p_n)$ if and only if each sequence $\{p_{mi}\}_{m=1}^{\infty}$ converges in $X_i$ for $i = 1, \ldots, n$.

(b) Prove that all three metrics for $X$ have the same Cauchy sequences.

(Equivalent metrics do NOT necessarily have the same Cauchy sequences so this is not an immediate consequence of the previous point. It is however essentially the same proof as one uses for the previous point.)
(c) Prove that $X$ is complete in any of the three metrics if and only if it is complete in the others, and this happens if and only if each one of the spaces $X_1, \ldots, X_n$ is complete.

(d) Explain how all this applies to $\mathbb{R}^n$. In particular, show that $\mathbb{R}^n$ is complete.

4. Let $d$ be the usual metric in $\mathbb{N}$; $d(n,m) = |n - m|$ for all $n, m \in \mathbb{N}$. Let $d_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined by $d(n,m) = \frac{1}{n} - \frac{1}{m}$. Prove that $d_1 \sim d_2$ (for example, show both are discrete) but $(\mathbb{N}, d_1)$ is complete while $(\mathbb{N}, d_2)$ is not complete.

5.** Let $(X, d)$ be a metric space, assume it is not complete. Let $X$ be the set of all Cauchy sequences of points in $X$; that is, $p \in X$ means $p = \{p_n\}_{n=1}^{\infty}$ where $\{p_n\}$ is a Cauchy sequence in $X$. We define an equivalence relation in $X$ by $p \sim q$ if and only if $\lim_{n \to \infty} d(p_n, q_n) = 0$. (Of course, $\{d(p_n, q_n)\}$ is a sequence of non-negative real numbers.) Prove this is indeed an equivalence relation. Let $\hat{X} = X/ \sim$ be the set of all equivalence classes. We will denote the elements of $\hat{X}$ by $[p], p \in X$; $[p] = [q]$ if and only if $p \sim q$. Prove: If $p, q \in X$, then the sequence of real numbers $\{d(p_n, q_n)\}$ is a Cauchy sequence in $\mathbb{R}$, hence converges. Moreover, If $p \sim p', q \sim q'$ then $\lim_{n \to \infty} d(p_n, q_n) = \lim_{n \to \infty} d(p'_n, q'_n)$. This allows us to define $\hat{d} : \hat{X} \times \hat{X} \to \mathbb{R}$ by $\hat{d}([p], [q]) = \lim_{n \to \infty} d(p_n, q_n)$. Prove that $(\hat{X}, \hat{d})$ is a complete metric space.

Here the notation may get a bit fuzzy, so one should be careful. If $p \in X$, define $\tilde{p} \in X$ by $\tilde{p}_n = p$ for all $n \in \mathbb{N}$. Next define a map $\phi : X \to \hat{X}$ by $\phi(p) = [\tilde{p}]$. Show this map is one-to-one and preserves the metric; i.e., $\hat{d}([\tilde{p}], [\tilde{q}]) = d(p, q)$. This allows one to identify $X$ with the metric space $\phi(X)$. Show that $\phi(X)$ is a dense subset of $\hat{X}$.

The space $\hat{X}$ is the completion of $X$. One can show (and perhaps we will) that it is unique up to metric isomorphism.