The best definition of this very important number is probably

\[ e = \sum_{n=0}^{\infty} \frac{1}{n!}. \]

But, because we do not officially know anything about series yet, we have to be a bit more explicit. We will define a sequence \( \{s_n\}_{n=0}^{\infty} \) by

\[ s_0 = 1, \ s_1 = 1 + 1 = 2, \ s_2 = 1 + 1 + \frac{1}{2!} = 2.5, \ldots, \ s_n = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}, \ldots \]

briefly,

\[ s_n = \sum_{k=0}^{n} \frac{1}{k!}. \]

We now have the rather simple lemma:

**Lemma 1** The sequence \( \{s_n\} \) is strictly increasing and bounded above, hence converges.

**Proof.** We have

\[ s_{n+1} - s_n = \left(1 + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!}\right) - \left(1 + \cdots + \frac{1}{n!}\right) = \frac{1}{(n+1)!} > 0; \]

it follows that the sequence is strictly increasing. To show that it is bounded we will use a rather crude estimate; we use the fact that \( k! > 2^k \) for \( k = 4, 5, \ldots \). Crude because, for example, \( 10! = 3,628,800 \) while \( 2^{10} = 1024 \); the difference in size gets to be quite notable. The proof of this estimate is a trivial exercise in mathematical induction. From it we get, if \( n \geq 4 \):

\[ s_n = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-4}}\right) \leq 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \cdot \frac{1}{1 - \frac{1}{2}} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{67}{24} \approx 2.7917. \]

We used here the maybe well known formula for sum of a geometric progression: If \( r \neq 1 \), then

\[ 1 + r + r^2 + \cdots + r^m = \frac{1 - r^{m+1}}{1 - r}. \]

With \( r = 1/2, m = n - 4 \), one gets

\[ \frac{1}{2^4} + \cdots + \frac{1}{2^n} = \frac{1}{2^4} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-4}}\right) = \frac{1}{2^4} \cdot \frac{1 - \frac{1}{2^{n-4}}}{1 - \frac{1}{2}}. \]

The sequence is bounded. In fact,

\[ s_n \leq \frac{67}{24} < 3 \quad \text{for} \quad n = 0, 1, 2, \ldots \]

Because of the lemma we can officially define the number \( e \) by

\[ e = \lim_{n \to \infty} s_n. \]
Since \( \{s_n\} \) is strictly increasing and \( s_1 = 2 \), it is clear that \( e > 2 \). In view of (1) we also see that \( e \leq 67/24 \) thus \( e < 3 \). That is,

\[
3 \quad 2 < e < 3.
\]

We will now prove what I think is a rather non-obvious fact, namely that we also have

\[
4 \quad e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n.
\]

The proof is adapted (copied?, plagiarized?) from Rudin’s *Principles of Mathematical Analysis* and is presented here mostly to show how one uses the concepts of lim sup and lim inf. Let us set

\[
t_n = \left( 1 + \frac{1}{n} \right)^n, \quad n = 0, 1, 2, \ldots
\]

We now need to remember the binomial formula, another formula that isn’t too hard to prove by induction. Namely,

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = b^n + n a b^{n-1} + \frac{n(n-1)}{2} a^2 b^{n-2} + \cdots + a^n.
\]

We apply it to \( t_n \), so we use it with \( a = 1, \ b = 1/n \). The general term in the binomial expansion is then

\[
\binom{n}{k} \frac{1}{n^k} = \frac{n!}{k!(n-k)!n^k} = \frac{n(n-1) \cdots (n-k+1)(n-k)!}{k!(n-k)!n^k} = \frac{n(n-1) \cdots (n-k+1)}{k!n^k} = \frac{n(n-1) \cdots (n-k+1)}{n^k} \cdot \frac{1}{k!} = \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-k+1}{n} \cdot \frac{1}{k!}.
\]

The factors preceding \( 1/k! \) are all less than 1, so is their product and it follows that this general term is \( \leq 1/k! \). That is,

\[
5 \quad t_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \leq \sum_{k=0}^{n} \frac{1}{k!} = s_n, \quad n = 0, 1, 2, \ldots
\]

Actually, a careful reader would notice that our expression for the \( k \)-th term assumes implicitly that \( k \geq 2 \); however one sees directly that for \( k = 0, 1 \) he terms in the expansion of \( t_n \) are equal to those in the expression for \( s_n \) (they are both equal to 1), so (5) follows without a problem. In fact, we proved \( t_n < s_n \) if \( n \geq 2 \).

Here is where limsup and liminf come into the picture. We don’t know whether \( \{t_n\} \) converges yet, so we can’t talk of its limit. But its limsup and liminf always exist. We apply the following result; the proof is left as an exercise:

**Lemma 2** Let \( \{a_n\}, \{b_n\}, \{c_n\} \) be sequences of real numbers. Assume \( \{a_n\} \) converges; \( \lim_{n \to \infty} a_n = a \).

1. If \( b_n \leq a_n \) for all \( n \), then \( \limsup_{n \to \infty} b_n \leq a \).
2. If \( c_n \geq a_n \) for all \( n \), then \( \liminf_{n \to \infty} c_n \geq a \).
Applying the lemma with \( a_n = s_n \) and \( b_n = t_n \), we get \( \limsup_{n \to \infty} t_n \leq \lim_{n \to \infty} s_n \); that is
\[
\limsup_{n \to \infty} t_n \leq e. \tag{6}
\]
Let us return to the binomial expansion of \( t_n \). We’ll use the expression we got above for the \( k \)-th term; that is for \( \binom{n}{k} \frac{1}{n^k} \). We repeat the expression here in a slightly different way, namely
\[
\binom{n}{k} \frac{1}{n^k} = \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \cdots \left( 1 - \frac{k-1}{n} \right) \cdot \frac{1}{k!},
\]
noticing this time that this expression is only valid for \( k \geq 2 \). Let us fix for a while \( m \in \mathbb{N} \), \( m \geq 2 \) and et \( n \geq m \). Then
\[
t_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \geq \sum_{k=0}^{m} \binom{n}{k} \frac{1}{n^k} \\
= 1 + 1 + \left( 1 - \frac{1}{n} \right) \frac{1}{2!} + \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \frac{1}{3!} + \cdots + \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \cdots \left( 1 - \frac{m-1}{n} \right) \cdot \frac{1}{m!}.
\]
Here is the tricky thing. We keep \( m \) fixed and let \( n \to \infty \). The expression on the right hand side above; that is
\[
1 + 1 + \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \cdots \left( 1 - \frac{m-1}{n} \right) \cdot \frac{1}{m!},
\]
is a sum of \( m + 1 \) terms and it should be clear that, because we are dealing with a finite sum of terms!, that
\[
\lim_{n \to \infty} \left[ 1 + 1 + \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \cdots \left( 1 - \frac{m-1}{n} \right) \cdot \frac{1}{m!} \right] = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} = s_m.
\]
By Lemma 2, applied now with
\[
a_n = 1 + 1 + \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \cdots \left( 1 - \frac{m-1}{n} \right) \cdot \frac{1}{m!}, \quad c_n = t_n,
\]
we get
\[
\liminf_{n \to \infty} t_n \geq s_m, \tag{7}
\]
and since \( m \in \mathbb{N} \), \( m \geq 2 \) was arbitrary, (7) holds for all \( m \in \mathbb{N} \), \( m \geq 2 \). But now we use the fact that if \( s_m \leq \tau \) for all \( m \) and \( \{s_m\} \) converges, then \( \lim_{m \to \infty} s_m \leq \tau \). Applying this with \( \tau = \liminf_{n \to \infty} t_n \) and considering that \( \lim_{m \to \infty} s_m = e \), we proved
\[
\liminf_{n \to \infty} t_n \geq e. \tag{8}
\]
Together with (6), we proved
\[
\limsup_{n \to \infty} t_n \leq e \leq \liminf_{n \to \infty} t_n. \tag{9}
\]
Since ne always has that \( \liminf_{n \to \infty} t_n \leq \limsup_{n \to \infty} t_n \), we actually proved that
\[
\liminf_{n \to \infty} t_n = e = \limsup_{n \to \infty} t_n
\]
hence that \( \lim_{n \to \infty} t_n = e \).