Weakly integrally closed domains: minimum polynomials of matrices

James Brewer and Fred Richman
Florida Atlantic University
Boca Raton, FL 33431

17 November 1999

Abstract

Must the coefficients of the minimum polynomial of a matrix over a domain lie in that domain? This question leads to the notion of a weakly integrally closed domain, over which the answer is "yes" for 3-by-3 matrices. It is shown that certain subalgebras of $k[t]$ are weakly integrally closed, as are rings consisting of quadratic algebraic numbers.

1 Minimum polynomials of matrices

Let A be an n-by-n matrix with entries in a commutative integral domain R with quotient field F. We are interested in the question of when the (monic) minimum polynomial of A over F has all its coefficients in R. This is true if R is integrally closed because the minimum polynomial is a factor of the characteristic polynomial. It is also true in the following two simple cases:

- If the characteristic polynomial of A is the minimum polynomial.
- If A is a scalar matrix.

These are the only cases for $n = 2$, so every 2-by-2 matrix has this property. For a 3-by-3 matrix there is one other case: The characteristic polynomial of A is $(X - a)^2(X - b)$ and the minimum polynomial is $(X - a)(X - b)$ with

$a, b \in F$ integral over R (possibly $a = b$). In [1, Example 2.3] William C. Brown considered the matrix

$$A = \begin{pmatrix} 0 & t^3 & t^4 \\ t^7 & 0 & t^6 \\ t^6 & t^4 & 0 \end{pmatrix}$$

with entries in the ring $R = k[[t^3, t^4]]$, where k is the two-element field. The minimum polynomial of A is $X^2 - t^5 X$, and its coefficients do not lie in R. The motivation for this paper was to understand Brown's example.

We saw that the question for 3-by-3 matrices reduces to the case where the minimum polynomial is quadratic. Let us say that R is **3-closed** if the minimum polynomial of every 3-by-3 matrix over R has its coefficients in R, and that R is **quadratically closed** if that is true for quadratic minimum polynomials of any square matrix over R. So if R is quadratically closed, then R is 3-closed.

2 Brown's example extended

Let D be a domain and M a submonoid of $t^{\mathbf{N}} = \{t^n : n \in \mathbf{N}\}$, where t is an indeterminate. Consider the monoid algebra $R = D[M]$. We can represent M by an infinite binary string, like $10111\ldots$ for the monoid generated by t^2 and t^3. Brown's example of the preceding section can be elaborated upon.

Theorem 1 *If the binary string of the monoid M contains the pattern* 11011*, then the monoid algebra $R = D[M]$ is not* 3*-closed.*

Proof. Suppose the pattern 11011 starts at position k (for that to happen, k must be at least 3). Consider the matrix

$$A = t^k \begin{pmatrix} 0 & 1 & t \\ t^4 & 0 & t^3 \\ t^3 & t & 0 \end{pmatrix} = t^k B.$$

Note that $B^2 = t^2 B + 2t^4 I$, so

$$A^2 = t^{2k} B^2 = t^{2k+2} B + 2t^{2k+4} I = t^{k+2} A + 2t^{2k+4} I.$$

Thus the minimum polynomial of A is $X^2 - t^{k+2} X - 2t^{2k+4}$, and $t^{k+2} \notin R$. ■

Similarly, replacing t by t^n in B, we see that any string containing k, $k+n$, $k+3n$, $k+4n$ and missing $k+2n$, is not 3-closed. So the pattern

$$1\cdot\cdot 1\cdot\cdot 0\cdot\cdot 1\cdot\cdot 1$$

is excluded.

For the purpose of deciding whether or not $D[M]$ is 3-closed, we might as well assume that the string for M ends in all 1's. Otherwise, $M=(M')^n$, where $n>1$, and $D[M]\cong D[M']$.

2.1 The ring $\mathbf{Z}[\sqrt[3]{16}]$

If we look at Brown's example from another angle, we can construct a number ring that is not 3-closed. Instead of thinking of $k[t^3,t^4]$, we can think of $k[x,y]/(x^4-y^3)$. In the latter ring we can identify t as $y/x=t^4/t^3$. So we look for a number ring with elements x and y so that $x^4=y^3$.

Let $\alpha=\sqrt[3]{16}$, so $\alpha^3=2^4$, whence $(\alpha/2)^3=2$. Let $t=\alpha/2$, so $t^3=2$ and $t^4=\alpha$ are in $\mathbf{Z}[\alpha]$, and $t^5=\alpha^2/2$ is not in $\mathbf{Z}[\alpha]$. Our matrix is

$$\begin{pmatrix} 0 & 2 & \alpha \\ 2\alpha & 0 & 4 \\ 4 & \alpha & 0 \end{pmatrix}$$

which satisfies the quadratic polynomial $X^2-t^5X-2t^{10}$. That is, $X^2-(\alpha^2/2)X-8\alpha$.

This summarizes what we know of the negative side of the problem. We now turn to the positive side.

3 Weakly integrally closed domains

Let $R\subset R'$ be integral domains. We say that an element $x\in R'$ is **strongly integral** over R if there is a nonzero finitely generated ideal J of R such that $xJ\subset J^2$. In particular, if x is strongly integral over R, then x is in the quotient field of R, and x is integral over R. We say that R is **weakly integrally closed** in R' if each element of R' that is strongly integral over R is in R. By the preceding observation, if R is an integrally closed domain, then R is weakly integrally closed in R'.

Theorem 2

1. *(Intersections) If each member in a family of subrings R_i of R' is weakly integrally closed in R', then $\bigcap_i R_i$ is weakly integrally closed in R'.*

2. *(Transitivity) If R is weakly integrally closed in R', and R' is weakly integrally closed in R'', then R is weakly integrally closed in R''.*

3. *(Weakness) If R is integrally closed in R', then R is weakly integrally closed in R'.*

4. *(Localization)*

   (a) *If R is weakly integrally closed in R', then R_S is weakly integrally closed in R'_S for any multiplicatively closed subset of R not containing zero.*

   (b) *If R_M is weakly integrally closed in R'_M for every maximal ideal M of R, then R is weakly integrally closed in R'.*

Proof. Suppose J is a nonzero finitely generated ideal of $\bigcap_i R_i$, and $xJ \subset J^2$ for some $x \in R'$. Let $J_i = R_i J$, a nonzero finitely generated ideal of R_i. Then $xJ_i \subset J_i^2$ for each i, so $x \in \bigcap_i R_i$. This proves (1).

To show (2), let J be a nonzero finitely generated ideal of R such that $xJ \subset J^2$ for $x \in R''$. Then $J' = R'J$ is a nonzero finitely generated ideal of R' such that $xJ' \subset J'^2$, so $x \in R'$, so $x \in R$.

Let J be a nonzero finitely generated ideal of R and $xJ \subset J^2$ for some $x \in R'$. Then $xJ \subset J$, so x is integral over R, whence $x \in R$. This proves (3).

To show (4a), suppose R is weakly integrally closed in R'. Let $x \in R'_S$ and J be a nonzero finitely generated ideal of R_S. There is $s \in S$ such that sJ is a finitely generated ideal of R, and $sx \in R'$. If $xJ \subset J^2$, then $sxsJ \subset (sJ)^2$, so $sx \in R$ whence $x \in R_S$. Therefore R_S is weakly integrally closed in R'_S.

For (4b), suppose R_M is weakly integrally closed in R'_M for every maximal ideal M of R. Let $x \in R'$ and J be a nonzero finitely generated ideal of R such that $xJ \subset J^2$. Let $C = \{r \in R : rx \in R\}$. If M is any maximal ideal of R, then $x \in R_M$, so $rx \in R$ for some $r \in R \setminus M$. Therefore C is not contained in M for any maximal ideal M, so $C = R$, whence $x \in R$. ∎

Because of (1) we can talk about the **weak integral closure** of R in R'. If R is weakly integrally closed in its quotient field, then we say simply that

R is **weakly integrally closed**. We shall see later that the weak integral closure need not coincide with the set of strongly integral elements (Example 1).

Theorem 3 *If R is weakly integrally closed, then R is quadratically closed. That is, if A is a nonzero square matrix with entries in R that satisfies a monic quadratic polynomial with coefficients in the quotient field of R, then those coefficients are in R.*

Proof. Suppose

$$A^2 + xA + yI = 0$$

and we want to show that $x \in R$, and so $y \in R$. The matrix $A - rI$, with $r \in R$, satisfies the polynomial equation

$$X^2 + (x + 2r)X + y + r^2 + xr = 0$$

so we may assume that some diagonal entry of A is zero. Let J be the ideal generated by the entries of A. Note that $y \in J^2$. Then $xJ \subset J^2$, so $x \in R$. ■

So $D[t^3, t^4]$ is not weakly integrally closed, because it is not 3-closed, being a monoid algebra with string $100110111\ldots$.

What are some examples of rings that are weakly integrally closed but not integrally closed? For monoid algebras, the following is our best positive result. First a lemma.

Lemma 4 *If D is an integrally closed domain with quotient field K, and M is a submonoid of $t^{\mathbf{N}}$, then $D[M]$ is weakly integrally closed if and only if $K[M]$ is.*

Proof. We have $D[M] = K[M] \cap D[t]$, and $D[t]$ is integrally closed. Thus if $K[M]$ is weakly integrally closed, so is $D[M]$. Conversely, $K[M]$ is a localization of $D[M]$. ■

Theorem 5 *Let D be an integrally closed domain and t an indeterminate. Then the domain*

$$R_{i,m} = D[t^i, t^m, t^{m+1}, t^{m+2}, \ldots]$$

is weakly integrally closed for each pair of natural numbers i and m.

Proof. From Lemma 4, we may assume that D is a field.

The theorem is clearly true for $i = 1$ or $m = 1$. Fix $i > 1$, write R_m for $R_{i,m}$, and proceed by induction on m. Note that

$$D[t] = R_1 \supset R_2 \supset R_3 \supset \cdots$$

Note that R_1 is integrally closed in the common quotient field $D(t)$. We will show that R_m is weakly integrally closed in R_{m-1}, for $m > 1$. Let x be an element of R_{m-1} and J a finitely generated nonzero ideal of R_m such that $xJ \subset J^2$. If $m \equiv 1 \bmod i$, then $R_m = R_{m-1}$, so we may assume that $m \not\equiv 1 \bmod i$.

For any element $e \in D[t]$ let $e = e_0 + e_1 t + e_2 t^2 + \cdots$ and define the value $v_t(e)$ to be the least j such that $e_j \neq 0$. Let n be the minimum of $v_t(e)$, as e ranges over J. As $v_t(xe) \geq 2n$ for each $e \in J$, it follows that $v_t(x) \geq n$.

We pause for an observation. Let $a \in R_{m-1}$ and $b \in R_m$. Suppose $v_t(a) \geq n$ and $v_t(b) \geq n$, where $n < m$, and i divides n. Then

$$(ab)_{m-1+n} = a_{m-1}b_n + \sum_{0<j\leq m-1-n} a_{m-1-j}b_{n+j}.$$

As i does not divide $m-1+n$, either i does not divide $m-1-j < m-1$, or i does not divide $n+j < m-1$. Thus $a_{m-1-j}b_{n+j} = 0$ for each j, so

$$(ab)_{m-1+n} = a_{m-1}b_n.$$

Now suppose $n < m$, so n is a multiple of i. Recall that $v_t(x) \geq n$. If e and e' are in J, then $(ee')_{m-1+n} = e_{m-1}e'_n = 0$, by the observation above. So if f is any element of J^2, then $f_{m-1+n} = 0$. Therefore, if $e \in J$, then $(xe)_{m-1+n} = 0$.

If $e \in J$ has value n, then $0 = (xe)_{m-1+n} = x_{m-1}e_n$ from the observation again. So $x_{m-1} = 0$, whence $x \in R_m$. ■

We can get a simple characterization of when $D[t^m, t^n]$ is weakly integrally closed (or 3-closed) when D is integrally closed.

Theorem 6 *Let $R = D[t^m, t^n]$ with D a domain and $1 \leq m < n$. If m divides $2n$, and D is integrally closed, then R is weakly integrally closed. If m does not divide $2n$, then R is not 3-closed.*

Proof. As the hypothesis and the conclusion are invariant under dividing m and n by a common factor, we may assume that m and n are relatively prime. If m divides $2n$, then m divides 2, so $m = 1$ or $m = 2$. If $m = 1$, then $R = D[t]$ is integrally closed, hence weakly integrally closed. If $m = 2$, then $R = D[t^2, t^n]$ where n is odd. Then Theorem 5 shows that R is weakly integrally closed.

So suppose that $m \geq 3$ is relatively prime to $n > m$. We will show that R is not 3-closed by finding the pattern 11011 in the binary string of the monoid M generated by m and n (Theorem 1). Consider the m-by-n array

$$\begin{array}{|r|cccr|}
\hline
0 & 1 & 2 & \cdots & n-1 \\
n & & \cdots & \cdots & 2n-1 \\
2n & & \cdots & \cdots & 3n-1 \\
\vdots & & & & \vdots \\
(m-2)n & & \cdots & \cdots & (m-1)n-1 \\
\cline{2-5}
(m-1)n & & \cdots & \cdots & mn-1 \\
\hline
\end{array}$$

Each column contains an element divisible by m, because $(m, n) = 1$. So every integer from $(m-1)n$ on is in M, because we can subtract a multiple of n from it and get a multiple of m.

If $n = m + 1$, then the first column of the smaller array, and no other, consists entirely of numbers that are not divisible by m. As the first entry of the last column is equal to m, this means that $(m-2)n + 1$ is not in M, while the four integers closest to it are in M (this uses $m > 2$): the pattern 11011.

If $n > m + 1$, then there is a column of the smaller array, other than the first, which consists of numbers that are not divisible m. Pick the rightmost such column. Its last entry is the 0 of 11011. ∎

The weak integral closure does not always coincide with the set of strongly integral elements, which need not be closed under addition (but is closed under multiplication).

Example 1 *Let D be an integrally closed domain. The weak integral closure of $R = D[t^3, t^4]$ is $R' = D[t^3, t^4, t^5]$. The set of elements in R' that are strongly integral over R is $R \cup M'$, where $M' = \{f \in R' : f(0) = 0\}$.*

Proof. We see from Theorem 5 that R' is weakly integrally closed. Let $M = M' \cap R$. If $x \in M'$, then $xM \subset M^2$, while if $x \in R$, then $xR \subset R^2$, so the elements of $R \cup M'$ are strongly integral over R, and $R' = D \oplus M'$ is the weak integral closure of R.

Now suppose $x \in R'$ but not in $R \cup M'$, and J is a nonzero ideal of R. If J is not contained in M, then xJ is not contained in R. If $J \subset M$, let the smallest t-value of an element of J be $k \geq 3$. The smallest t-value of an element of J^2 is then $2k > k$, so xJ cannot be contained in J^2. ■

In particular, although 1 and t^5 are strongly integral over $D[t^3, t^4]$, the sum $1 + t^5$ is not.

3.1 Sufficient conditions

Recall that the **integral closure** of an ideal I in a ring R consists of those elements $x \in R$ that satisfy an equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$$

with $a_i \in I^i$. The intersection of integrally closed ideals is integrally closed. Maximal ideals are integrally closed. Any ideal in a Dedekind domain is integrally closed. If I is integrally closed in R, then I_S is integrally closed in R_S.

The **conductor** of R in R' is $\{x \in R : xR' \subset R\}$. It is the largest R'-ideal contained in R. Here is a general sufficient condition for R to be weakly integrally closed in R'.

Theorem 7 *Let $R \subset R'$ be integral domains, and C the conductor of R in R'. Suppose that*

- $C = \{r \in R : rx \in R\}$ *for each* $x \in R' \setminus R$,
- C *is integrally closed in* R'.

Then R is weakly integrally closed in R'.

Proof. Let J be a finitely generated nonzero ideal of R, and $x \in R'$ such that $xJ \subset J^2$. Suppose $x \notin R$. As $C = \{r \in R : rx \in R\}$, we have $J \subset C$. As $xJ \subset J^2$, the element x is integral over C, hence in $C \subset R$. ■

Note that the first condition in Theorem 7 holds if C is a maximal ideal in R.

Corollary 8 *If the conductor of R in R' is a finite intersection of maximal ideals of R, and is integrally closed in R', then R is weakly integrally closed in R'.*

Proof. By Theorem 2 it suffices to prove that R_M is weakly integrally closed in R'_M for each maximal ideal M of R. Let C be the conductor of R in R'. If M does not contain C, then $R'_M = R_M$ because $rR' \subset R$ for any $r \in C \setminus M$. If M contains C, then $CR_M = MR_M$ because C is a finite intersection of maximal ideals. Moreover, the maximal ideal CR_M is the conductor of R_M in R'_M. Finally, CR_M is integrally closed as an ideal of R'_M because C is integrally closed as an ideal of R'. The result now follows from Theorem 7. ■

Corollary 9 *If R' is a Dedekind domain, and the conductor of R in R' is a finite intersection of maximal ideals of R, then R is weakly integrally closed in R'.*

Corollary 9 covers the rings $K[t^m, t^{m+1}, t^{m+2}, \ldots]$, for K a field (and so for K integrally closed), and also the rings $\mathbf{Z}[\sqrt{d}]$ where d is square-free and congruent to 1 modulo 4, as we will see in the next section. We can use Theorem 7 to extend Theorem 5 via the next theorem.

Theorem 10 *Let M be a submonoid of $t^{\mathbf{N}}$ and t^m the smallest nonunit power of t in M. Let D be an integrally closed domain, $R = D[M \setminus \{t^m\}]$ and $R' = D[M]$. If R' is weakly integrally closed, then so is R.*

Proof. From Lemma 4 we may assume that D is a field. The conductor C of R in R' is the linear span of $M \setminus \{1, t^m\}$, a maximal ideal in R. Moreover, the obvious discrete valuation v on R' has the property that $C = \{r \in R' : vr > m\}$, so C is integrally closed in R'. The result now follows from Corollary 8. ■

Iterating this theorem gives yet another proof that $D[t^m, t^{m+1}, t^{m+2}, \ldots]$ is weakly integrally closed, if D is integrally closed, by deleting $t, t^2, \ldots, t^{m-1}$ successively from $D[t]$. We can also apply it to the other rings in Theorem 5 to show, for example, that $D[t^4, t^6, t^9, t^{11}]$ is weakly integrally closed (from $D[t^2, t^9]$ by deleting t^2).

4 Quadratic extensions of Dedekind domains

Let $1, \theta$ be an integral basis for the integers in a quadratic field $\mathbf{Q}(\theta)$. Any subring of the integers of $\mathbf{Q}(\theta)$ has the form $R = \mathbf{Z}[n\theta] = \mathbf{Z} \oplus n\mathbf{Z}\theta$. In particular, when d is square-free and congruent to 1 modulo 4, then $\mathbf{Z}[\sqrt{d}] = \mathbf{Z}[2\theta]$ where $\theta = (1+\sqrt{d})/2$. The conductor of R in its integral closure $\mathbf{Z}[\theta]$ is $C = n\mathbf{Z}[\theta]$, and R/C is isomorphic to $\mathbf{Z}/n\mathbf{Z}$. So the conductor is maximal in R if and only if n is a prime, in which case R is weakly integrally closed by Corollary 9.

What about $\mathbf{Z}[4i]$? It suffices to show that $\mathbf{Z}[4i]$ is weakly integrally closed in $\mathbf{Z}[2i]$. The conductor is the unique maximal ideal $M = 2\mathbf{Z} \oplus 4\mathbf{Z}i = 2\mathbf{Z}[2i]$ of $\mathbf{Z}[4i]$ that contains 2. Is M integrally closed in $\mathbf{Z}[2i]$? No because $(2i)^2 = -2^2 \in M^2$ shows that $2i$ is integral over M. So this is not covered by Theorem 7. Nevertheless, $\mathbf{Z}[4i]$ is weakly integrally closed, as we shall see.

If D is a subring of an integral domain R, we say that R is a **separable quadratic extension** of D if R is a subring of a separable quadratic extension field of the quotient field K of D. By the primitive element theorem, it suffices to require that each element of R satisfies a separable quadratic polynomial over K.

Theorem 11 *Any separable quadratic integral extension of a discrete valuation domain is weakly integrally closed.*

Proof. Let D be a discrete valuation domain with prime p, and R a separable quadratic integral extension of D. Let $1, \theta$ be an integral basis for the integral closure $D[\theta]$ of R (see [2, Corollary 2, page 265]). Then either $R = D[\theta]$, which is integrally closed, or $R = D[p^{k+1}\theta]$ for some nonnegative integer k. By induction and transitivity, it suffices to show that $R = D[p^{k+1}\theta]$ is weakly integrally closed in $R' = D[p^k\theta]$.

Note that $D[\theta]$ is a principal ideal domain with at most two primes. Note also that R is a local ring, and its unique maximal ideal, pR', is the conductor of R in R'.

Write $\theta^2 = s + t\theta$ where $s, t \in D$. Suppose J is a proper finitely generated ideal of R. For $x \in D$, let vx denote the value of x. We can generate J as a D-module by $a + b\theta$ and c, where $va \geq 1$ and $vb \geq k+1$. We may assume that J is not principal, so $c \neq 0$. If $vc \leq va$, then we may, and do, take $a = 0$. Note that J^2 is generated by $a^2 + sb^2 + (2ab + tb^2)\theta$, and c^2 and $ca + cb\theta$.

Suppose $(r + p^k\theta)J \subset J^2$. Then $a + b\theta \in J$ and

$$(r + p^k\theta)(a + b\theta) = ra + sp^kb + (rb + p^ka + tp^kb)\theta \in J^2.$$

If $va \le k$, so $vc > va$, then $J^2 \subset p^{2va}D \oplus p^{va+k+1}D\theta$, so

$$v(ra + sp^k b) \ge 2va \quad \text{and} \quad v(rb + p^k a + tp^k b) \ge va + k + 1.$$

The second inequality gives $vr + vb = va + k$ so $vr < va$. So the first gives $vr + va = vs + k + vb \ge 2k + 1$, so $va > k$. What if $a = 0$? Want $vc > k$. If $vc \le k$, then $J = p^{vc}D \oplus p^{vb}D\theta$ and $J^2 = p^{2vc}D \oplus p^{vc+vb}D\theta$. Consider $(r + p^k\theta)c \in J^2$. So $k + vc \ge vc + vb$, a contradiction.

So we may assume that $va > k$ and $vc > k$. Note that p^{k+1} does not divide $r + p^k\theta$. Let π be the prime in $D[\theta]$ that divides $r + p^k\theta$ the least. Let n be the π-value of p. Then the π-value of $r + p^k\theta$ is less than $n(k+1)$. As $va > k$ and $vc > k$, we have $J \subset \pi^m D[\theta] \setminus \pi^{m+1} D[\theta]$ for some $m \ge n(k+1)$. So $J^2 \subset \pi^{2m} D[\theta]$ but $(r + p^k\theta)J$ is not contained in $\pi^{2m} D[\theta]$. ■

We can eliminate the requirement that the extension be integral in Theorem 11.

Theorem 12 *Any separable quadratic extension of a discrete valuation domain is either integral, a field, or a principal ideal domain.*

Proof. Suppose D is a discrete valuation domain with prime p and quotient field K, and R is a separable quadratic extension of D. Suppose $D[\theta] = D \oplus D\theta$ is integrally closed, where $\theta \in KR = K[\theta]$, the quotient field of R, and $\theta^2 = s\theta + t$ with $s, t \in D$. Let λ be an element of $R \subset K[\theta]$. We will show that if λ is not in $D[\theta]$, then $D[\lambda]$ contains either K or θ. Thus either $R \subset D[\theta]$, in which case R is integral over D; or R contains K, in which case $R = KR$ is a field; or R contains θ, in which case R is between $D[\theta]$, a principal ideal domain, and its quotient field $K[\theta]$, so is a principal ideal domain.

Multiplying λ by an appropriate power of p we may assume that

$$\lambda = \frac{a + b\theta}{p}$$

where p does not divide both a and b. If p does not divide b, then $\theta = (p\lambda - a)/b \in D[\lambda]$, so we may assume that p divides b and that p does not divide a. Adding an element of D to a multiple of λ, we may assume that $a = 1$. Then

$$p\lambda^2 = \frac{1 + b^2 t + (2b + b^2 s)\theta}{p}$$

so

$$p\lambda^2 - (2 + bs)\lambda = \frac{1 + b^2t - (2 + bs)}{p} = \frac{-1 + b(bt - s)}{p}$$

whence $1/p \in D[\lambda]$, so $D[\lambda]$ contains K. ■

Having done the discrete valuation domain case, we proceed to the global case.

Theorem 13 *Let R' be a separable quadratic extension of a domain R. Let $\mathcal{F}$ be a family of prime ideals of R such that R_P is a discrete valuation domain for each $P \in \mathcal{F}$. If $R' = \bigcap_{P \in \mathcal{F}} R'_P$, then R' is weakly integrally closed.*

Proof. It suffices to show that R'_P is weakly integrally closed for each P in $\mathcal{F}$. But R'_P is a separable quadratic extension of the discrete valuation domain R_P. ■

Corollary 14 *Let R' be a separable quadratic extension of a Krull domain R. If R is a Dedekind domain, or if R' is a free R-module, then R' is weakly integrally closed.*

Proof. Let $\mathcal{F}$ be the set of height-one prime ideals of R. In both cases, localization at an ideal in $\mathcal{F}$ gives a discrete valuation domain. When R is Dedekind, these ideals are maximal, so R' is equal to the indicated intersection. When R' is a free R-module, this equality follows from the fact that $R = \bigcap_{P \in \mathcal{F}} R_P$. ■

Corollary 15 *If R is a Dedekind domain with quotient field K, and L is a separable quadratic extension of K, then any ring between R and L is weakly integrally closed. In particular, any ring of quadratic algebraic numbers is weakly integrally closed.*

Recall from Section 2.1 that the ring $\mathbf{Z}[\sqrt[3]{16}]$, which is a free cubic extension of $\mathbf{Z}$, is not weakly integrally closed.

The conductor, $M = pR'$, of R in R' need not be integrally closed for rings of quadratic integers. For $\mathbf{Z}[pi] \subset \mathbf{Z}[i]$, the conductor is maximal and integrally closed. For $\mathbf{Z}[p^2i] \subset \mathbf{Z}[pi]$, the conductor is maximal but not integrally closed as $(pi)^2 = p^2 \in M^2$. So Theorem 7 does not apply. For $\mathbf{Z}[p^2i] \subset \mathbf{Z}[i]$ the conductor $p^2\mathbf{Z}[i]$ is integrally closed but not maximal.

As a nonintegral example of Corollary 15, consider the ring $\mathbf{Z}[(3+4i)/5]$. If we localize at a prime other than 2 or 5 we get $\mathbf{Z}_{(p)}[i]$, which is integrally closed. Localizing at 2 gives $\mathbf{Z}_{(2)}[4i]$, which is weakly integrally closed. If we localize at 5 we get $\mathbf{Z}[i]$ localized at the prime ideal $(2+i)$, a discrete valuation domain, so integrally closed.

5 Inheritance by polynomial rings

We address the question as to whether $R[X]$ is weakly integrally closed whenever R is. First we observe that all of our constructions of weakly integrally closed domains R have the property that $R[X]$ is also weakly integrally closed.

If R is a separable quadratic integral extension of a discrete valuation domain D, then R is free over D, so $R[X]$ is a free separable quadratic extension of the Krull domain $D[X]$. So $R[X]$ is weakly integrally closed. This also works for other separable quadratic extensions R of discrete valuation domains as these are either fields or principal ideal domains. If D is a Dedekind domain, then $R_P[X]$ is weakly integrally closed for each maximal ideal P of D, so $R[X]$ is weakly integrally closed.

As for the monoid rings, if $R = D[M]$ where D is integrally closed, then $R[X] = D[X][M]$ and $D[X]$ is integrally closed.

Here are some general facts about being strongly integral over $R[X]$ which fall short of showing that $R[X]$ is weakly integrally closed whenever R is.

Lemma 16 *Let $R \subset R'$ be integral domains. If f in $R'[X]$ is strongly integral over $R[X]$, then $f(r)$ is strongly integral over R for each element r in R.*

Proof. Let J be a nonzero finitely generated ideal of $R[X]$ and $f \in R'[X]$ such that $fJ \subset J^2$. For r in R, let J_r be the image of J in R under evaluation at r. Then J_r is a finitely generated ideal of R. Write $J = (X-r)^m I$ where I is finitely generated, $m \geq 0$, and $I_r \neq 0$. Then $f \cdot (X-r)^m I \subset (X-r)^{2m} I^2$, so $fI \subset I^2$. As $f(r)I_r \subset I_r^2$, it follows that $f(r)$ is strongly integral over R. ■

Theorem 17 *Let R be a domain that contains an infinite set U, all of whose nonzero differences are units. If R is weakly integrally closed in R', then $R[X]$ is weakly integrally closed in $R'[X]$.*

Proof. If $f \in R'[X]$ is strongly integral over $R[X]$, then it follows from the lemma that $f(u) \in R$ for each u in U. Using $1 + \deg f$ of these elements, a Vandermonde determinant argument shows that $f \in R[X]$. ■

Corollary 18 *If R is a weakly integrally closed domain that contains an infinite field, then $R[X]$ is weakly integrally closed.*

The ring R need not contain an infinite field to satisfy the condition of Theorem 17. Consider the subring R of the field $\mathbf{Q}(X)$ consisting of quotients a/b of polynomials in $\mathbf{Z}[X]$ where b is primitive. We can let U be the set $\{1, 1+X, 1+X+X^2, \dots\}$. Of course R is integrally closed, being a unique factorization domain, but the ring $R[2i]$ is not, and $R[2i]$ is weakly integrally closed because it is a free quadratic extension of the Krull domain R.

Theorem 19 *If R is a Noetherian domain, and f is strongly integral over $R[X]$, then the leading coefficient of f is strongly integral over R.*

Proof. Suppose $fJ \subset J^2$ for J a nonzero finitely generated ideal of $R[X]$. Let I be the ideal in R of leading coefficients of J. As R is Noetherian, I is finitely generated. The ideal of leading coefficients of J^2 is I^2. So if a is the leading coefficient of f, then $aI \subset I^2$, whence a is strongly integral over R. ■

Corollary 20 *If R is a Noetherian weakly integrally closed domain, and f is strongly integral over $R[X]$ with $\deg f \leq 2$, then $f \in R[X]$.*

Proof. Let $f = a_0 + a_1X + a_2X^2$. Then $a_2 \in R$ as is $f(0) = a_0$ and $f(1) = a_0 + a_1 + a_2$. ■

6 Open problems

- Characterize the weakly integrally closed algebras $k[M]$ where M is a submonoid of the free monoid on one generator, and k is a field. At least do this for k the two-element field. Is $k[t^4, t^5, t^{11}]$ weakly integrally closed? Is it enough to exclude the patterns of Theorem 1 and the remark following?
- Are weakly integrally closed domains 4-closed? Is $k[t^2, t^3]$ 4-closed?
- Same question for cubically closed (which is possibly stronger).
- Are there 3-closed domains that are not weakly integrally closed?

- When is the property of being weakly integrally closed inherited? For example, are polynomial rings over weakly integrally closed domains weakly integrally closed? It is not inherited by integral quadratic extensions:

  $$k[t^3, t^6, t^7, t^8, \dots] \subset k[t^3, t^4].$$

  What about integral quadratic extensions of integrally closed domains? Of (nondiscrete) valuation domains?

- What can be said about completions? (Brown's example was $k[[t^3, t^4]]$).

- Does the weak integral closure of a domain consist of sums of strongly integral elements?

References

[1] Brown, William C., Null ideals and spanning ranks of matrices, *Communications in Algebra* **26** (1998), 2401–2417

[2] Zariski, Oscar and Pierre Samuel, *Commutative algebra*, Van Nostrand 1958.