Weak Markov's principle, strong extensionality, and countable choice

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Abstract

Ishihara showed, using countable choice, that weak Markov's principle is equivalent to all real functions on a complete metric space being strongly extensional. In this note we show that weak countable choice suffices, and that the theorem fails in sheaf models of the real numbers.

In [4] Mandelkern calls a real number c **pseudopositive** if for all real x either $\neg(x \leq 0)$ or $\neg(c \leq x)$. In [3, Lemma 16.2] he shows that if $c \leq 1$ is pseudopositive, then there exists an increasing function $f : [0, 1] \to [0, 1]$ such that $f(0) = 0$ and $f(c) = 1$. Clearly the converse is also true: we may extend f in a trivial way to all of $\mathbf{R}$, and either $f(x) > f(0)$ or $f(x) < f(c)$. This idea is related to a weak version of Markov's principle, which Mandelkern calls WLPE but we will call **WMP**: *every pseudopositive number is positive.* Markov's principle itself, which Mandelkern calls LPE, says that, for all x, if $\neg(x = 0)$, then $x \neq 0$.

Both Markov's principle and WMP were formulated in the context of the countable axiom of choice. In this context, they may be interpreted in terms of binary sequences; indeed, Markov's principle was originally stated in terms of sequences. In the absence of countable choice, each principle has a sequential version and a real version. The sequential version can be obtained by restricting to *Cauchy* reals. We will be concerned only with WMP here, and we will mean the real version.

WMP is related to strong extensionality. A function f between metric spaces is **strongly extensional** if $d(x, y) > 0$ whenever $d(f(x), f(y)) > 0$.

Looking at Mandelkern's function f above, we see that if f is strongly extensional, then the pseudopositive number c is positive. Hence if every function from $\mathbf{R}$ to $\mathbf{R}$ is strongly extensional, then WMP holds. Conversely, Ishihara [2, Lemma 4] proved, using countable choice, that if f maps a complete metric space to a metric space, and $f(x) \neq f(y)$, then $d(x, y)$ is pseudopositive—so if WMP holds, then every map from a complete metric space to a metric space is strongly extensional. What happens without countable choice?

To analyze Ishihara's lemma in a choice-free environment, we first make three observations (Theorems 1,2,3). Call two elements x and y of a set **pseudoapart** if for each z in the set either $\neg(z = x)$ or $\neg(z = y)$. For example, a pseudopositive number is pseudoapart from 0.

Theorem 1 *If x and y are points in a metric space, and $d(x, y)$ is pseudopositive, then x is pseudoapart from y.*

Proof. Either $\neg(d(x, z) \leq 0)$ or $\neg(d(x, y) \leq d(x, z))$. In the first case $\neg(x = z)$, in the second $\neg(y = z)$. ■

Theorem 2 *If x is pseudoapart from y in a metric space, then $\{x, y\}$ is a closed set.*

Proof. Suppose z is in the closure of $\{x, y\}$. If $z \neq y$, then $z = x$, so $\neg(z = x)$ implies $z = y$. Similarly $\neg(z = y)$ implies $z = x$. ■

Theorem 3 *If f be a function. If $f(x)$ and $f(y)$ are pseudoapart, then x and y are pseudoapart. (Every function is strongly pseudoextensional.)*

Proof. If $\neg(f(z) = f(x))$, then $\neg(z = x)$. ■

As a result of these observations, we see that if f is a function between metric spaces, and $d(f(x), f(y)) > 0$, then the set $\{x, y\}$ is closed; if the domain of f is complete, then $\{x, y\}$ is complete, and $\neg(x = y)$. So the complete metric space in Ishihara's lemma might as well consist of two unequal points, in which case the lemma would read: If $\neg(x = y)$, and $\{x, y\}$ is a complete metric space, then $d(x, y)$ is pseudopositive. The map taking x to 0 and y to $d(x, y)$ is an isometry, so we might as well assume that $x = 0$ and $y = c$ are real numbers such that $\neg(c \leq 0)$. Then Ishihara's lemma becomes:

($*$) If $\{0, c\}$ is complete and $\neg(c \leq 0)$, then c is pseudopositive.

Note that the converse of $(*)$ is true by Theorems 1 and 2. We will show that $(*)$ holds in the presence of a weak axiom of countable choice, that $(*)$ fails in the standard sheaf model of the real numbers, and that WMP holds in that model (so WMP does not by itself entail that every function from a compact subset of $\mathbf{R}$ to $\mathbf{R}$ is strongly extensional).

The **sheaf model** of $\mathbf{R}$ on a topological space X is obtained by identifying $\mathbf{R}$ with the (classical) ring of continuous real valued functions on X (see [5]). The Heyting algebra of open subsets of X is the algebra of truth values for this model. The truth value of $a < b$ is $\{x : a(x) < b(x)\}$. We say that a statement holds at $x \in X$ if x is in its truth value. If a statement can be proved constructively without appeal to the axiom of countable choice, then its truth value is X, that is, it holds at every x. These sheaf models are interesting because they give concrete pictures of distinctions that disappear in the presence of countable choice. For example, if X is the unit interval, then the Cauchy reals correspond to the constant functions on X.

Theorem 4 *In the sheaf model of* $\mathbf{R}$ *on* $[0,1]$*, the statement "*$\{0,c\}$ *is complete and* $\neg(c \le 0)$*" holds at* x *if and only if either* $c(x) > 0$*, or* x *is an endpoint and* c *is positive on a punctured neighborhood of* x*.*

Proof. The condition $\neg(c \le 0)$ at x is precisely that c is positive on a dense subset of a neighborhood of x. A number a is in the closure of $\{0,c\}$ at x exactly when $a(y) \in \{0, c(y)\}$ for all y in a neighborhood of x. If x is not an endpoint, and $c(x) = 0$, we can take $a(y) = 0$ for $y \le x$ and $a(y) = c(y)$ for $y \ge x$. Then a is in the closure of $\{0,c\}$ at x, but it is not true that $a = 0$ or $a = c$ in a neighborhood x. If x is an endpoint, and c vanishes arbitrarily close to x, then we can let a be c on intervals arbitrarily close to x, and a be 0 on intervals arbitrarily close to x. If x is an endpoint and c is positive on a punctured neighborhood of x, then if a is in the closure of $\{0,c\}$ at x, and a is not zero in a neighborhood of x, then a must be c in a neighborhood of x. ■

Theorem 5 *Let* X *be any metric space. Then WMP holds in the sheaf model of* $\mathbf{R}$ *on* X*.*

Proof. We want to show that, if c is pseudopositive at x_0, then $c(x_0) > 0$. If x_0 is an isolated point there is no problem, so suppose $1 \ge r_1 > r_2 > \cdots$ converges to 0 where $r_i = d(x_i, x_0)$. We prove the contrapositive. Suppose

$c(x_0) = 0$ and let f be a continuous function from $(0,1]$ to $[0,1]$ such that $f(t) = 0$ for $t \in (r_{4n+2}, r_{4n})$ and $f(r_{4n+3}) = 1$. Set

$$a(x) = f(d(x,x_0))(c(x) + d(x,x_0))$$

for $x \neq x_0$, and $a(x_0) = 0$. Then $a(x_{4n+3}) > c(x_{4n+3})$, and $a(x) = 0$ on the open set

$$\{x \in X : r_{4n+2} < d(x,x_0) < r_{4n}\}$$

which contains x_{4n+1}. So $\neg(0 \geq a)$ and $\neg(a \geq c)$ both fail at x_0. ■

A weak axiom of countable choice was introduced in [1]. It is implied by the law of excluded middle, and suffices for many standard applications of countable choice in constructive mathematics. In particular, given a real number c, it allows the construction of a binary sequence λ_n such that if $\lambda_n = 0$, then $c < 1/2n$, and if $\lambda_n = 1$, then $c > 1/(2n+1)$. It is

WCC. Let A_n be a sequence of nonempty sets. Suppose that for $n \neq n'$, either A_n or $A_{n'}$ is a singleton. Then there is a choice sequence $a_n \in A_n$.

Theorem 6 *Assume WCC. Let c be a real number such that $\neg(c \leq 0)$ and $\{0, c\}$ is complete. Then c is pseudopositive.*

Proof. Let t be a real number. By a lemma in [1], there is a binary sequence λ_n such that if $\lambda_n = 0$, then $c < 1/2n$, and if $\lambda_n = 1$, then $c > 1/(2n+1)$. Similarly μ_n for t. Define a Cauchy sequence s_n in $\{0, c\}$ by set $s_n = 0$ unless $\lambda_m = 0$ and $\mu_{m+2} = 1$ for some $m \leq n$, in which case set $s_n = c$.

Suppose s_n converges to c. If $t \leq 0$, then $\mu_n = 0$ for all n. So $s_n = 0$ for all n, a contradiction. Hence $\neg(t \leq 0)$. On the other hand, suppose s_n converges to 0. Then $s_n = 0$ for all n. Suppose now that $t \geq c$. If $\lambda_m = 1$, then $\mu_{m+1} = 1$ lest $c > 1/(2m+1)$ and $t < 1/(2m+2)$. So $\lambda_m = 0$ for all m, a contradiction. Therefore $\neg(t \geq c)$. ■

In the absence of the axiom of countable choice, we must distinguish between a subset of the real numbers being closed (that is, complete) and being sequentially closed. However in the presence of WCC, the two concepts coincide on finitely enumerable subsets like $\{0, c\}$.

Theorem 7 *If WCC holds, then the sequential closure of any finitely enumerable subset of a metric space coincides with its closure.*

Proof. Let $c_1, \ldots, c_k$ be points in a metric space, and a in the closure of $\{c_1, \ldots, c_k\}$. For $i = 1, \ldots, k$, let λ_n^i be a binary sequence such that if $\lambda_n^i = 0$, then $d(a, c_i) < 1/2n$, and if $\lambda_n^i = 1$, then $d(a, c_i) > 1/(2n+1)$. Let j_n be the smallest index i such that $\lambda_n^i = 0$ (there must be one because a is in the closure), and set $s_n = c_{j_n}$. Then $d(s_n, a) < 1/2n$. ■

References

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