Assume \( b \) (Weierstrass) Let \( a \) that for some ties:

\[
\text{Properties 1 and 3 hold for } S
\]

This establishes the claim. We conclude the proof showing that

\[
\text{Properties 1, 2, 3 hold with } n
\]

is infinite, set \( a \) is done inductively as follows. We begin setting

\[
\epsilon/
\]

Because

\[
\text{We will use the famous Weierstrass bisection argument to prove this}
\]

Proof. Let \( d \) be the metric of \( X \). Since there is a convergent subsequence, it is legal to say let \( \{p_n\} \) be a convergent subsequence and let \( p = \lim_{n \to \infty} p_n \). We have to prove \( \lim_{n \to \infty} p_n = p \). For this purpose, let \( \epsilon > 0 \) be given. Because the sequence is Cauchy, there exists \( N \) such that if \( n, m \geq N \), then \( d(p_n, p_m) < \epsilon/2 \).

Because \( \{p_n\} \) converges to \( p \), there is \( K \) such that if \( k \geq K \), then \( d(p_{nk}, p) < \epsilon/2 \). Assume \( n \geq N \). Select \( k \geq \max(N, K) \). Then \( d(p_{nk}, p) < \epsilon/2 \). Moreover, \( k \geq N \) implies \( n_k \geq K \geq N \), thus \( d(p_n, p_{nk}) < \epsilon/2 \). It follows that

\[
d(p_n, p) \leq d(p_n, p_{nk}) + d(p_{nk}, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Our next objective is to prove that \( \mathbb{R} \) is a complete metric space. In the process we’ll prove a few other useful (and important) results about \( \mathbb{R} \). Most of these results are not valid in general metric spaces.

Proposition 2 (Weierstrass) Let \( S \) be a bounded subset of \( \mathbb{R} \). If \( S \) is an infinite set, then \( S \) has an accumulation (cluster) point.

Proof. We will use the famous Weierstrass bisection argument to prove this result.

Because \( S \) is bounded, there exist real numbers \( a, b \), \( a < b \) such that \( S \subset [a, b] \). We will construct two sequences \( \{a_n\}, \{b_n\} \) having the following properties:

1. \([a_n, b_n] \cap S \) is an infinite set for each \( n \in \mathbb{N} \).
2. \( a_{n-1} \leq a_n < b_n \leq b_{n-1} \) for each \( n \in \mathbb{N}, n \geq 2 \).
3. \( b_n - a_n = (b - a)/2^{n-1} \) for each \( n \in \mathbb{N} \).

This is done inductively as follows. We begin setting \( a_1 = a, b_1 = b \). It is clear that Properties 1 and 3 hold for \( n = 1 \), while 2 holds vacuously. Assume now that for some \( n \geq 1 \) we have satisfied Properties 1 and 3 and, if \( n \geq 2 \), also Property 2. Let \( c = (a_n + b_n)/2 \). Since \([a_n, b_n] \cap S \) is an infinite set and since

\[
[a_n, b_n] \cap S = ([a_n, c] \cap S) \cup ([c, b_n] \cap S),
\]

at least one of the sets \([a_n, c] \cap S \), \([c, b_n] \cap S \) must be infinite. If \([a_n, c] \cap S \) is infinite, set \( a_{n+1} = a_n, b_{n+1} = c \). Otherwise, set \( a_{n+1} = c, b_{n+1} = b_n \). It is now easy to verify that Properties 1, 2, 3 hold with \( n \) replaced by \( n + 1 \).

Now that we have the sequences we notice that \( \{a_n\} \) is increasing and bounded above (by any \( b_n \)) and \( \{b_n\} \) is decreasing and bounded below (by all the \( a_n \)’s). It follows that both sequences converge. Claim that they converge to the same limit. In fact, we see that \( \lim_{n \to \infty} (b_n - a_n) = \lim_{n \to \infty} (a - b)/2^n = 0 \), thus setting \( p = \lim_{n \to \infty} a_n \), we get

\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n + \lim_{n \to \infty} (b_n - a_n) = p + 0 = p.
\]

This establishes the claim. We conclude the proof showing that \( p \) is an accumulation point of \( S \). Let \( \epsilon > 0 \) be given. There is then \( N_1 \) such that
\[ |a_n - p| < \epsilon \text{ if } n \geq N_1, \text{ such that } |b_n - p| < \epsilon \text{ whenever } n \geq N_2. \]

It follows that \([a_n, b_n] \subset (p - \epsilon, p + \epsilon)\) and since \([a_n, b_n] \cap S\) is infinite, we also have that \(S \cap (p - \epsilon, p + \epsilon)\) is an infinite set; in particular it contains \(x \in S, x \neq p. \)

**Proposition 3** Every bounded sequence of real numbers contains a convergent subsequence.

**Proof.** Let \(\{a_n\}\) be a bounded sequence of real numbers and let \(S = \{a_n : n \in \mathbb{N}\}\) be the range of the sequence.

We have two cases. Case 1: \(S\) is a finite set. By necessity there is then a constant subsequence, which of course converges.

Case 2: \(S\) is an infinite set. We saw last time that Cauchy sequences are bounded, thus \(S\) is bounded. By Proposition 2, \(S\) has an accumulation point \(p\).

We need to show that there is a subsequence converging to \(a\). This is not the same thing (though it almost is) as saying there is a sequence in \(S\) converging to \(a\). A sequence in \(S\) is not necessarily a subsequence of the sequence \(\{a_n\}\) (why not?)

We define a strictly increasing sequence of positive integers \(1 \leq n_1 < n_2 < \cdots\) as follows: Set \(n_1 = 1\). Assuming \(n_k\) defined for some \(k \geq 1\), the interval \((a - \frac{1}{k}, a + \frac{1}{k})\) must contain an infinite number of elements of \(S\); there is thus an index \(n > n_k + 1\) such that \(a_n - a| < 1/k\). Set \(n_{k+1}\) equal to the first such \(n\). Then \(\{a_{n_k}\}\) is a subsequence of \(\{a_n\}\) converging to \(a\).

**Theorem 4** The metric space \(\mathbb{R}\) is complete.

**Proof.** If \(\{a_n\}\) is a Cauchy sequence in \(\mathbb{R}\) then it is bounded thus, by Proposition 3 has a convergent subsequence. By Proposition 1 it converges.

**Proposition 5** Assume \(X\) is a complete metric space. A subspace \(Y\) of \(X\) is complete if and only if \(Y\) is a closed subset of \(X\).

**Proof.** Assume first \(Y\) is a closed subset of \(X\) and let \(\{p_n\}\) be a Cauchy sequence in \(Y\). Then it is a Cauchy sequence in \(X\), hence converges to a limit \(p \in X\). But since \(Y\) is closed, \(p \in Y\).

Conversely, assume \(Y\) is complete. If \(\{y_n\}\) is a sequence of points in \(Y\) converging to \(p \in X\), then \(\{p_n\}\), being convergent, is a Cauchy sequence. It follows that it must converge in \(Y\); since limits are unique we conclude that \(p \in Y\).