Introductory Analysis 1–Fall 2007
Homework #3–Solutions

Some sayings to live by, or to ignore. Mathematics is a very powerful but
delicate tool. If you use it wrong, it breaks easy.

Mathematics can be very treacherous. Once you start wading through its
darker waters there will be many things you think are true, but which turn out
to be false. The only way to avoid the worst pitfalls is from the beginning to
acquire some good work habits, such as:

• Develop your intuition. When you come to a new definition, create ex-
  amples for yourself. Try to make sure you actually understand it. As an
  example, in this problem set you encountered the definition of lim sup
  and of lim inf. Did you try out the concept on some sequences to get a
  feeling for it? At least you should have tried a constant sequence, a simple
  convergent sequence such as \{1/n\}, and at least one divergent sequence,
  say \{(-1)^n\}.

• Prove everything. Don’t take anything for granted. That doesn’t mean
  write down every single little step, but make sure you prove it in your
  mind to your satisfaction.

• Don’t give up too soon. A lot of mathematics is problem solving and
  chances are that even the greatest problem solver there ever was (Erdős?)
  would have the same reaction as the worst one on first seeing a problem: I
  have no idea what to do. But one thinks about it, one runs a few examples,
  and with a bit of luck things begin to fall into place. One of the greatest
  satisfactions in mathematics is to have solved a hard problem; you will
  never know that satisfaction if you give up too soon.

• Know when to give up.

1. (Rosenlicht, Ch. 3, #9). As almost all of you did, I’ll define \{q_n\} by
   \[q_{2n-1} = p_n, \quad q_{2n} = p_n.\] Then the exercise can be stated as: Prove that
   \[\lim_{n \to \infty} p_n = p\] if and only if \{q_n\} converges.

   Assume first that \[\lim_{n \to \infty} p_n = p.\] Let \(\epsilon > 0\) be given. There exists \(N\)
   such that \(n \geq N\) implies \(d(p_n, p) < \epsilon.\) Let \(N_1 = 2N - 1.\) Assume \(n \geq N_1.\)
   If \(n\) is even, then \(q_n = p,\) hence \(d(q_n, p) = 0 < \epsilon.\) If \(n\) is odd, then
   \(n = 2m - 1\) and \(q_n = p_m.\) Now \(n \geq N_1 = 2N - 1\) implies \(m \geq N,\) hence
   \(d(q_n, p) = d(p_m, p) < \epsilon.\) In either case, \(d(q_n, p) < \epsilon.\) It follows that \{q_n\}
   converges (to \(p\)).

   Conversely, assume that \{q_n\} converges. Then every subsequence of \{q_n\}
   converges to the same limit as \{q_n\}. Since the subsequence of even terms
   converges to \(p,\) the subsequence of odd terms, namely \{p_n\} must also
   converge to \(p.\)

2. (Rosenlicht, Ch. 3, #10). Let \(S = \{p_n : n \in \mathbb{N}\} \cup \{p\}.\) We have to
   prove \(S\) is closed in the metric space (let’s give it a name) \(X.\) There are
   many possible approaches to this problem. Here are two.

   Proof 1. By the definition, we prove \(U = X \setminus S\) is open. Assume \(q \in U;\)
   that is \(q \notin S.\) Then \(q \neq p, p_1, p_2, \ldots.\) In particular, \(d(p, q) > 0\) and
   because the sequence converges to \(p,\) there is \(N\) such that \(n \geq N\) implies
   \(d(p_n, p) < d(p, q)/2.\) Let \(r = \min(d(q, p_1), \ldots, d(q, p_{N-1}), d(p, q)/2).\) If
$x \in X$ and $d(x, q) < r$, then $x \neq p_1, \ldots, p_{N-1}$, since it is closer to $q$ than any of these points. We also have $x \neq p_n$ for $n \geq N$, since then
\[
d(p_n, x) \geq d(p_n, q) - d(q, x) \geq d(p, q) - d(p, p_n) - d(q, x) > d(p, q) - \frac{d(p, q)}{2} = 0,
\]
thus $x \neq p_n$. Finally, $d(x, q) < d(p, q)/2$ implies $x \neq p$. Thus $B(q, r) \subset U$. This proves $U$ is open.

**Proof 2.** Proving $S' \subset S$. We have to be careful here. It is tempting but wrong to think that $p$ is an accumulation point of $S$; wrong because the sequence $S$ could be, for example, a constant sequence and then $S$ has no accumulation points. What we need to see is that IF $q$ is an accumulation point of $S$, then $q = p$. So assume $q$ is an accumulation point of $S$. Let $\epsilon > 0$ be given. Because the sequence converges to $p$, there is $N$ such that $n \geq N$ implies $d(p_n, p) < \epsilon/2$. Because $p$ is an accumulation point of $S$, the set $B(q, \epsilon/2) \cap S$ is an infinite set, there must thus exist an infinity of indices $n$ such that $p_n \in B(q, \epsilon/2)$; in particular, there is such $n$ with $n \geq N$. There is thus $n$ such that simultaneously we have $d(p_n, p) < \epsilon/2$ and $d(p_n, q) < \epsilon/2$. By the triangle inequality, $d(p, q) < \epsilon$. Since $\epsilon > 0$ was arbitrary, this proves that $q = p \in S$.

3. (Rosenlicht, Ch. 3, #11). Because the sequence of real numbers \{a_n\} converges, it is bounded. Then \{|a_n - a|\} is also bounded; there is $M \in \mathbb{R}$, $M > 0$ such that $|a_n - a| \leq M$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$ be given. Because the sequence converges to $a$, there is $N_1 \in \mathbb{N}$ such that $|a_n - a| < \epsilon/2$ if $n \geq N_1$. Let $N = \max(N_1, 2N_1M/\epsilon)$ (or, if you insist in $N \in \mathbb{N}$, let it be the first integer larger than or equal max($N_1, 2N_1M/\epsilon$)). If $n \geq N$, then
\[
\frac{|a_1 + \cdots + a_n - a|}{n} = \frac{|a_1 + \cdots + a_n - na|}{n} = \frac{|(a_1 - a) + \cdots + (a_n - a)|}{n} \leq \frac{|a_1 - a| + \cdots + |a_n - a|}{n} = \frac{|a_1 - a| + \cdots + |a_{N_1} - a|}{n} + \frac{|a_{N_1} - a| + \cdots + |a_n - a|}{n} < \frac{N_1M}{n} + \frac{(n - N_1)\epsilon}{n} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
(Used that $N_1M \leq N\epsilon/2 \leq n\epsilon/2$ and $n - N_1 \leq n$.)

The result follows.

4. (Rosenlicht, Ch. 3, #13). Omitted.

5. Rosenlicht, Ch. 3, #18. Please, try to read also the comments I added at the end to this exercise. The concepts of lim inf and lim sup are important ones.

That $\liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n$ is easy, once one spent enough time to make sure one understands the definition. First of all let us notice (as one should) that neither of the sets
\[
\{x \in \mathbb{R} : a_n > x \text{ for an infinite number of integers } n\},
\{x \in \mathbb{R} : a_n < x \text{ for an infinite number of integers } n\}
\]
is empty. In fact, because the assumption is that the sequence is bounded, say $|a_n| \leq M$ for all $n \in \mathbb{N}$, then every $x < -M$ is in the first set (if $x < -M$ then $a_n > x$ for all $n$) and the whole interval $(-M, \infty)$ is in the
second set. The first set is bounded above, for example by $M$; if $a_n > x$ for as much as a single $n$, then $M \geq a_n > x$. Similarly, the second set is bounded below. Thus the first set has a least upper bound in $\mathbb{R}$, the second set has a greatest lower bound in $\mathbb{R}$. Let us denote them by $\overline{L}, \underline{L}$, respectively; that is

$$\overline{L} = \text{l.u.b.} \{x \in \mathbb{R} : a_n > x \text{ for an infinite number of integers } n\},$$

$$\underline{L} = \text{g.l.b.} \{x \in \mathbb{R} : a_n < x \text{ for an infinite number of integers } n\}.$$

Let $\epsilon > 0$ be given. Then $\overline{L} + \epsilon$ cannot possibly be in the first set. What does this mean? Well, it obviously means that it isn’t true that there is an infinite number of $n \in \mathbb{N}$ such that $a_n > \overline{L} + \epsilon$. A thing to know (and if you didn’t know it before to know from now onwards) is that the negation of a statement of the form: \textit{There exists an infinite number of positive integers } $n$ \textit{such that } $P$ \textit{holds}, where $P$ is some property, is: \textit{There is } $N \in \mathbb{N}$ \textit{such that } $P$ \textit{fails to hold for all } $n \geq N$. In fact, a direct negation is that $P$ holds for at most a finite number of integers, hence it ceases to hold from some point onwards.

Let us return to the exercise. Because $\overline{L} + \epsilon$ cannot be in the first set, there is $N_1 \in \mathbb{N}$ such that $a_n \leq \overline{L} + \epsilon$ for all $n \geq N_1$. Similarly, there will exist $N_2 \in \mathbb{N}$ such that $a_n \geq \underline{L} - \epsilon$ for all $n \geq N_2$. Taking $n = \max(N_1, N_2)$ (or anything larger), we get

$$\underline{L} - \epsilon \leq a_n \leq \overline{L} + \epsilon,$$

thus $\underline{L} - \overline{L} \leq 2\epsilon$.

Since $\epsilon$ is arbitrary, we conclude that $\underline{L} - \overline{L} \leq 0$; i.e., $\underline{L} \leq \overline{L}$.

We proved (or noticed) in the process something which is actually a useful property of these limit analogues, namely:

**LU:** Let $\epsilon > 0$ be given. There exists $N \in \mathbb{N}$ such that $a_n \leq \overline{L} + \epsilon$ for all $n \geq N$.

**LI:** Let $\epsilon > 0$ be given. There exists $N \in \mathbb{N}$ such that $a_n \geq \underline{L} - \epsilon$ for all $n \geq N$.

To see that we have equality if and only the sequence converges, assume first that $\{a_n\}$ converges to a limit $a$. Let us establish here something that could be useful later on:

Given a sequence of real numbers $\{a_n\}, x \in \mathbb{R}$, the following two statements are equivalent:

(a) $a_n > x$ for an infinite number of indices $n$.

(b) The sequence has a subsequence all of whose terms are $> x$.

The equivalence should be sort of obvious. Or fully obvious. Or close to obvious. If the first statement holds, we can determine integers $n_1, n_2, \ldots$ such that $1 \leq n_1 < n_2 < \cdots$ inductively as follows. We define $n_1$ as the first element of the set of $n \in \mathbb{N}$ such that $a_n > x$. Assuming $n_k$ defined, we define $n_{k+1}$ as being the first element of the necessarily non-empty set of all $n \in \mathbb{N}$ such that $n > n_k$ and $a_n > x$. Then $\{a_{n_k}\}$ is a subsequence all of whose terms are $> x$. Conversely, if there is such a subsequence, there obviously is an infinite number of indices $n$ for which $a_n > x$. 

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Assume now, with the sequence converging to \( a \), that \( x \) is in the first set; i.e., (in our new formulation), there is a subsequence \( \{a_{n_k}\} \) of \( \{a_n\} \) such that \( a_{n_k} > x \) for all \( k \in \mathbb{N} \). Since this sequence also converges to \( a \) we conclude that \( a \geq x \). That is, \( a \) is an upper limit of the first set, hence \( a \geq \bar{L} \). In exactly the same way, \textit{mutatis mutandis}, one sees that \( a \leq L \). Since \( L \leq \bar{L} \), we conclude that \( L = \bar{L} = a \).

Conversely, assume \( L = \bar{L} \). We know from what we just did that if the sequence has a limit, that limit equals the common value of \( L, L \), so let us try to prove that the sequence converges to this common value. It’s either that, or nothing. By properties \( \text{LU, LI} \) above, given \( \epsilon > 0 \) we can find \( N_1, N_2 \) such that \( a_{n_k} \leq L + \epsilon \) for all \( n \geq N_1 \) and \( a_{n_k} \geq L - \epsilon \) for all \( n \geq N_2 \). But since now \( L = \bar{L} \), this means that for \( n \geq \max(N_1, N_2) \) we have

\[
\bar{L} - \epsilon = \bar{L} - \epsilon \leq a_n \leq L + \epsilon; \quad \text{i.e., } \left| a_n - \bar{L} \right| \leq \epsilon.
\]

The exercise is over.

Some comments and extensions. The reason why the lim sup and the lim inf are important is that they always exist, specially if we extend the reals by adding \( \pm \infty \) to the set, as detailed in [Homework n] that lets infinity into your life.

If we do this, so that the l. u. b. of a set that isn’t bounded above is \( \infty \), the g. l. b. of a set that is not bounded below is \( -\infty \), and l. u. b. \( \emptyset = -\infty \), g. l. b. \( \emptyset = \infty \), then one can prove the following additional properties of the lim sup and of the lim inf:

(a) \( \{a_n\} \) is bounded above if and only if \( \limsup_{n \to \infty} a_n < \infty \).
(b) \( \{a_n\} \) is bounded below if and only if \( \liminf_{n \to \infty} a_n > -\infty \).
(c) The sequence \( \{a_n\} \) converges, or diverges to \( \infty \), or diverges to \( -\infty \) if and only if \( \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n \), in which case we also have \( \lim_{n \to \infty} a_n = \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n \).
(d) Let \( S \) be the set of all real numbers \( x \) such that there is a subsequence of \( \{a_n\} \) converging to \( x \). Then

\[
\liminf_{n \to \infty} a_n = \text{g. l. b. } S, \quad \limsup_{n \to \infty} a_n = \text{l. u. b. } S.
\]

A standard way of proving that a sequence \( \{a_n\} \) converges is to prove that \( \liminf_{n \to \infty} a_n \geq \limsup_{n \to \infty} a_n \). The converse inequality being always true, one would be done.

In conjunction with all this, here are a few other exercises to think about. Could one of them show up on the exam? Only time will tell.

(i) Prove: Let \( \{p_n\} \) be a sequence in a metric space and let \( S = \{p_n : n \in \mathbb{N}\} \) be the range of the sequence. Prove: If \( p \) is an accumulation point of \( S \), then there is a subsequence of \( \{p_n\} \) converging to \( p \). Is the converse true? That is, if there is a subsequence of \( \{p_n\} \) converging to \( p \), must \( p \) be an accumulation point of \( S \)?
Suppose $S$ has a single accumulation point $p$. Is it true or false that $\lim_{n \to \infty} p_n = p$?

(ii) Let $\{p_n\}$ be a sequence in a metric space $X$ and this time let $X$ consist of the set of all points $q \in S$ such that there exists a subsequence of $\{p_n\}$ converging to $q$. Prove: $S$ is closed.

(iii) Define $\limsup$, $\liminf$ as Rosenlicht does in Exercise 18 of Chapter 3. Prove the following additional properties for a sequence $\{a_n\}$ of real numbers. Assume it bounded if you want to avoid working with $\infty$ or $-\infty$.

(a) There exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\lim_{k \to \infty} a_{n_k} = \limsup_{n \to \infty} a_n$. Similarly, there exists a subsequence $\{a_{m_k}\}$ of $\{a_n\}$ such that $\lim_{k \to \infty} a_{m_k} = \liminf_{n \to \infty} a_n$.

(b) For every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ we have

$$\liminf_{k \to \infty} a_{n_k} \leq \liminf_{n \to \infty} a_n \leq \limsup_{k \to \infty} a_{n_k} \leq \limsup_{n \to \infty} a_n.$$ 

(c) If $a_n \leq b$ for all $n$, and $\lim_{n \to \infty} b_n = b$, then $\limsup_{n \to \infty} a_n \leq b$. Similarly, if $a_n \geq b$ for all $n$, and $\lim_{n \to \infty} b_n = b$, then $\liminf_{n \to \infty} a_n \geq b$.

(d) $\limsup_{n \to \infty} a_n = -\liminf_{n \to \infty} (-a_n)$.

(iii) Define a sequence $\{a_n\}$ of real numbers as follows. Consider the decimal representation of $n$,

$$n = d_1 \ldots d_k, \quad \text{where } d_1, \ldots, d_k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}; d_1 \neq 0.$$ 

Just to be sure this is understood, if $n = 1234$, then $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 4$; if $n = 99$, then $d_1 = 9, d_2 = 9$. In other words, $d_1, \ldots, d_k$ are the digits of $n$. Now define: If $k$, the number of digits of $k$ is odd, set $a_n = -1$. If $k$ is even, let

$$a = d_1 \ldots d_{\frac{k}{2}}, \quad b = d_{\frac{k}{2}+1} \ldots d_k$$

and set $a_n = b/a$. It is possible for $b$ to be zero, but $a \neq 0$. Again to make sure we understand how this works, here are some terms of this sequence:

$$a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = -1; \quad a_{10} = 0, a_{11} = 1, a_{12} = 2,$$

$$a_{13} = 3, a_{19} = 9, a_{20} = 0, a_{21} = \frac{1}{2}, a_{125} = -1, a_{4364} = \frac{64}{43}, \text{ etc.}$$

Determine the set of all numbers $x \in \mathbb{R}$ such that there exists a subsequence of $\{a_n\}$ converging to $x$.

6. Rosenlicht, Chapter 3, #19. If $\{a_n\}$, $\{b_n\}$ are bounded sequences, so is $\{a_n + b_n\}$. Let $\alpha = \limsup_{n \to \infty} a_n$, $\beta = \limsup_{n \to \infty} b_n$.

Let $\epsilon$ be a positive real number. By the definition of $\alpha$, there is only a finite number of $n \in \mathbb{N}$ such that $a_n > \alpha + \epsilon$; there is thus $N \in \mathbb{N}$ such that $a_n \leq \alpha + \epsilon$ if $n \geq N$.

Let $x \in \mathbb{R}$ be such that there is an infinite number of $n \in \mathbb{N}$ such that $x < a_n + b_n$. There is then also an infinite number of $n \in \mathbb{N}$ such that $n \geq N$ and $x < a_n + b_n$; for such $n$ we also have $x < \alpha + \epsilon + b_n$. In other words, there is an infinite number of $n \in \mathbb{N}$ such that $x - \alpha - \epsilon < b_n$; by
the definition of lim sup we get \( x - \alpha - \epsilon \) is in the set of which \( \beta \) is the l. u. b., hence \( x - \alpha - \epsilon \leq \beta \), or \( x \leq \alpha + \beta + \epsilon \). But this means that \( \alpha + \beta + \epsilon \) is an upper bound of the set of which \( \limsup_{n \to \infty} (a_n + b_n) \) is the l. u. b., hence

\[
\limsup_{n \to \infty} (a_n + b_n) \leq \alpha + \beta + \epsilon.
\]

This being true for all \( \epsilon > 0 \), we proved

\[
\limsup_{n \to \infty} (a_n + b_n) \leq \alpha + \beta = \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n.
\]

Assume now one of the sequences converges, say \( \lim_{n \to \infty} a_n = \alpha \). By the previous exercise we know that the limit and the limit sup coincide in this case. Now let \( x \in \mathbb{R} \) be a point in the set of which \( \beta \) is the l. u. b.; i.e., such that \( b_n > x \) for an infinite number of \( n \in \mathbb{N} \). That means there is a subsequence \( \{b_{n_k}\} \) such that \( b_{n_k} > x \) for all \( k \in \mathbb{N} \). If now \( \epsilon > 0 \) is given, there exists \( N \in \mathbb{N} \) such that \( a_n > \alpha - \epsilon \) for ALL \( n \geq N \). The set \( \{n_k : k \in \mathbb{N}, n_k \geq N\} \) is infinite and \( a_{n_k} + b_{n_k} > \alpha + \beta - \epsilon \) for all such \( n_k \). This implies that \( \limsup_{n \to \infty} (a_n + b_n) \geq \alpha + \beta - \epsilon \); \( \epsilon > 0 \) being arbitrary, we proved \( \limsup_{n \to \infty} (a_n + b_n) \geq \alpha + \beta \). The converse inequality being always true, we are done.