Trace-class operators

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Abstract

In this paper we give a direct definition of the von Neumann-Schatten classes $\mathcal{C}_p$ in terms of the existence of a certain supremum. The theory is developed without appeal to separability, or to the existence of an orthonormal basis, and without using countable choice or the law of excluded middle. We construct the singular values of compact operators, and characterize compact operators, and the classes $\mathcal{C}_p$, in terms of their singular values.

1 Introduction

A constructive development of the theory of trace-class and Hilbert–Schmidt operators was begun in [3]. In that paper it was required that the operators have adjoints (which is not automatic in constructive mathematics), and that the Hilbert space be separable. Extensive use was made of the countable axiom of choice, both directly and through appeal to theorems whose published proofs depend on its validity.

In this paper we define trace-class and Hilbert-Schmidt operators—the von Neumann-Schatten classes $\mathcal{C}_1$ and $\mathcal{C}_2$—without assuming the existence

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of an adjoint or even an absolute value. In fact, an operator is in the class $\mathcal{C}_p$, for $p \in [1, \infty)$, exactly when a certain supremum exists. We prove that such operators are compact, hence have adjoints. The theory is developed without appeal to separability, or to the existence of an orthonormal basis, and without using countable choice. We construct the singular values of compact operators, and characterize compact operators, and the classes $\mathcal{C}_p$, in terms of their singular values.

By **constructive mathematics** we mean mathematics in the context of intuitionistic logic. One advantage of using intuitionistic logic is that results can be interpreted both traditionally and computationally, either informally or in computational models such as recursive analysis [7] and Weihrauch's TTE [10]. They can also be interpreted in various topos models, including sheaf models if countable choice is avoided. For background information on constructive mathematics, see [1], [2], and [4].

Another advantage of using intuitionistic logic, and rejecting the countable axiom of choice, is an attribute of any good generalization: it forces you to look at things correctly and see their essence. The use of global bases to analyze an operator masks the fact that its action can be studied with respect to finite-dimensional subspaces without distinguishing, and having to construct, a special aligned family of them. As it is no longer *possible* to define something in terms of a basis, and then show that it is independent of the particular basis chosen, you are forced to employ a basis-free definition, which is probably what you should have done anyway. And of course you might have done that, even working with classical logic and countable choice, but you didn't have to.

2 Norms and singular values

A **k-frame** e in a Hilbert space H is a family $e_1, \ldots, e_k$ of orthonormal vectors. Denote the set of all k-frames by $F_k(H)$. There are various norms on a bounded operator A from H to a Hilbert space K that can be defined in terms of k-frames. For $1 \leq p < \infty$ denote the ℓ_p-norm of an k-tuple of complex numbers by

$$\|(c_1, \ldots, c_k)\|_p = \sqrt[p]{|c_1|^p + \cdots + |c_k|^p}$$

For fixed $e \in F_k(H)$ and $f \in F_k(K)$, the expression $\|(\langle Ae_1, f_1\rangle, \ldots, \langle Ae_k, f_k\rangle)\|_p$ defines a seminorm on linear operators from H to K, so

$$\|A\|_p = \sup_{\substack{e\in F_k(H)\\ f\in F_k(K)\\ k\in\omega}} \|(\langle Ae_1, f_1\rangle, \ldots, \langle Ae_k, f_k\rangle)\|_p$$

defines a norm, at least if we allow $\|A\|_p = \infty$. Constructively the situation is more complicated because merely bounding the terms in the indicated supremum does not guarantee that the supremum exists. We will use the notation $\|A\|_p$ even if the indicated supremum does not exist. That is, we treat $\|A\|_p$ as a generalized real number in the sense of [5] and [8]. For example, it makes sense to write $\|A\| \leq r$ or to write $r \leq \|A\|$ for any real number r.

If the supremum does exist, we say that $\|A\|_p$ exists or $\|A\|_p$ is **located**. It is easy to see that the defining properties of a norm hold in any event:

- $\|A + B\|_p \leq \|A\|_p + \|B\|_p$,
- $\|cA\|_p = |c|\,\|A\|_p$ if $\|A\|_p$ is bounded or $c \neq 0$.

The **operator norm** is

$$\|A\| = \sup_{\substack{e\in F_1(H)\\ f\in F_1(K)}} |\langle Ae_1, f_1\rangle|$$

and can be thought of as $\|A\|_\infty$ but we will have no occasion to do that. The **trace-class norm** is $\|A\|_1$ and the **Hilbert-Schmidt norm** is $\|A\|_2$. If $\|A\|_1$ exists, then we say that A is of **trace class**. If $\|A\|_2$ exists, then we say that A is a **Hilbert-Schmidt operator**. In general, if $\|A\|_p$ exists, we say that A is an operator in the von Neumann-Schatten class $\mathcal{C}_p$ [9].

We can refine the norm $\|A\|_p$ by considering the norms

$$\|A\|_p^{(n)} = \sup_{\substack{e\in F_k(H)\\ f\in F_k(K)\\ k\leq n}} \|(\langle Ae_1, f_1\rangle, \ldots, \langle Ae_k, f_k\rangle)\|_p$$

for each positive integer n. So

$$\|A\|_p = \sup_{n\in\omega} \|A\|_p^{(n)}$$

A priori, $\|A\|_p$ can be located without any $\|A\|_p^{(n)}$ being located. It turns out, however, that if $\|A\|_p$ exists (is located), then each $\|A\|_p^{(n)}$ exists.

Clearly

$$\|A\| = \|A\|_p^{(1)} \leq \|A\|_p^{(2)} \leq \cdots \leq \|A\|_p$$

and $\|A\|_p^{(n)} = \|A\|_p$ if either H or K has finite dimension at most n. Notice that all these norms are defined in terms of $\langle Ax, y\rangle$, so everything could be phrased in terms of a bilinear form on $H \times K$, rather than an operator from H to K. This would put the domain and the codomain on more equal footing.

We say that A **has singular values** if $\|A\|_1^{(n)}$ exists for each n. The **singular values** of A are then defined by $\sigma_n = \|A\|_1^{(n)} - \|A\|_1^{(n-1)}$, so

$$\|A\|_1^{(n)} = \sigma_1 + \sigma_2 + \cdots + \sigma_n$$

It is not difficult to see, from the nature of the suprema, that $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq 0$. The $\|A\|_1^{(n)}$ are all continuous in the operator norm as $\|A\|_1^{(n)} \leq n\|A\|$, so the singular values are continuous in the operator norm.

The set of operators with singular values is closed in the operator norm. In fact, if $\|A\|_1^{(n)}$ exists and $\|A - B\| < \varepsilon$, then

$$\|A\|_1^{(n)} - n\varepsilon < \|B\|_1^{(n)} < \|A\|_1^{(n)} + n\varepsilon$$

Here we have a graphic instance of showing that a norm exists by locating it.

Each of these norms is unitarily invariant: if U and V are unitary, then A and UAV have the same norm.

3 Singular value decomposition

An operator between finite-dimensional spaces can be represented by a matrix. The singular values of a positive diagonal matrix are the diagonal entries, at least within an arbitrary ε. This last proviso is necessary because we need not be able to arrange the diagonal entries in a linear order.

Lemma 1 *Let A be an m-by-n matrix, and $t = \min(m, n)$. If A is a diagonal matrix, with entries $d_1 \geq d_2 \geq \cdots \geq d_t \geq 0$, then*

$$\left(\|A\|_p^{(k)}\right)^p = \sum_{i=1}^{k} d_i^p$$

for $k = 1, \ldots, t$.

Proof. Clearly $\left(\|A\|_p^{(k)}\right)^p \geq \sum_{i=1}^k d_i^p$. Let e and f be k-frames in the domain and codomain of the operator A. Denote the coordinates of a vector v by v_j. Then $(Ae_i)_j = d_j e_{ij}$ so

$$\langle Ae_i, f_i \rangle = \sum_{j=1}^t d_j e_{ij} \bar{f}_{ij}$$

and, if $1/p + 1/q = 1$, then

$$|\langle Ae_i, f_i \rangle| \leq \sum_{j=1}^t d_j \left|e_{ij} f_{ij}\right|^{1/p} \left|e_{ij} f_{ij}\right|^{1/q}$$

Hölder's inequality gives

$$|\langle Ae_i, f_i \rangle| \leq \left(\sum_{j=1}^t d_j^p \left|e_{ij} f_{ij}\right|\right)^{1/p} \left(\sum_{j=1}^t \left|e_{ij} f_{ij}\right|\right)^{1/q}$$

The second factor on the right is at most 1 because e_i and f_i are unit vectors, so $\sum_{j=1}^n \left|e_{ij}\right|^2 = 1$ and $\sum_{j=1}^m \left|f_{ij}\right|^2 = 1$. Thus

$$\sum_{i=1}^k |\langle Ae_i, f_i \rangle|^p \leq \sum_{i=1}^k \sum_{j=1}^t d_j^p \left|e_{ij} f_{ij}\right|$$

Now $\sum_{i=1}^k \left|e_{ij}\right|^2 \leq 1$ and $\sum_{i=1}^k \left|f_{ij}\right|^2 \leq 1$ because e and f are k-frames. So $\sum_{i=1}^k \left|e_{ij} f_{ij}\right| \leq 1$ for each j, and $\sum_{i=1}^k \sum_{j=1}^t \left|e_{ij} f_{ij}\right| \leq k$, from which the desired conclusion follows easily. ■

The classical singular-value decomposition theorem for finite-dimensional Hilbert spaces says that if A is a matrix, then we can construct unitary matrices U and V so that UAV is a diagonal matrix with nonnegative real entries. The diagonal entries of UAV are the singular values of A (Lemma 1). We can't quite do this constructively, but we can construct unitary matrices U and V so that UAV is arbitrarily close to such a diagonal matrix.

Lemma 2 *Let A be an operator and let e and f be k-frames, $k \leq n$, such that*

$$\left(\|A\|_p^{(n)}\right)^p < \sum_{i=1}^{k} |\langle Ae_i, f_i\rangle|^p + \delta^p.$$

If u is a unit vector orthogonal to the span of e, then

$$|\langle Au, f_i\rangle|^2 < \delta(2\,|\langle Ae_i, f_i\rangle| + \delta)$$

for each i. If v is a unit vector orthogonal to the span of f, then

$$|\langle Ae_i, v\rangle|^2 < \delta(2\,|\langle Ae_i, f_i\rangle| + \delta)$$

for each i.

Proof. We shall prove the first inequality. We may assume that $i = 1$. If $|\langle Au, f_1\rangle|$ is sufficiently small, then we are done. If $|\langle Ae_1, f_1\rangle|$ is sufficiently small, then we are done by induction on k. So we may assume that $\langle Ae_1, f_1\rangle$ and $\langle Au, f_1\rangle$ are positive. The maximum value of $(\cos t)\langle Ae_1, f_1\rangle + (\sin t)\langle Au, f_1\rangle$ is

$$b = \sqrt{\langle Ae_1, f_1\rangle^2 + \langle Au, f_1\rangle^2}$$

so

$$b^p + \sum_{i=2}^{k} |\langle Ae_i, f_i\rangle|^p \leq \left(\|A\|_p^{(n)}\right)^p < \sum_{i=1}^{k} |\langle Ae_i, f_i\rangle|^p + \delta^p$$

because we can replace e_1 by $(\cos t)e_1 + (\sin t)u$. Therefore $b^p < \langle Ae_1, f_1\rangle^p + \delta^p$ so

$$\sqrt{\langle Ae_1, f_1\rangle^2 + \langle Au, f_1\rangle^2} = b < \langle Ae_1, f_1\rangle + \delta$$

The first inequality in the conclusion follows from squaring both sides of this inequality.

If we think of this as a result about bilinear forms, the second inequality is a consequence of the first. Otherwise, we can simply repeat the calculation with $\langle Ae_1, f_1\rangle$ and $\langle Ae_1, u\rangle$. ■

Theorem 3 (singular-value decomposition) *Let $A : H \to K$ be an operator between Hilbert spaces H and K of dimensions m and n. Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)}$ be the singular values of A. For each $\varepsilon > 0$ there exist orthonormal bases (e_i) of H and (f_j) of K such that the matrix $\langle Ae_i, f_j\rangle$ is within ε of the m-by-n diagonal matrix with diagonal entries $\sigma_1, \sigma_2, \ldots \sigma_{\min(m,n)}$. Hence $\|Ae_i - \sigma_i f_i\| < \varepsilon\sqrt{n}$ for each $i \leq \min(m,n)$.*

It follows that $\left(\|A\|_p^{(k)}\right)^p = \sigma_1^p + \sigma_2^p + \cdots + \sigma_k^p$ for each $k \leq \min(m,n)$.

Proof. It suffices to arrange for $\langle Ae_i, f_j\rangle$ to be within ε of some positive diagonal matrix because such a matrix can be perturbed slightly so that the diagonal entries are distinct, and then multiplied by a permutation matrix so that the diagonal entries decrease. Lemma 1 says that the entries in that diagonal matrix will be approximately the singular values of A.

Proceed by induction on m, which we may assume is at least 1. Choose a unit vector e_1 in H so that $\|Ae_1\| > \|A\| - \delta$, where δ is to be determined. If $\|Ae_1\| < \varepsilon - \delta$, then $\|A\| < \varepsilon$, so the matrix $\langle Ae_i, f_j\rangle$ is within ε of the zero matrix for any orthonormal bases (e_i) and (f_j). So we may assume that $Ae_1 \neq 0$, hence $Ae_1 = \|Ae_1\| f_1$ for some unit vector f_1 in K. From the case $n = 1$ and $p = 1$ of Lemma 2 it follows that $|\langle Au, f_1\rangle|^2 < \delta(2\|Ae_1\| + \delta)$ for each unit vector u orthogonal to e_1.

Let π be projection on the span of e_1, and λ projection on the span of f_1. By induction we can extend e_1 and f_1 to orthonormal bases (e_i) and (f_i) so that the matrix $\langle Ae_i, f_j\rangle$, for $2 \leq i \leq m$ and $2 \leq j \leq n$, is within ε of a positive diagonal matrix. But $\langle Ae_1, f_j\rangle = 0$ for $j \neq 1$, and $|\langle Ae_i, f_1\rangle|^2 < \delta(2\|Ae_i\| + \delta)$ if $i \neq 1$, so for sufficiently small δ, the m-by-n matrix $\langle Ae_i, f_j\rangle$ is within ε of a positive diagonal matrix.

The last claim follows from the first and Lemma 1 ■

Corollary 4 (Courant-Fischer) *Let $A : H \to K$ be an operator between Hilbert spaces H and K of dimensions m and n. Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)}$ be the singular values of A. Let A_V denote the operator A restricted to the subspace V. Then*

$$\sigma_k = \inf_{\dim V = m-k+1} \|A_V\|$$

Proof. For $\varepsilon > 0$, let (e_i) and (f_j) be as in Theorem 3. If V is the span of $e_k, \ldots, e_m$, then $\|A_V\| \leq \sigma_k + \varepsilon\sqrt{mn}$. It suffices to show that $\sigma_k \leq$

$\|A_V\| + \varepsilon\sqrt{mn}$ for any V of dimension $m - k + 1$. If $\dim V = m - k + 1$, then there are unit vectors u in V that are arbitrarily close to the span of $e_1, \ldots, e_k$. As $\|Au\| \geq \sigma_k - \varepsilon\sqrt{mn}$ for such u, we have $\sigma_k \leq \|A_V\| + \varepsilon\sqrt{mn}$. ■

The other half of the Courant-Fischer theorem says that we can write

$$\sigma_k = \sup_{\dim V = k} \inf_{e \in F_1(V)} \|Ae_1\|$$

This formula, which we will establish in Corollary 10, could be used to define σ_k directly, as a supremum, in the general case.

4 Finite-dimensional subspaces

The set of finite-dimensional subspaces of a Hilbert space plays a central role in our development. From a constructive point of view, this set is not a lattice, or even a directed set. So it will be important to construct a finite-dimensional subspace that *almost* contains two given finite-dimensional subspaces.

- Let U and V be subspaces of an inner product space, and $\varepsilon > 0$. We say that U is **ε-contained** in V, and write $U \subset_\varepsilon V$, if for each u in the unit ball of U, there is $v \in V$ such that $\|u - v\| < \varepsilon$.

Note that if V is the range of a projection, π_V, then $U \subset_\varepsilon V$ if and only if $\|u - \pi_V u\| < \varepsilon$ for each u in the unit ball of U.

Theorem 5 *Let U and V be finite-dimensional subspaces of an inner product space, and $\varepsilon > 0$. Then there is a finite-dimensional subspace W, of dimension at most $\dim U + \dim V$, such that $U \subset W$ and $V \subset_\varepsilon W$.*

Proof. The unit ball of V is totally bounded, so either $V \subset_\varepsilon U$, or there is e in the unit ball of V such that $\|(1 - \pi_U)e\| > 0$. In the former case we can take $W = U$, so we may assume that there is a unit vector e in V that is bounded away from U.

Let E be the span of e, and $U' = U + E$, which is a direct sum (not necessarily orthogonal), hence finite-dimensional. Then $U' = E \oplus (U' \cap E^\perp)$ and $V = E \oplus (V \cap E^\perp)$, so $\dim U' \cap E^\perp = \dim U$ and $\dim V \cap E^\perp = \dim V - 1$.

By induction on $\dim V$, there is a finite-dimensional subspace W' of $E^{\perp}$, containing $U' \cap E^{\perp}$ and ε-containing $V \cap E^{\perp}$, such that $\dim W' \leq \dim U + \dim V - 1$. Let $W = E \oplus W'$. ■

Next we need a technical lemma about orthogonalizing approximately orthonormal sets of vectors.

Lemma 6 *For each $\varepsilon > 0$ and $k \in \omega$, there exists $\delta > 0$ such that if $v_1, \ldots, v_k$ are vectors in the unit ball of a Hilbert space of dimension at least k, and $|\langle v_i, v_j \rangle| \leq \delta$ for $i \neq j$, then there is a k-frame f such that $\|v_i - \langle v_i, f_i \rangle f_i\| \leq \varepsilon$ for $i = 1, \ldots, k$.*

If, in addition, $\|v_i\|^2 \geq 1 - \delta$, then $\|v_i - f_i\| \leq \varepsilon$.

Proof. If $\sup_i \|v_i\| \leq \varepsilon$, then choose any k-frame f. So we may assume that $\|v_k\| \geq \varepsilon/2$. Write $v_k = \|v_k\| f_k$ for a unit vector f_k, and set $v_i' = v_i - \langle v_i, f_k \rangle f_k$ for $i = 1, \ldots, k-1$. Note that v_i' is in the unit ball of the orthogonal complement of f_k. Then

$$\|v_i' - v_i\| = |\langle v_i, f_k \rangle| \leq \frac{\delta}{\|v_k\|} \leq \frac{2\delta}{\varepsilon}$$

and

$$\left|\langle v_i', v_j' \rangle\right| = |\langle v_i, v_j \rangle - \langle v_i, f_k \rangle \langle f_k, v_j \rangle| \leq \delta + 4\frac{\delta^2}{\varepsilon^2}$$

for $i \neq j$. Choose $\delta \leq \varepsilon^2/4$ so $\|v_i' - v_i\| \leq \varepsilon/2$. By induction on k, if δ is small enough, there exist orthonormal vectors $f_1, \ldots, f_{k-1}$ in the orthogonal complement of f_k such that $\|v_i' - \langle v_i', f_i \rangle f_i\| \leq \varepsilon/2$, so

$$\begin{aligned} \|v_i - \langle v_i, f_i \rangle f_i\| &\leq \|v_i - \langle v_i', f_i \rangle f_i\| \\ &\leq \|v_i' - v_i\| + \|v_i' - \langle v_i', f_i \rangle f_i\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for $i = 1, \ldots, k-1$.

That proves the first part of the lemma. To verify the second part, note that we can arrange for $\langle v, f \rangle$ to be positive, if it is nonzero, by multiplying f by a suitable complex unit. Then

$$\|v - f\|^2 = \|v\|^2 + 1 - 2\langle v, f \rangle$$

and

$$\langle v, f \rangle = \sqrt{\|v\|^2 - \|v - \langle v, f \rangle f\|^2},$$

so if $\|v\|$ is close to 1 and $\|v - \langle v, f\rangle f\|$ is close to 0, then $\|v - f\|$ is close to 0. ■

We will use Lemma 6 in the following form.

Lemma 7 *For each $\varepsilon > 0$ and $k \in \omega$, there exists $\delta > 0$ such that if V and V' are subspaces of an inner product space with $V \subset_\delta V'$, and $e \in F_k(V)$, then there is $e' \in F_k(V')$ with $\|e_i - e'_i\| < \varepsilon$ for $i = 1, \ldots, k$.*

Proof. Choose $v'_1, \ldots, v'_k \in V'$ so that $\|v'_i - e_i\| < \delta$. Then

$$\langle v'_i, v'_j\rangle - \langle e_i, e_j\rangle = \langle v'_i - e_i, e_j\rangle + \langle v'_i - e_i, v'_j - e_j\rangle + \langle e_i, v'_j - e_j\rangle$$

so

$$\left|\langle v'_i, v'_j\rangle - \langle e_i, e_j\rangle\right| \le 2\delta + \delta^2$$

Now apply Lemma 6, for suitable $\delta < \varepsilon/2$, to find $e' \in F_k(V')$ so that $\|v'_i - e'_i\| < \varepsilon/2$. ■

Let $\pi : H \to H$ and $\lambda : K \to K$ be projections with images V and W. Denote by ${}_W A_V$ the operator from V to W that is the restriction of λA to V. This is not the same as, although obviously closely related to, $\lambda A \pi$, because the latter is an operator from H to K rather than from V to W.

Theorem 8 *Let $p \in [1, \infty)$. For all $\varepsilon > 0$ and $n \in \omega$, there exists $\delta > 0$ such that if $V \subset_\delta V'$ and $W \subset_\delta W'$, then*

$$\|{}_W A_V\|_p^{(n)} < \|{}_{W'} A_{V'}\|_p^{(n)} + \varepsilon.$$

Proof. Choose $k \le n$ and $e \in F_k(V)$ and $f \in F_k(W)$ such that

$$\|{}_W A_V\|_p^{(n)} < \|\langle Ae_1, f_1\rangle, \ldots, \langle Ae_k, f_k\rangle\|_p + \varepsilon/2.$$

For sufficiently small δ, Lemma 7 gives $e' \in F_k(V')$ and $f' \in F_k(W')$ such that $\|A\|\,\|f_i - f'_i\| < \varepsilon/4k$ and $\|A\|\,\|e_i - e'_i\| < \varepsilon/4k$. As $\|c_1, \ldots, c_k\|_p \le k \sup |c_i|$ we have

$$\|{}_W A_V\|_p^{(n)} < \|\langle Ae'_1, f'_1\rangle, \ldots, \langle Ae'_k, f'_k\rangle\|_p + \varepsilon \le \|{}_{W'} A_{V'}\|_p^{(n)} + \varepsilon.$$

■

We can now prove an infinite form of the singular-value decomposition.

Theorem 9 *Let* $A : H \to K$ *be an operator with singular values* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_t$*. For each* $\varepsilon > 0$ *there exists* $k \leq t$*, and* k*-frames* e *in* H *and* f *in* K *such that* $\sigma_{k+1} < \varepsilon$ *if* $k < t$*, and* $\|Ae_i - \sigma_i f_i\| < \varepsilon$ *for* $i = 1, \ldots, k$*. Moreover we arrange it so that* $\|Au\| < \sigma_{k+1} + \varepsilon$ *for each unit vector* u *orthogonal to the span of* e*.*

Proof. Fix $\delta > 0$ subject to later revision. For $n = 1, \ldots, t$ we can find finite-dimensional subspaces V_n of H and W_n of K such that $\|A\|_1^{(n)} < \|{}_{W_n}A_{V_n}\|_1^{(n)} + \delta$. By Theorem 5 we can find finite-dimensional subspaces V of H and W of K so that $V_n \subset_\delta V$ and $W_n \subset_\delta W$ for each n. So Theorem 8 says that $\|A\|_1^{(n)} < \|{}_W A_V\|_1^{(n)} + \varepsilon'$ for $n = 1, \ldots, t$ and ε' as small as we please. Therefore the first t singular values $\sigma'_1, \ldots, \sigma'_t$ of ${}_W A_V$ are close to the first t singular values of A. Let $k = \min(\dim V, \dim W, t)$. If $k < t$ and $\sigma_{k+1} > 0$, then by decreasing δ we can increase k. So we may assume that σ_{k+1} is small if $k < t$. Theorem 3 says there exist k-frames e of V and f of W so that ${}_W A_V e_i$ is close to $\sigma'_i f_i$ for $i = 1, \ldots, k$. Lemma 2 says that $\langle Ae_i, w\rangle$ is small for any unit vector w orthogonal to the span of f; so ${}_W A_V e_i$ is close to Ae_i. The first conclusion follows.

Clearly $\|A\|_1^{(k)} < \sum_{i=1}^{k} |\langle Ae_i, f_i\rangle| + \varepsilon''$, for ε'' as small as we please. Lemma 2 says that $\langle Au, f_i\rangle$ is small for any unit vector u orthogonal to the span of e. If w is a unit vector orthogonal to the span of f, then

$$\sum_{i=1}^{k} |\langle Ae_i, f_i\rangle| + |\langle Au, w\rangle| \leq \|A\|_1^{(k)} + \sigma_{k+1},$$

so $|\langle Au, w\rangle| \leq \sigma_{k+1} + \varepsilon''$. It follows that we can arrange for the second conclusion to hold. ■

Corollary 10 *Let* $A : H \to K$ *be an operator with singular values* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_t$*. Then*

$$\sigma_t = \sup_{\dim V = t} \inf_{u \in F_1(V)} \|Au_1\|$$

where V *ranges over* t*-dimensional subspaces of* H*.*

Proof. For $\varepsilon > 0$, let $e \in F_k(H)$ and $f \in F_k(K)$, with $k \leq t - 1$, be as in Theorem 9 (with $t - 1$ in place of t). If V is a t-dimensional subspace of H, then there is a unit vector u in V that is arbitrarily close to the orthogonal complement of the span of e. So $\|Au\| < \sigma_{k+1}+\varepsilon$. If $k < t-1$, then $\sigma_{k+1} < \varepsilon$. Thus

$$\sup_{\dim V=t} \inf_{u\in F_1(V)} \|Au_1\| < \sigma_t + 2\varepsilon$$

for any $\varepsilon > 0$.

Now let $e \in F_k(H)$ and $f \in F_k(K)$, with $k \leq t$, be as in Theorem 9, and let V be the span of e. If $k < t$, then $\sigma_t \leq \sigma_{k+1} < \varepsilon$. If $k = t$, and u is a unit vector in V, then $\|Au\| \geq \sigma_t - \varepsilon\sqrt{t}$. Thus

$$\sup_{\dim V=t} \inf_{u\in F_1(V)} \|Au_1\| \geq \sigma_t - \varepsilon\sqrt{t}$$

for any $\varepsilon > 0$.■

We can use Corollary 10 to define singular values as suprema for any operator whatsoever. Note that Corollary 10 implies that the singular values increase in the sense that the singular values of A dominate the singular values of ${}_WA_V$.

A set $\mathcal{V}$ of finite-dimensional subspaces of H is **cofinal** in H if for each $\varepsilon > 0$ and finite-dimensional subspace U of H, there is $V \in \mathcal{V}$ such that $U \subset_\varepsilon V$. So Theorem 5 says that if V_0 is a finite-dimensional subspace of H, then the set of finite-dimensional subspaces of H that contain V_0 is cofinal in H. More generally, the set of finite-dimensional subspaces that respect a given projection is cofinal.

Theorem 11 *If π is a projection on H, then the set of finite-dimensional subspaces V of H such that $\pi V \subset V$ is cofinal in H.*

Proof. Let $\varepsilon > 0$ and let U be the span of $e \in F_n(H)$. Let U' be the span of $e_1, \ldots, e_{n-1.}$ By induction on n there is a finite-dimensional subspace V' of H so that $\pi V' \subset V'$ and $U' \subset_\varepsilon V'$. We will construct a finite-dimensional subspace V'' of H so that $\pi V'' \subset V''$ and $U \subset_{2\varepsilon} V''$.

If e_n is within ε of V', set $V'' = V'$. So we may assume that e_n is bounded away from V'. Then either πe_n or $(1-\pi)e_n$ is bounded away from V'. Let $v \in \{\pi e_n, (1-\pi)e_n\}$ be bounded away from V', and set W equal to the span of V' and v. Note that W is finite-dimensional and $\pi W \subset W$. Again, either e_n is within ε of W, in which case we set $V'' = W$, or e_n is bounded away from W. In the latter case, let V'' be the span of e_n and W, so V'' is the span of V', πe_n and $(1-\pi)e_n$. Note that V'' is finite-dimensional. Clearly $U \subset V$ and $\pi V'' \subset V''$. ■

Theorem 12 *Let A be an operator from H to K, and $\mathcal{V}$ and $\mathcal{W}$ cofinal sets of finite-dimensional subspaces of H and K. Then*

$$\|A\|_p^{(n)} = \sup_{\substack{W\in\mathcal{W}\\ V\in\mathcal{V}}} \|{}_W A_V\|_p^{(n)}.$$

Proof. Clearly $\|A\|_p^{(n)} \geq \sup_{W\in\mathcal{W}, V\in\mathcal{V}} \|{}_W A_V\|_p^{(n)}$. For $\varepsilon > 0$, let $e \in F_k(H)$ and $f \in F_k(K)$ be such that

$$\|A\|_p^{(n)} \leq \|\langle Ae_1, f_1\rangle, \ldots, \langle Ae_k, f_k\rangle\|_p + \varepsilon.$$

For any $\delta > 0$, there exist $W \in \mathcal{W}$ and $V \in \mathcal{V}$ such that the span of e is δ-contained in V and the span of f is δ-contained in W. So, for sufficiently small δ,

$$\|\langle Ae_1, f_1\rangle, \ldots, \langle Ae_k, f_k\rangle\|_p \leq \|{}_W A_V\|_p^{(n)} + \varepsilon$$

whence $\|A\|_p^{(n)} \leq \|{}_W A_V\|_p^{(n)} + 2\varepsilon$. ■

Theorem 13 *Let H and K be Hilbert spaces, and $A : H \to K$ an operator. Suppose that $A = \lambda A \pi$ for projections λ of K and π of H. Then $\|A\|_p^{(n)} = \|{}_{\lambda K} A_{\pi H}\|_p^{(n)}$.*

Proof. First suppose that H and K are finite-dimensional. It suffices to show, for each $\varepsilon > 0$, that

$$\left(\|A\|_p^{(n)}\right)^p \leq \left(\|{}_{\lambda K} A_{\pi H}\|_p^{(n)}\right)^p + \varepsilon$$

But, from Theorem 3,

$$\left(\|A\|_p^{(n)}\right)^p = \left(\|A\|_p^{(n-1)}\right)^p + \sigma_n^p$$

so, by induction on n,

$$\left(\|A\|_p^{(n)}\right)^p = \left(\|{}_{\lambda K} A_{\pi H}\|_p^{(n-1)}\right)^p + \sigma_n^p \leq \left(\|{}_{\lambda K} A_{\pi H}\|_p^{(n)}\right)^p + \sigma_n^p$$

whence we are done if $\sigma_n^p < \varepsilon$. Therefore we may assume that $\sigma_n > 0$.

Given any $\delta > 0$, Theorem 3 says that there exist n-frames e in H and f in K such that $\|Ae_i - \sigma_i f_i\| < \delta$ and $\|A^* f_i - \sigma_i e_i\| < \delta$ for $i \leq n$. As

$$\|\sigma_i f_i - \sigma_i \lambda f_i\| \leq \|\sigma_i f_i - Ae_i\| + \|\lambda Ae_i - \sigma_i \lambda f_i\| < 2\delta,$$

we have $\|f_i - \lambda f_i\| < 2\delta/\sigma_n$. For small enough δ, Lemma 7 says that there is an n-frame f' in λK, with $\|f'_i - f_i\|$ as small as we please. Similarly, there is an n-frame e' in πH with $\|e'_i - e_i\|$ as small as we please. It follows that, for any $\eta > 0$, we can, by choosing δ sufficiently small, arrange for

$$\sum_{i=1}^{n} |\langle Ae'_i, f'_i \rangle|^p > \left(\|A\|_p^{(n)} \right)^p - \eta.$$

Now suppose H and K are arbitrary. Let $\mathcal{V}$ and $\mathcal{W}$ be the sets of finite-dimensional subspaces V of H and W of K such that $\pi V \subset V$ and $\lambda W \subset W$. From the finite-dimensional case, if $V \in \mathcal{V}$ and $W \in \mathcal{W}$, then

$$\|{}_W A_V\|_p^{(n)} = \|{}_{\lambda W} A_{\pi V}\|_p^{(n)}.$$

From Theorems 11 and 12

$$\|A\|_p^{(n)} = \sup_{\substack{W \in \mathcal{W} \\ V \in \mathcal{V}}} \|{}_W A_V\|_p^{(n)}$$

and

$$\|{}_{\lambda K} A_{\pi H}\|_p^{(n)} = \sup_{\substack{W \in \mathcal{W} \\ V \in \mathcal{V}}} \|{}_{\lambda W} A_{\pi V}\|_p^{(n)}$$

■

- If π is a finite-dimensional projection, then $\|\pi\|_p$ exists and is equal to the dimension of π. This follows from Theorem 13 and Lemma 1. Conversely, suppose π is a projection such that $\|\pi\|_p$ exists and is equal to s. If the range of π contains an n-frame, then $s \geq n$, and if $n - 1 < s$, then the range of π contains such a set. A real number s with the property that $n - 1 < s$ implies $n \leq s$ is equal to an integer, and that will be the dimension of the range of π.

- An operator A on a finite-dimensional Hilbert space H is of trace class. It suffices to show that $\|A\|_1^{(n)}$ exists, for n the dimension of H. The n-dimensional subspace $\{(x, Ax) : x \in H\}$ of $H \oplus K$ is ε-contained in $H \oplus \lambda K$ for some finite-dimensional projection λ of K. The operator ${}_{\lambda K}A$ is between finite-dimensional spaces, $\|{}_{\lambda K}A\|_1^{(n)}$ exists, and Theorem 13 says that $\|{}_{\lambda K}A\|_1^{(n)} \leq \|A\|_1^{(n)} < \|{}_{\lambda K}A\|_1^{(n)} + \varepsilon$, so $\|A\|_1^{(n)}$ is located.

5 Compact operators

A **compact** operator may be defined by any of the following three equivalent conditions.

Theorem 14 *Let $A : H \to K$ be an operator between inner product spaces. Then the following are equivalent.*

1. *The image of the unit ball of H under A is totally bounded.*
2. *For each $\varepsilon > 0$ there are finite-dimensional projections $\pi : H \to H$ and $\lambda : K \to K$ such that $\|A - \lambda A \pi\| < \varepsilon$.*
3. *For each $\varepsilon > 0$ there is a finite-dimensional subspace V of H so that $\|Ae\| < \varepsilon$ for each unit vector e orthogonal to V.*

Proof. We may assume that $\|A\| \leq 1$. Clearly (3) follows from (2) by taking V to be the image of π, so $\pi e = 0$. To prove (1) from (3) we note that if S is an ε-approximation to the unit ball V_1 of V, then $A(S)$ is a 2ε-approximation to $A(H_1)$. It remains to prove (2) from (1).

From (1) we can construct a finite-dimensional projection $\lambda : K \to K$ such that $\|A - \lambda A\| < \varepsilon$, so we may assume that K is finite-dimensional. Suppose first that K is one-dimensional, so A is a linear functional.

As $A(H_1)$ is totally bounded, we can find $e \in H_1$ so that $|Ae| \geq \|A\| - \delta$. If $|Ae| < \varepsilon - \delta$, then take $\pi = 0$. If $e \neq 0$, then take π to be the projection on the span of e. If u is a unit vector orthogonal to e, then Lemma 2 says that $|Au| < \varepsilon$ if δ is small. So $\|A(1 - \pi)\| < \varepsilon$.

Now suppose K has dimension n, and let $\lambda_1, \ldots, \lambda_n$ be the projections with respect to some orthonormal basis. The operators $\lambda_i A$ satisfy (1), so, from the one-dimensional case, there exist projections π_i of H, of dimension either 0 or 1, so that

$$\|\lambda_i A - \lambda_i A \pi_i\| < \varepsilon$$

By tweaking the π_i, we may assume that the images of the π_i generate a finite-dimensional space with projection π. Then

$$\|\lambda_i A - \lambda_i A \pi\| < \varepsilon$$

and, since $A = \sum \lambda_i A$,

$$\|A - A\pi\| < n\varepsilon$$

■

It is clear from (1) that if A is compact, then $\|A\|$ exists. Ishihara [6], using characterization (1), proved the following.

Corollary 15 *The sum of two compact operators is compact.*

Proof. We use characterization 3. Let A and B be compact. Then there exist finite-dimensional subspace U and V so that $\|Ae\| < \varepsilon$ for each unit vector e orthogonal to U, and $\|Be\| < \varepsilon$ for each unit vector e orthogonal to V. Let W be a finite-dimensional subspace such that $U \subset W$ and $V \subset_\varepsilon W$. If e is a unit vector orthogonal to W, then e is orthogonal to U, so $\|Ae\| < \varepsilon$. Let λ be the projection on V. There is $w \in W$ such that $\|\lambda e - w\| < \varepsilon$. So

$$\|\lambda e\|^2 = \langle \lambda e, e\rangle = \langle \lambda e - w, e\rangle \leq \varepsilon.$$

Thus

$$\|Be\| \leq \|B(1-\lambda)e\| + \|B\lambda e\| \leq \varepsilon + \|B\| \sqrt{\varepsilon}$$

so $\|(A+B)e\| \leq 2\varepsilon + \|B\| \sqrt{\varepsilon}$ if e is a unit vector orthogonal to W. ■

We can also characterize compact operators in terms of singular values. First a lemma.

Lemma 16 *Let $A : H \to K$ be an operator, $e \in F_k(H)$ and $f \in F_k(K)$. If*

$$\left(\|A\|_p^{(k)}\right)^p < \sum_{i=1}^{k} |\langle Ae_i, f_i\rangle|^p + \delta^p$$

and

$$\left(\|A\|_p^{(k+1)}\right)^p < \sum_{i=1}^{k} |\langle Ae_i, f_i\rangle|^p + (\delta')^p,$$

then $\|Au\| < \delta' + \sqrt{k\delta(2\|A\| + \delta)}$ for any unit vector u orthogonal to the span of e.

Proof. From the first inequality, and Lemma 2, we get $|\langle Au, f_i\rangle|^2 < \delta(2\|A\| + \delta)$. So $Au - \sum_{i=1}^{k}\langle Au, f_i\rangle f_i$ is orthogonal to $f_1, \ldots, f_k$ and within $\sqrt{k\delta(2\|A\| + \delta)}$ of Au. If $Au - \sum_{i=1}^{k}\langle Au, f_i\rangle f_i$ is small, then we're done. If it is nonzero, then we can write it as rf_{k+1} with f_{k+1} a unit vector orthogonal to $f_1, \ldots, f_k$. Setting $e_{k+1} = u$ we get

$$\sum_{i=1}^{k} |\langle Ae_i, f_i\rangle|^p + |r|^p \leq \left(\|A\|_p^{(k+1)}\right)^p$$

so $|r| < \delta'$, from the second inequality, and the conclusion follows. ∎

Theorem 17 *An operator is compact if and only if it has singular values that converge to zero.*

Proof. If A is a compact operator, then A can be approximated in the operator norm by operators of the form $\lambda A\pi$, with λ and π finite-dimensional projections. As $\|\lambda A\pi\|_1^{(k)}$ exists (by Theorem 13) so does $\|A\|_1^{(k)}$, so A has singular values. We will show that an operator A with singular values (σ_k) is compact if and only if $\sigma_k \to 0$.

If A is compact, then there is a subspace V, of finite dimension $k-1$, such that A has norm at most ε on $V^\perp$. Either $\sigma_k \leq 2\varepsilon$, or $\sigma_k > 0$. If $\sigma_k > 0$, then we can find k-frames e and f so that $\sum_{i=1}^k |\langle Ae_i, f_i\rangle|$ is close to $\sigma_1 + \cdots + \sigma_k$. Let π be the projection onto the span E of e, and λ the projection onto the span of f. Then the singular values σ_i' of $\lambda A\pi$ are close to σ_i for $i = 1, \ldots, k$. In particular, we may arrange for $\sigma_k' \geq \sigma_k/2$. As $\dim V = k-1$ and $\dim E = k$, there is a unit vector u in E that is as close as we please to $V^\perp$. From Theorems 13 and 3, we get $\|Au\| \geq \sigma_k' \geq \sigma_k/2$, so $\sigma_k \leq 2\varepsilon$.

Conversely, suppose $\sum_{i=1}^k |\langle Ae_i, f_i\rangle|$ is within δ of $\sigma_1 + \cdots + \sigma_k$, and E is the span of e. Lemma 16 says that the norm of A is less than $\sigma_{k+1} + \delta + \sqrt{k\delta(2\sigma_1 + \delta)}$ on $E^\perp$. ∎

Theorem 18 *If $\|A\|_p$ exists, then A is compact.*

Proof. If $\sum_{i=1}^k |\langle Ae_i, f_i\rangle|^p$ is within δ^p of $\|A\|_p^p$, and E is the span of $e_1, \ldots, e_k$, then Lemma 16 says the norm of A is less than $\delta + \sqrt{k\delta(2\|A\| + \delta)}$ on $E^\perp$. ∎

We know what a trace-class operator is, but what is its trace? That is, if A is an operator from H to H, what does it mean to say that $\operatorname{tr} A = t$? We define this in terms of traces of finite-dimensional operators.

- We say that $\operatorname{tr} A = t$ if, for each finite-dimensional subspace F', there is a finite-dimensional subspace $F \supset F'$ such that $|\operatorname{tr} \pi A \pi - t| < \varepsilon$ for each finite-dimensional projection π such that $\pi F = F$.

What is the criterion for $\operatorname{tr} A$ to exist?

- For each finite-dimensional subspace F' there is a finite-dimensional subspace $F \supset F'$ such that $|\operatorname{tr} \pi A \pi| < \varepsilon$ for each finite-dimensional projection π such that $\pi F = 0$.

This criterion conforms to the general definition of what it means for a metric space to be complete. In the absence of countable choice, it is not enough to require that every Cauchy sequence converge. What you need is that (roughly) whenever you have a consistent method for approximating a point, then there is some point that you are approximating. Why do we meet this criterion in the case of an operator of trace class? Choose a finite-dimensional projection φ, with range F, so that $\|A - \varphi A \varphi\|$ is small. We have to verify that we can find one such that F contains a prespecified F'. That follows from Theorem 5. The rest is straightforward.

6 The von Neumann-Schatten ideals

The operators A such that $\|A\|_p$ exists can be characterized in terms of their singular values.

Theorem 19 *Let A be an operator with singular values (σ_i). Then $\|A\|_p^p = \sum \sigma_i^p$. Thus $\|A\|_p$ exists if and only if $\sum \sigma_i^p$ converges.*

Proof. We have

$$\|A\|_p^p = \sup_{\substack{e \in F_n(H) \\ f \in F_n(H) \\ n \in \omega}} \sum_{i=1}^{n} |\langle Ae_i, f_i \rangle|^p .$$

For any t, and any $\varepsilon > 0$, Theorem 9 says that there exist k-frames e and f, with $k \leq t$, such that $\|Ae_i - \sigma_i f_i\| < \varepsilon$ for $i = 1, \ldots, k$, and that $\sigma_{k+1} < \varepsilon$ if $k < t$. So

$$\|A\|_p^p \geq \sum_{i=1}^{t} \sigma_i^p - (t - k)\varepsilon^p$$

whence $\|A\|_p^p \geq \sum \sigma_i^p$.

To get the inequality the other way round, we will show that

$$\sum_{i=1}^{k} |\langle Ae'_i, f'_i\rangle|^p \leq \sum_{i=1}^{k} \sigma_i^p$$

for any k-frames e and f. Let E and F be the spans of e and f, and let $\tau_1, \ldots, \tau_k$ be the first k singular values of ${}_F A_E$. From Corollary 10, we have $\tau_i \leq \sigma_i$. Then

$$\sum_{i=1}^{k} |\langle Ae_i, f_i\rangle|^p \leq \left(\|{}_F A_E\|_p^{(k)}\right)^p = \tau_1^p + \cdots + \tau_k^p \leq \sigma_1^p + \cdots + \sigma_k^p$$

■

It follows that if $\|A\|_p$ exists, then so does $\|A\|_{p'}$ for $p' > p$.

Theorem 20 *Let $A : H \to K$ be compact with singular values (σ_i), and $B : K \to L$ bounded. If (τ_i) are the singular values of the compact operator BA, then $\tau_i \leq \|B\| \sigma_i$.*

Proof. This is immediate from Corollary 10. ■

It follows that if $\|A\|_p$ exists, then so does $\|BA\|_p$.

Lemma 21 *If $\|A\|_p$ exists, then for each $\varepsilon > 0$ there are finite-dimensional projections λ and π such that $\|A - \lambda A\pi\|_p < \varepsilon$. Conversely, the set of operators A for which the norm $\|A\|_p$ exists is closed in that norm.*

Proof. Suppose $\|A\|_p$ exists. Given δ and n, we can find k-frames e and f, with $k \leq n$, such that

$$\|A\|_p^p < \sum_{i=1}^{k} |\langle Ae_i, f_i\rangle|^p + \delta + \sum_{i=n+1}^{\infty} \sigma_i^p$$

Let π be projection on the span of e, and λ projection on the span of f. As

$$\|A\|_p^p \geq \|\lambda A\pi\|_p^p + \|(1-\lambda)A(1-\pi)\|_p^p$$

we have

$$\|(1-\lambda)A(1-\pi)\|_p^p \leq \|A\|_p^p - \|\lambda A\pi\|_p^p \leq \delta + \sum_{i=n+1}^{\infty} \sigma_i^p.$$

From Lemma 2

$$\|\lambda A(1-\pi)\| + \|(1-\lambda)A\pi\| < 2\sqrt{k\delta(2\,\|A\| + \delta)}$$

so

$$\|\lambda A(1-\pi)\|_p + \|(1-\lambda)A\pi\|_p < 2k\sqrt{k\delta(2\,\|A\| + \delta)}$$

and the first conclusion follows for small enough δ.

Now suppose $\|A-B\|_p < \varepsilon$ where $\|B\|_p$ exists. Then

$$\|B\|_p - \varepsilon \leq \|A\|_p \leq \|B\|_p + \varepsilon$$

so if such B exists for each $\varepsilon > 0$, then $\|A\|_p$ is located. ■

Theorem 22 *If* $\|A\|_p$ *and* $\|B\|_p$ *exist, then so does* $\|A+B\|_p$.

Proof. The lemma provides finite-dimensional projections λ, π, λ', and π' so that $\|A - \lambda A\pi\|_p$ and $\|B - \lambda' A\pi'\|_p$ are small. So

$$\|A + B - (\lambda A\pi + \lambda' B\pi')\|_p \leq \|A - \lambda A\pi\|_p + \|B - \lambda' A\pi'\|_p$$

is small. From the second part of the lemma, it suffices to show that $\|C\|_p$ exists, for $C = \lambda A\pi + \lambda' B\pi'$. From Theorem 5 we can find $\lambda'' \supset \lambda'$ such that $\|(1-\lambda'')\lambda\| < \varepsilon$. Then $\|\lambda'' C - C\|_p < \varepsilon\,\|A\|\dim\pi$. It remains to show that $\|\lambda'' C\|_p$ exists. It suffices to show that C is compact as the singular values of $\lambda'' C$ are zero beyond the dimension of λ, but that follows from Corollary 15 because C is a sum of compact operators. ■

So the class $\mathcal{C}_p$ of operators A such that $\|A\|_p$ exists forms a left ideal in the algebra of bounded operators on H. It is an ideal in the algebra of bounded operators on H with adjoints because $\|B^*A^*\|_p$ exists, so $\|AB\|_p$ exists. However it is possible for A to be trace class, B bounded, and AB not even compact. Let H be a Hilbert space with basis (e_n). Let $Be_n = a_n e_1$, where (a_n) is a binary sequence with at most one 1. Let $A = \pi_1$, the projection on the span of e_1. Then $AB = B$ is not compact. More precisely, if AB is compact, then either $a_n = 1$ for some n, or $a_n = 0$ for all n.

7 A counterexample

In [3] there are two claims on page 15 about an arbitrary trace-class operator A. One is, for a_n an orthonormal basis and $q > p$, that

$$\sum_{p+1}^{q} |\langle Aa_n, a_n\rangle| \leq \sum_{p+1}^{q} \langle |A|\, a_n, a_n\rangle$$

The other is that $|A| - A$ is a positive operator. Let H be two dimensional and consider the operators given by the matrices

$$\pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Set

$$A = U\pi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Then U is unitary and $\pi = |A|$. Let e be the column vector $(1\ 1)^t$. Then $Ae = \frac{1}{\sqrt{2}}e$ and $|A|\, e = (0\ 1)^t$, so

$$\langle Ae, e\rangle = \sqrt{2} > 1 = \langle |A|\, e, e\rangle$$

In particular, the operator $|A| - A$ is not positive. Of course $|A| - A$ can only be positive if A is a difference of positive operators, hence Hermitian.

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