1 Resources

The Penguin dictionary of curious and interesting numbers, by David Wells.

Number freak: from 1 to 200—the hidden language of numbers revealed, by Derrick Niederman, Penguin Group (2009) 291 pages. There's another 2009 edition with 304 pages called *Number freak: a mathematical compendium from 1 to 200*, put out by G. Duckworth, and a 2011 edition of the same name and publisher with 320 pages. You can see parts of the 291-page edition on Google Books. He is described as "a mathematician and securities analyst investment writer".

> "Number fanatic Derrick Niederman has a mission - to bring numbers to life. In Number Freak he explores the unique properties of the most fascinating numbers from 1 to 200, wherever they may crop up: from mathematics to sport, from history to the natural world, from language to pop culture. Packed with illustrations, amusing facts, puzzles, brainteasers and anecdotes, Number Freak is an enthralling and thought-provoking numerical voyage through the history of mathematics, investigating problems of logic, geometry and arithmetic along the way. Entertaining and accessible, it is a must for trivia addicts, maths-lovers and arithmophiles."

The lure of the integers, by Joe Roberts. An MAA publication whose chapters are devoted to individual numbers:

> "For a long time I have collected, in a rather haphazard way, interesting properties of various integers. If it came to my attention that, for example, 17 is the largest integer not the sum of three pairwise relatively prime integers eachlarger than 1, then this would be recorded on a slipt of paper, along with its source, and put in my desk."

Factorization using the elliptic curve method, by Dario Alpern, www.alpertron .com.ar/ECM.HTM. This is a fantastic resource. You can test for the primality of very large numbers and factor them. These numbers can be entered in forms like 93! + 1 or 2^257 - 1.

The *Prime Curios!* web site: primes.utm.edu/curios. A cooperative enterprise (but not a wiki) with lots of facts about composites also. It is now in book form also, called *Prime Curios! The dictionary of prime number trivia*, by Chris K. Caldwell and G. L. Honaker, Jr.

The *Number Gossip* web site by Tanya Khovanova, www.numbergossip.com. "Enter a number and I'll tell you everything you wanted to know about it but were afraid to ask."

Erich Friedman's web site, *What's special about this number?*, is located at http://www2.stetson.edu/~efriedma/numbers.html. Lists one fact each for a bunch of numbers.

A little off the mark, but possibly interesting, is *Wonders of Numbers: Adventures in Mathematics, Mind, and Meaning*, by Clifford A. Pickover, Oxford University Press, 2001. Has a larger scope than small positive integers. You can see some of it on Google Books. He refers to the cannonball problem (see below) as "grenade stacking".

2 Unfinished stuff

- The number 80 is the largest number n such that no prime factor of n or $n+1$ is bigger than 5. Check this.

  Maybe this is okay but the proof has eluded me.

- That 76 is the *only* number n that is the sum of distinct primes in exactly n ways. It looks like everybody after is a sum of distinct primes in more ways, and everybody before in fewer. Also looks like the number of ways strictly increases after 76 (actually somewhat before that).

3 Figurate numbers

This was the most referenced contents of figures.txt.

Bow tie 1 5 11 19 29 41 55 71 89
Wings 1 7 17 31 49 71 97
Centered triangular 1 4 10 19 31 46 64 85
Centered square 1 5 13 25 41 61 85
Centered pentagonal 1 6 16 31 51 76
Centered hexagonal 1 7 19 37 61 91
Centered heptagonal 1 8 22 43 71
Triangular 1 3 6 10 15 21 28 36 45 55 66 78 91
Square 1 4 9 16 25 36 49 64 81 100
Pentagonal 1 5 12 22 35 51 70 92
Hexagram 1 37 73 121

The standard figurate numbers from six on, hexagonal, heptagonal, etc., don't really have good pictures. I'm not sure what their history is. The corresponding centered figurate numbers, on the other hand, have nice pictures.

Wings are two squares of the same size joined at a common vertex. So they are of the form $2n^2 - 1$. The *bow ties* are two triangles of the same size joined at a common vertex. So they are of the form $n(n+1) - 1$. Some square and triangles have centers, so there are corresponding *punctured* figures. The squares with centers are the odd ones, the triangles with centers are 1, 10, 28, I guess it's every third one. The formula is $n(n+1)/2$ where $n \equiv 1 \mod 3$, and the punctured triangles have one less point. The (singly) trimmed triangles and square have their vertices removed. The number 77 is a trimmed square. You can trim more. The *handled* triangles are trimmed, and then three more points are removed right by the vertices, leaving the six points that surround each of them. A *tipped* triangle, like 39, has an extra point on each vertex.

4 Numbers that are equal to one less than twice their reversal.

I believe these are all of the form 799999999993. That is, $8 \cdot 10^n - 7$. The reversal is $4 \cdot 10^n - 3$ and the equation is clear. Why are these all of them? Here is a sketch, at least for emirps. The first digit must be at least twice the last digit or twice the reversal will be too big. It can't be more than one plus twice the last digit or the reversal will be too small. Twice the first digit must be equal to the last digit plus 1 modulo 10. If the numbers are prime, those digits are from $\{1, 3, 7, 9\}$. Is that important? The only possibility is first digit 7 and last digit 3.

Now suppose the numbers are $7xy3$ and $3yx7$. Then $2(10y + x) + 1$ has to equal $100 + 10x + y$. What does that say? $20y + 2x + 1 = 100 + 10x + y$ so $29y = 99 + 8x$. The solution to that modulo 29 is $x = 9$. So $x = 9$ and $y = 9$. And so on.

I believe I have strong computational evidence for this. We must be able to modify this sketch to avoid emirps.

The question remains, why can't 799999999993 and 399999999997 both be primes for some gigantic numbers (at least 41 digits). Either one can be prime, and when they are composite, there is no obvious pattern on the least prime factor.

5 The cannonball problem

This is for the interesting number 70. It states that the only pyramidal number (other than 1) that is also a square is 70^2. An elementary proof is in "The square pyramid puzzle", W. S. Anglin, *Monthly*, **97** (1990) 120–124. From Wolfram MathWorld.

6 The inverse φ-function

To implement this algorithm we need a prior bound on $\varphi^{-1}(n)$. We show here that if $\varphi(k) = n$, then $k \leq 2n^2$. That is $k \leq 2\varphi(k)^2$. Now $\varphi(k) = k \prod (1 - 1/p)$ where p ranges over the primes dividing k, so we are trying to prove that $1 \leq 2k \prod (1 - 1/p)^2$. Clearly it suffices to prove this for square-free k. In this case, $k = \prod p$, so

$$2k \prod (1 - 1/p)^2 = 2 \prod (p - 1)(1 - 1/p) = 2 \prod \frac{(p-1)^2}{p}$$

so we want

$$1 \leq 2 \prod \frac{(p-1)^2}{p}$$

Now for $p = 2$, the factor $(p-1)^2/p = 1/2$, but for the rest of the primes, it is greater than 1. So the right-hand side is at least 1. For k different from 2 or 6 we are supposed to be able to replace the 2 in $2n^2$ by 1. Probably easy enough to work that out. If fact, if some prime other than 2 or 3 appears, then it will take care of the factor of 1/2 from $p = 2$. Or if k is divisible by 4 or 9. The remaining cases are $k = 2$ or $k = 6$. We take care of those by modifying the inequality to read $k \leq \max\left(n^2, 6\right)$. That is what is now implemented. (This is well known.)

7 The functions $\varphi(n)$ and $\pi(n)$

We want to show that $\varphi(n) > \pi(n)$ for $n > 90$. Here $\pi(n)$ is the number of primes that do not exceed n.

Euler's constant is $\gamma < 0.57722$.

A bound on φ for $n > 2$ is

$$\varphi(n) > \frac{n}{e^{\gamma} \log\log n + 3/\log\log n}$$

A bound on π for $n > 1$ is $\pi(n) < 1.25506n/\log n$.

So, asymptotically, we want

$$\frac{1}{e^{\gamma} \log\log n + 3/\log\log n} > \frac{1.25506}{\log n}$$

that is

$$e^{\gamma} \log\log n + \frac{3}{\log\log n} < \frac{\log n}{1.25506}$$

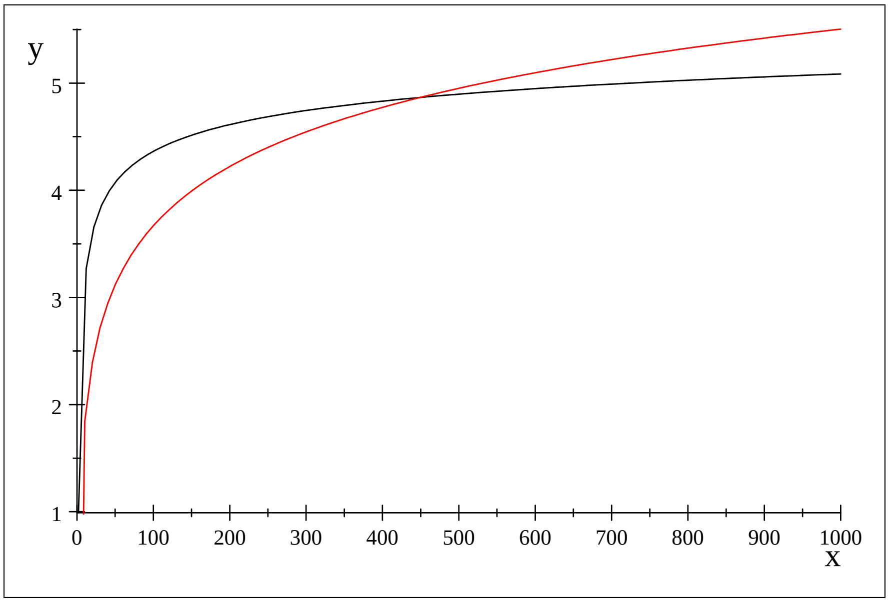
$e^{\gamma} < 1.7811$

$3/\log(\log(500)) < 1.643$

Thus it suffices to show that

$$1.7811 \log\log n + 1.643 < \frac{\log n}{1.25506}$$

for $n > 500$. Here is the graph with the left side in black and the right in red.



We conclude that $\varphi(n) > \pi(n)$ for $n \geq 500$. We check directly that this inequality holds for $91 \leq n \leq 1000$. Of course we knew that $\varphi(90) = \pi(90) = 24$—that's what this was all about.

8 Sum of the digits plus the product of the digits

This is a dynamical system $\varphi : \mathbf{Z}^+ \to \mathbf{Z}^+$ that receives some attention. The number 99 is the largest fixed point. (The other fixed points are 19, 29, ..., 89. The numbers 1, 2, ..., 9 get taken to their doubles; all the other two-digit numbers go to smaller numbers.) Here is a proof. If n is a k digit number, then $\varphi(n) \leq 9k + 9^k$. A computer search shows that $\varphi(n) < n$ for every three-digit number n. Now induct on k for $k \geq 3$. If one of the digits is 0 there is no problem. We need to prove, for $k+1$ digits $1 \leq d_0 \leq d_1 \leq \cdots \leq d_k$, that

$$\prod_{i=0}^{k} d_i + \sum_{i=0}^{k} d_i < d_0 10^k + d_1 10^{k-1} + \cdots + d_{k-1} 10 + d_k$$

where the expression on the right is the smallest $k+1$ digit number that you can construct from these digits. By induction

$$\prod_{i=1}^{k} d_i + \sum_{i=1}^{k} d_i < d_1 10^{k-1} + \cdots + d_{k-1} 10 + d_k$$

Adding $d_0 10^k$ to the right and $(d_0 - 1) \prod_{i=1}^{k} d_i + d_0$ to the left gives us the inequality we want, so it suffices to show that

$$(d_0 - 1) \prod_{i=1}^{k} d_i + d_0 \leq d_0 10^k$$

But the left side of this is no greater than $d_0 \prod_{i=1}^{k} d_i < d_0 10^k$.

9 The number 6, a triangle between twin primes

Let $t = m(m+1)/2$ be a triangle number. Then

$$t - 1 = \frac{m(m+1)}{2} - 1 = \frac{m^2 + m - 2}{2} = \frac{(m-1)(m+2)}{2}$$

is divisible by $(m-1)/2$, if m is odd, and by $(m+2)/2$, if m is even. So the only triangle number that is one more than a prime is 6, corresponding to $m = 3$.

10 The number 26, between a square and a cube

I couldn't prove this. Erich Friedman says that he can't find the reference, but it's been proved, and he remembers that the proof was difficult. I tend to believe him.

11 The number 97 and sums of four squares

Prime Curios says that the number 97 is the largest prime p that can be written as a sum of four squares using any square that is less than p. (Prime Curios, "as a corollary to work by J. H. Conway") This seems to be false: the primes 257 and 313 also have this property.

12 Is 81 the largest number that is equal to the square of the sum of its digits?

The biggest the square of the sum of the digits of a five digit number can be is $(5 \cdot 9)^2 = 2025$, a four digit number. So we only have to check up to 9999. The largest number such that square of the sum of its digits exceeds it is 399, and only for 1 and 81 so we get equality.

13 Pythagorean triples

Each odd number $2n+1$ with $n \geq 1$, is contained in the primitive pythagorean triple

$$[2n+1,\ 2n(n+1),\ 2n(n+1)+1]$$

If $m \geq 2$, then 2^m is contained in the primitive pythagorean triple

$$\left[2^m,\ 2^{2(m-1)} - 1,\ 2^{2(m-1)} + 1\right]$$

If k is odd, then $4k$ is contained in the primitive pythagorean triple

$$\left[4k,\ \left|k^2-4\right|,\ k^2+4\right]$$

So the only guys who might not be in a primitive pythagorean triple are 1 and those of the form $2k$ where k is odd. And these are not because you can't write $k = uv$ where one of u or v is even.

14 Znam's problem

There are 96 sets of 8 numbers that are greater than 1 where each is a proper divisor of 1 plus the product of all the others. The problem of finding such sets, and determining how many there are, is known as *Znám's problem* for the number 8. An example of one of the 96 sets is $\{2, 5, 7, 11, 17, 157, 961, 4398619\}$. Note that $2 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 157 \cdot 961 + 1 = 1974\,979\,931 = 449 \cdot 4398619$. See http://oeis.org/A075441 and the link to "all results for $a\,(8)$" (Zhao Hui Du).

15 When is $p^{86} - 2$ composite?

We don't have to look at $p^{86}+2$ which is divisible by 3 for any odd number p. If p is 3 or 4 modulo 7, then $p^{86} - 2$ is divisible by 7. I tested the odd primes less than 86 on the elliptic curve factoring site getting the following results: for 61 the smallest factor was 12220038807117289, for 43 it was 24928566706649, for 19 it knew right away it was composite, but after chugging along for 8 minutes, all it could say was that the smallest factor was at least 20 digits long. For the rest of the primes, the smallest factor is listed below.

83	79	71	47	41	37	29	23	13	7	5
23	359	1217	79	199	23	17	353	41	51721	17

All I really need here is the probabilistic primality tester which this site obviously has. It tells you right away if it knows the number is composite, then it tries to find all of its factors. I'd like to find out if any of the odd numbers up to 100 give (probable) primes. I assume the program will tell you immediately if something is a probable prime. http://www.alpertron.com.ar/ECM.HTM

16 There are 83 right truncatable primes

This is an easy program. See also I.O. Angell and H.J. Godwin, "On truncatable primes", *Mathematics of Computation*, **31** (1977) 265–267 which is on JSTOR.

17 Is 77 the largest number that can't be written as a sum of distinct number greater than 1 whose reciprocals sum to 1

Ron Graham's paper, *A theorem on partitions* (1963), contains a proof of this. He says D. H Lehmer has an unpublished proof of the fact that 77 cannot be so written. I'm interested in that part.

Suppose $n = \sum x_i$ and $\sum 1/x_i = 1$. Because $\sum 1/x_i$ is an integer, if a prime divides one of the x_i it has to divide two of them. So $3p \leq n$ if p divides one of the x_i. Continuing in this vein, we want to show that if $p > 5$, then $6p \leq n$. That's true if three of the x_i are divisible by p because the smallest numbers we could use would be p, $2p$, and $3p$. Now if there were only two, and $6p > n$, then they would appear in one of the combinations

$$\frac{1}{ap} + \frac{1}{bp}$$

where (a, b) is equal to $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 3)$. These sums are

$$\frac{3}{2p}, \frac{4}{3p}, \frac{5}{4p}, \frac{5}{6p}$$

so if $p > 5$, then p would have to divide another term. So for 77, we have $p \leq 11$.

What happens with 11 in the case of 77? If it appears, then 11, 22, and 33 all must appear. That sums to 66, so we must partition $77 - 66 = 11$ and have the sum of the reciprocals add to $1 - 1/11 - 1/22 - 1/33 = 5/6$. You can't use 7 because it would have to appear twice (as 6 is not divisible by 7). You can't use 5 for the same reason. How can you write 11 as a sum of a subset of $\{2, 3, 4, 6, 8, 9\}$? I guess $2 + 9$, $2 + 3 + 6$, and $3 + 8$. None the reciprocal sums adds to 5/6. Actually, my program looks at stuff like this, so maybe you don't have to eliminate 11 this way.

Here are the results of the program when I ran it on the numbers from 1 to 77:

11 = 2 3 6	24 = 2 4 6 12	30 = 2 3 10 15	31 = 2 4 5 20
32 = 2 3 9 18	37 = 2 3 8 24	38 = 3 4 5 6 20	43 = 2 4 10 12 15
45 = 2 4 9 12 18	50 = 3 4 6 10 12 15	52 = 3 4 6 9 12 18	53 = 2 5 6 10 30
54 = 2 3 7 42	55 = 2 4 7 14 28	57 = 3 4 5 10 15 20	59 = 3 4 5 9 18 20
60 = 2 6 9 10 15 18	61 = 2 4 6 21 28	62 = 3 4 6 7 14 28	64 = 3 4 5 10 12 30
65 = 2 6 8 10 15 24	66 = 2 3 12 21 28	67 = 2 5 6 9 45	69 = 2 3 14 15 35
71 = 3 4 9 10 12 15 18	73 = 3 5 6 9 12 18 20	74 = 2 5 9 10 18 30	75 = 3 4 5 8 15 40
76 = 2 5 7 14 20 28			

18 Is 69 the smallest number n such that $100^n - n$ is prime?

How to check this? There is an online factoring routine http://www.alpertron.com.ar/ECM.HTM.

This routine claimed that $100^{69} - 69$ is prime with an elapsed time of 16.3 seconds, and wrote it as a sum of three squares, two of them equal, all with the same number of digits. Nice routine to know about. Found it at Prime Curios. Written by Dario Alejandro Alpern.

For the smallest part, certainly n cannot be divisible by 2 or 5. Also, n cannot be 1 modulo 3. What does that leave? 3, 9, 11, 17, 21, 23, 27, 29, 33, 39, 41, 47, 51, 53, 57, 59, 63. Below is a table of smallest prime factors for various n.

Scientific WorkPlace will factor for n equal to $3, 9, 11$. It also finds the factor 367 of $100^{63} - 63$, and the factor 1367 of $100^{59} - 59$. At least it did once.

63. 367
59. 1367
57. 7
53. 7
51. 283
47. 37
41. 5086612178741
39. 31
33. 31
29. 13
27. 13
23. 11
21. 263134112870431780253
17. 4507

Checking with SWP: $\left(100^{21} - 21\right)/263134112870431780253 = 3800\,343\,441\,188\,120\,424\,743$ and

$\left(100^{63} - 63\right)/367 = 2724\,795\,640\,326\,975\,476\,839\,237\,057\,220\,708\,446\,866\,485\,013\,623\,978\,201\,634\,877\,384\,196\,185\,286\,103\,542\,234\,332\,425\,068\,119\,891\,008\,174\,386\,920\,980\,926\,430\,517\,711$

19 Writing a number as the sum of odd primes.

The claim is that 35 is the unique number n such that n can be written in n ways as a sum of odd primes. To verify this, we compute a lower bound on the number of ways to write n as a sum of 3s, 5s, 7s, and 11s. That bound is

$$\sum_{7i+11j \le n} \left\lfloor \frac{n - 7i - 11j}{15} \right\rfloor$$

The bound is strictly increasing for $n \geq 53$ and is equal to 82 at $n = 81$. So there are no numbers greater than 80 that can work. But the numbers from 1 to 80 are easily checked directly, and the only one that works is 35.

Why is the bound strictly increasing for $n \geq 53$? It is clear that the bound is always weakly increasing. The term we pick up at 54 is from $54 - 7 \cdot 4 - 11 = 15$.

At 55 it is from $55-7-11\cdot 3=15$. You also get another one from $55-7\cdot 2-11=30$. We can show more quickly that the displayed sum is strictly increasing for $n\geq 81$, which is all that we need. Look at the numbers $m-11j$, for $11j\leq m$ until you get one that is divisible by 7. If you get to look at least 7 numbers j, then one of $m-11j$ must be divisible by 7. So if m is at least 66 we, can write m as a combination of 7's and 11's. So if $n\geq 81$, we can write subtract a combination of 7's and 11's from n and get 15, which gives us an extra term in the displayed sum.

Here is the computer output for $n=81$. The entry 0-5 means that $\lfloor 81/15\rfloor = 5$. The entry 11-4 means that $\lfloor (81-11)/15\rfloor = 4$. The entry 18-4 means that $\lfloor (81-7-11)/15\rfloor = 4$.

81. 0-5 11-4 22-3 33-3 44-2 55-1 66-1 7-4 18-4 29-3 40-2 51-2 62-1 14-4 25-3 36-3 47-2 58-1 21-4 32-3 43-2 54-1 65-1 28-3 39-2 50-2 61-1 35-3 46-2 57-1 42-2 53-1 64-1 49-2 60-1 56-1 63-1 (82)

Here is the computer output for the actual number of ways to write n as a sum of odd primes for n from 1 to 80.

1. 0
2. 0
3. 1
4. 0
5. 1
6. 1
7. 1
8. 1
9. 1
10. 2
11. 2
12. 2
13. 3
14. 3
15. 3
16. 4
17. 5
18. 5
19. 6
20. 7
21. 7
22. 9
23. 10
24. 11
25. 12
26. 14
27. 15
28. 17
29. 20
30. 21

31. 24
32. 26
33. 29
34. 33
35. 35
36. 40
37. 44
38. 47
39. 53
40. 58
41. 64
42. 70
43. 77
44. 84
45. 91
46. 101
47. 110
48. 120
49. 130
50. 142
51. 155
52. 168
53. 184
54. 199
55. 215
56. 234
57. 254
58. 275
59. 298
60. 323
61. 348
62. 376
63. 407
64. 439
65. 474
66. 511
67. 551
68. 592
69. 638
70. 688
71. 739
72. 795
73. 854
74. 917
75. 984
76. 1057

77. 1134
78. 1215
79. 1303
80. 1395