SIMPLY PRESENTED TAG MODULES

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ABSTRACT. A tag module is a generalization, in any abelian category, of a **t**orsion **a**belian **g**roup. The theory of such modules is developed, it is shown that countably generated tag modules are simply presented, and that Ulm's theorem holds for simply presented tag modules. Zippin's theorem is stated and proved for countably generated tag modules.

1. TAG-MODULES

In the theory of torsion abelian groups, a distinguished role is played by the cyclic groups of prime power order. Such a group is a **uniserial module**, by which we will mean a module whose submodules form a *finite* chain. The following two conditions on a module M were introduced by Singh in [14].

(I) Finitely generated submodules of homomorphic images of M are direct sums of uniserials.

(II) If u and v are uniserial submodules of homomorphic images of M, and the length of u is no greater than the length of v, then any one-to-one map from a nonzero submodule of u to v can be extended to an isomorphism of u with a submodule of v.

Singh's condition (II) was slightly different from this one but equivalent to it. Modules satisfying these two conditions were called S_2-modules by Kahn [7] and **TAG-modules** by Benabdallah and Singh [3]. The idea is that a TAG-module is like a torsion abelian group. Certainly each torsion abelian group is a TAG-module, the uniserial submodules being cyclic groups of prime power order.

Rather than working in a category of modules over a ring, we will work in an abelian category whose objects we will call **modules**. Normally we will want this category to have infinite direct sums and products, and to satisfy Grothendieck's axiom **AB-5** (see [12]) on directed families B_i of submodules:

$$A \cap \sum B_i = \sum A \cap B_i.$$

It will also be convenient to have a distinguished monic $S \to M$ for each submodule of M so that submodules may be canonically considered as objects in the category.

This is certainly the case in the category of modules over a ring—the distinguished monic is the injection of the submodule into the module—and it is true in our other examples.

The reader not interested in the extra generality may simply think of modules over a ring. The difference is that arguments will not involve elements of the modules (nor of the ring), only submodules and homomorphisms. There is a virtue to this approach even in the case of modules over a ring. Elements can be a distraction—at the very least we have to decide whether to look at left modules or at right modules. If we view abelian groups, say, simply as objects in an abelian category, then we are not tempted to worry about what the height of $x + y$ is in terms of the heights of x and y—the issue does not arise.

It may be helpful to keep a sufficiently rich collection of TAG-modules in mind. Fix a field k, and consider the quiver (directed graph) on the positive integers $\mathbf{N}$ with an arrow from $n+1$ to n for each n in $\mathbf{N}$.

$$\cdots \to 3 \to 2 \to 1$$

Consider representations of this quiver by vector spaces over k. Such a representation is a sequence $A_1, A_2, \ldots$ of vector spaces over k, together with a linear transformation $A_{n+1} \to A_n$ for each n. We will denote any of these linear transformations by the letter p. Such representations form an abelian category in a natural way, and each representation is a TAG-module. An isomorphism class of uniserial modules is determined by positive integers $m \leq n$, setting $u_i = k$ for $m \leq i \leq n$ and $u_i = 0$ otherwise, and letting the map between adjacent nonzero u_i be the identity.

This example is very close to abelian group theory as the representations may be considered to be graded modules over the principal ideal domain $k[X]$ with multiplication by X a homomorphism of degree -1. Indeed, many (but not all) X-primary $k[X]$-modules admit such a grading, with multiplication by X being one-to-one on A_n for $n > 1$. This might be an interesting category for abelian p-group theorists to investigate. The grading is additional structure that is reflected, for example, in the Ulm invariants.

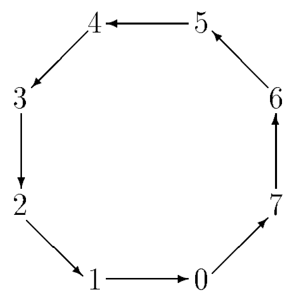
More generally, we may consider the integers $\mathbf{Z}$ as a quiver

$$\cdots \to 3 \to 2 \to 1 \to 0 \to -1 \to -2 \to -3 \to \cdots$$

and concentrate on representations of $\mathbf{Z}$ that are **torsion** in the sense that for each x we have $p^n x = 0$ for some n.

Representations of these quivers may be thought of as modules over an appropriate infinite matrix ring, but it is not clear that you want to think of them that way. To do that, not only do you have to introduce the ring, which may be excess baggage, but you have to consider the direct sum of the vector spaces A_n, whereas you really never look at anything but elements of the various A_n.

We can also consider torsion representations of the quiver $\mathbf{Z}_q$, the integers modulo q, for $q = 0, 1, 2, \ldots$. Here is $\mathbf{Z}_q$.



As before, we have a single arrow going from $n+1$ to n. Note that $\mathbf{Z}_0 = \mathbf{Z}$, and that the quiver $\mathbf{Z}_1$ consists of a single node with a loop. The quivers $\mathbf{Z}_q$ are exactly the connected quivers where each node has out-degree 1 and in-degree 1. Representations of the quiver $\mathbf{Z}_1$ are simply $k[X]$-modules.

2. Tag classes of uniserials

If $\mathcal{U}$ is a class of uniserial modules, let $\mathcal{U}^*$ denote the class of finite direct sums of modules in $\mathcal{U}$. The following theorem is true in an arbitrary abelian category, so its dual is also true.

Theorem 1. *Let $\mathcal{U}$ be a class of uniserial modules that is closed under images. Then the following are equivalent:*

1. *$\mathcal{U}^*$ is closed under images.*
2. *If u and u' are in $\mathcal{U}$, then any image of $u \oplus u'$ is in $\mathcal{U}^*$.*
3. *Any isomorphism between nonzero submodules of two modules in $\mathcal{U}$ extends to an isomorphism between one of them and a submodule of the other.*

Proof. Clearly (2) holds if (1) holds. To show that (3) follows from (2), suppose (2) holds and f is an isomorphism between a nonzero submodule w of $u \in \mathcal{U}$ and w' of $u' \in \mathcal{U}$. Consider the pushout

$$\begin{array}{ccc} w & \xrightarrow{f} & u' \\ \downarrow & & \downarrow \\ u & \rightarrow & P \end{array}$$

Then P is an image of $u \oplus u'$, so $P \in \mathcal{U}^*$. Suppose the lengths of w, u and u' are $n < m \leq m'$. As P contains a copy of u', it must have a summand v in $\mathcal{U}$ of length at least m'. The length of P is $m + m' - n$, so the complementary summand of v has length less than m. So the copies of u and u' in P both map isomorphically into v by projection, with the map from w to v being the same by either route. This provides us with an extension of f to a map from u to u'.

To show that (1) follows from (3), suppose (3) holds, $A \in \mathcal{U}^*$ and $B \subset A = \bigoplus_{i=1}^n u_i$. We want to show that $A/B \in \mathcal{U}^*$, and we may assume that B is nonzero. If $n = 1$ this follows from the fact that $\mathcal{U}$ is closed under images, so we may assume that $n > 1$ although this is not strictly necessary. We will induct on the length of A. Let $K_j = B \cap \sum_{i \neq j} u_i$. Then

$$\frac{A}{K_j} = u_j \oplus \frac{\sum_{i \neq j} u_i}{K_j}$$

is in $\mathcal{U}^*$ by induction. If K_j is nonzero, then A/B, which is an image of A/K_j, is in $\mathcal{U}^*$ by induction. So we may assume that the projection of B on each u_i is monic. Suppose u_1 is the shortest of the u_i. By (3), there are maps $f_i : u_1 \to u_i$ so that $\pi_i = f_i \pi_1$ on B. Then $(1 + \sum_{i=2}^n f_i)\pi_1$ is the identity on B, so $(1 + \sum_{i=2}^n f_i)u_1$ is a complementary summand of $\bigoplus_{i=2}^n u_i$ that contains B. Because $\mathcal{U}$ is closed under images, we see from this that A/B is in $\mathcal{U}^*$. So (1) holds. ■

The dual theorem is

Theorem 2. *Let $\mathcal{U}$ be a class of uniserial modules that is closed under submodules. Then the following are equivalent:*

1. *$\mathcal{U}^*$ is closed under submodules.*
2. *If u and u' are in $\mathcal{U}$, then any submodule of $u \oplus u'$ is in $\mathcal{U}^*$.*
3. *Any isomorphism between nonzero images of two modules in $\mathcal{U}$ lifts to an isomorphism between one of them and an image of the other.*

■

Let $\mathcal{U}$ be the set of uniserial submodules of images of a TAG-module M. As Condition (3) of Theorem 1 is Condition (II) in the definition of a TAG-module, $\mathcal{U}^*$ is closed under images. It is not necessary that $\mathcal{U}^*$ be closed under submodules—only those modules in $\mathcal{U}^*$ that actually appear as submodules of images of M need have their submodules in $\mathcal{U}^*$. For example, let M be uniserial of length four with composition factors 1213 from the top. Then M is a TAG-module but the pullback of $12 \to 1 \leftarrow 13$ is not in $\mathcal{U}^*$. You can easily construct such a uniserial representation of the quiver $3 \leftarrow 1 \leftrightarrows 2$.

If we require that $M \oplus M$ be a TAG-module (it follows that M is also a TAG-module), then $\mathcal{U}^*$ is also closed under submodules.

Theorem 3. *Suppose $M \oplus M$ is a TAG-module and $\mathcal{U}$ is the class of uniserial submodules of images of M. Then $\mathcal{U}^*$ is closed under submodules and images, and $\mathcal{U}$ is the class of uniserial submodules of images of $M \oplus M$.*

Proof. From Theorem 1 we know that $\mathcal{U}^*$ is closed under images. If u and u' are in $\mathcal{U}$, then $u \oplus u'$ is a submodule of some image of $M \oplus M$. So any submodule of $u \oplus u'$ is a direct sum of uniserials, each of which projects isomorphically onto a submodule of either u or u'. Hence Condition 2 of Theorem 2 holds, so $\mathcal{U}^*$ is closed under submodules. Finally, if v is a uniserial submodule of an image of $M \oplus M$, then v is a submodule of an image of an element of $\mathcal{U}^*$, hence is in $\mathcal{U}$. ■

In light of this theorem, we will call a class $\mathcal{U}$ of uniserial modules a **tag class** if $\mathcal{U}$ and $\mathcal{U}^*$ are closed under submodules and images. Note that if $\mathcal{U}^*$ is closed under submodules and images, then so is $\mathcal{U}$.

Henceforth $\mathcal{U}$ will denote a class of uniserial modules that is closed under submodules and images. A **$\mathcal{U}$-module** is a module that is generated by its submodules that lie in $\mathcal{U}$. If $\mathcal{U}$ is a tag class, then each finitely generated submodule of a $\mathcal{U}$-module is in $\mathcal{U}^*$. In particular, each uniserial submodule of a $\mathcal{U}$-module is in $\mathcal{U}$. A **tag module** is a $\mathcal{U}$-module for some tag class $\mathcal{U}$. Because of AB-5, each submodule of a tag module is a tag module.

If $\mathcal{U}$ is a tag class, it is convenient to have a set $\overline{\mathcal{U}}$ of representatives of the isomorphism classes of modules of $\mathcal{U}$. Partially order $\overline{\mathcal{U}}$ by setting $u \leq v$ if there exists a monic $u \to v$. Because $\mathcal{U}$ is a tag class, each element of $\overline{\mathcal{U}}$ has at most one immediate successor, at most one immediate predecessor, and a finite number of predecessors. That is, as a partially ordered set, $\overline{\mathcal{U}}$ is a disjoint union of well-ordered sets of type at most ω. In addition, for each u and v in $\overline{\mathcal{U}}$, we want a subset $\overline{\mathrm{Hom}}(u, v)$ of $\mathrm{Hom}(u, v)$ such that

- For each map in $\mathrm{Hom}(u, v)$ there is exactly one map in $\overline{\mathrm{Hom}}(u, v)$ with the same kernel,
- $\overline{\mathcal{U}}$ is a category if we let $\overline{\mathrm{Hom}}(u, v)$ be the set of maps between u and v. That is, the identity map is in $\overline{\mathrm{Hom}}(u, u)$, and if $f \in \overline{\mathrm{Hom}}(u, v)$ and $g \in \overline{\mathrm{Hom}}(v, w)$, then $gf \in \overline{\mathrm{Hom}}(u, w)$.

In particular, if u admits a monic, or epic, map into v then we have a distinguished such map, namely the one in $\overline{\text{Hom}}(u, v)$.

In our examples, the category $\overline{\mathcal{U}}$ occurs naturally. For torsion abelian groups it consists of the modules $\mathbf{Z}/n\mathbf{Z}$ where n is a power of a positive prime (including $n = 1$) and the homomorphism from $\mathbf{Z}/m\mathbf{Z}$ to $\mathbf{Z}/n\mathbf{Z}$ with kernel $d\mathbf{Z}/m\mathbf{Z}$, where $d > 0$ divides m, and m divides dn, takes 1 to dn/m. For k-representations of the quiver $\mathbf{Z}$, the elements of $\overline{\mathcal{U}}$ are those representations such that $A_i = k$, for i in some finite sequence of adjacent integers, and $A_i = 0$ otherwise, with the maps between adjacent nonzero A_i being the identity. The homomorphisms are precisely those that are either the identity or zero on each A_i.

3. Simply presented $\mathcal{U}$-modules

By a **torsion forest** we mean a set F (of nodes) with a distinguished point 0, together with a function $\pi : F \to F$ such that $\pi 0 = 0$, and for each $x \in F$ there exists $n > 0$ such that $\pi^n x = 0$. We are only interested in the nonzero nodes—the node 0 is sometimes a convenience, sometimes a nuisance. An alternative approach is to let π be a partial function such that $\pi^n x$ is eventually undefined for each x, see [1]. It is clear how to move back and forth between these two pictures, the main difference being whether to write "$\pi x \neq 0$" or "πx is defined". A **map** of torsion forests is a function $f : F \to F'$ such that $f(\pi x) = \pi f(x)$ for every x in F.

The function π is the **parent function** on the nonzero nodes: if $\pi x = y \neq 0$, then x is a **child** of y. If $x \neq 0$ and $\pi x = 0$, then we say that x is a **root**. If $y = \pi^n x \neq 0$ for some $n \geq 0$, then we say that y is an **ancestor** of x or that x is a **descendant** of y. Note that we consider x to be an ancestor of x. A childless nonzero node is called a **leaf**.

We may view a torsion forest as a directed graph with arrows going from πx to x. Or we can view it as a partially ordered set, with $\pi x < x$, if $x \neq 0$, hence as a small category—essentially the free category generated by the arrows other than $0 \to 0$. We will be interested in functors, often called **diagrams**, on these partially ordered sets.

A $\mathcal{U}$-**forest** is a functor u from a torsion forest to $\mathcal{U}$ so that the map $u_{\pi x} \to u_x$ maps $u_{\pi x}$ isomorphically onto the radical of u_x, and $u_x \neq 0$ if $x \neq 0$. It follows that $u_0 = 0$. If $\mathcal{U}$ is a tag class, and $\pi x' = \pi x \neq 0$, then $u_{x'} \cong u_x$ because of Condition (3) of Theorem 1. The set of uniserial submodules of a $\mathcal{U}$-module form a $\mathcal{U}$-forest in the obvious way (using the canonical monic associated with each submodule).

A **map** between $\mathcal{U}$-forests u and v is a map f of the underlying torsion forests together with maps $u_x \to v_{f(x)}$ so that the diagram

$$\begin{array}{ccc} u_x & \to & v_{f(x)} \\ \uparrow & & \uparrow \\ u_{\pi x} & \to & v_{\pi f(x)} \end{array}$$

commutes for each x. A map between $\mathcal{U}$-modules induces a map between the associated $\mathcal{U}$-forests, so this association is a *functor* that forgets the $\mathcal{U}$-module structure and remembers only the $\mathcal{U}$-forest structure. We shall describe the adjoint, S, of that forgetful functor.

Given a $\mathcal{U}$-forest u based on a torsion forest F, we get a $\mathcal{U}$-module $S(u)$ by taking the direct sum of the modules u_x and identifying $u_{\pi x}$ with its image in u_x. More precisely, for each x in F we have a map from $u_{\pi x}$ to $u_{\pi x} \oplus u_x$ given by the negative identity $u_{\pi x} \to u_{\pi x}$ and the map $u_{\pi x} \to u_x$. Let K_x be the image of that map, and let $K = \sum_{x \in F} K_x$, a submodule of $\bigoplus_{x \in F} u_x$. Then $S(u)$ is the cokernel of the inclusion of K in $\bigoplus_{x \in F} u_x$, giving an exact sequence

$$0 \to K \to \bigoplus_{x \in F} u_x \to S(u) \to 0$$

Thus $S(u)$ is the colimit [9] of the diagram $(u_x)_{x \in F}$ of elements of $\mathcal{U}$. A colimit is a generalized pushout: the maps $u_x \to \bigoplus_{x \in F} u_x$ give maps $u_x \to S(u)$ compatible with the maps $u_{\pi x} \to u_x$, and, given any maps $u_x \to A$ that are compatible with the maps $u_{\pi x} \to u_x$, we get a unique map $S(u) \to A$ that they all factor through. A $\mathcal{U}$-module is said to be **simply presented** if it is isomorphic to $S(u)$ for some $\mathcal{U}$-forest u. Note that an arbitrary direct sum of elements of $\mathcal{U}$ is simply presented.

Because all the maps $u_{\pi x} \to u_x$ are monic, the maps $u_x \to S(u)$ are monic and we can think of u_x as a submodule of $S(u)$. We prove this for our situation.

Theorem 4. *If u is a $\mathcal{U}$-forest, then the natural maps $u_x \to S(u)$ are monic.*

Proof. For the purposes of this proof, we drop the requirement that $u_{\pi x}$ map onto the radical of u_x. This is one of the situations where the zero in a torsion forest is a nuisance. With the notation as above, we must show that $u_x \cap K = 0$. As $K = \sum_{x \in F} K_x$, it suffices, by AB-5, to prove the theorem for a finite forest F. We may assume that F has precisely one root.

Let x_0 be the branch point of F nearest the root (if F has no branch points, then $S(u) = u_y$ for y the leaf). Clearly we may assume that x_0 is the root of F. Let $F' \subset F$ consist of the descendants of one child of x_0, together with x_0, and let $F'' \subset F$ consist of the descendants of the remaining children of x_0, together with x_0. Let u' and u''

be the $\mathcal{U}$-forests obtained by restricting u to F' and to F'' (setting $\pi x = 0$ when necessary—here the picture of a partial function π is better). By induction on the number of nodes in F, the maps $u_x \to S(u')$ for $x \in F'$, and $u_x \to S(u'')$ for $x \in F''$ are monic. Therefore in the pushout diagram

$$\begin{array}{ccc} u_{x_0} & \to & S(u') \\ \downarrow & & \downarrow \\ S(u'') & \to & S(u) \end{array}$$

the top and left maps are monics, so the other two maps are monics. Thus the map $u_x \to S(u)$ is the composition of two monics. ■

Let $\mathcal{U}$ be a tag class. Call a $\mathcal{U}$-forest u **standard** if u_x is a module in $\overline{\mathcal{U}}$, and $u_{\pi x} \to u_x$ is a map in $\overline{\mathcal{U}}$, for each x. It is straightforward to show that every $\mathcal{U}$-forest is isomorphic to a standard one. We may think of a standard $\mathcal{U}$-forest u as a torsion forest F with labels $\sigma_x(u)$ on its nonzero nodes, $\sigma_x(u)$ being the simple module in $\overline{\mathcal{U}}$, such that $u_x/u_{\pi x} \cong \sigma_x(u)$. We can reconstruct u_x from the finite sequence $\sigma_x(u), \sigma_{\pi x}(u), \sigma_{\pi^2 x}(u), \ldots$ because $\mathcal{U}$ is a tag class and u is standard. In fact, we can define a partial function on the simple modules in $\overline{\mathcal{U}}$ by setting $\pi s = s'$ when there is an element u in $\mathcal{U}$ with $\operatorname{rad} u \cong s'$ and $u/\operatorname{rad} u \cong s$. So we can write the condition on the simple modules $\sigma_x(u)$ as $\sigma_{\pi x}(u) = \pi\sigma_x(u)$.

In the case of abelian p-groups, all the simples are isomorphic so we don't have to label the nodes at all. In the case of k-representations of the quiver $\mathbf{N}$, the simples correspond to positive integers, so we label the nodes with positive integers σ_x in such a way that if $y = \pi x$, then $\sigma_y = \sigma_x - 1$.

If $\mathcal{U}$ is a tag class, then finite-length $\mathcal{U}$-modules are direct sums of modules in $\mathcal{U}$. In general, finite-length simply presented $\mathcal{U}$-modules need not be direct sums of uniserials.

We have defined simply presented $\mathcal{U}$-modules by a construction. There is a corresponding internal characterization. A **uniserial basis** of a module A is a set U of nonzero uniserial submodules of A such that

1. nonzero submodules of modules in U are in U,

2. U generates A,

3. If $u_0, u_1, \ldots, u_m \in U$, and $u_0 \subset \sum_{i=1}^{m} u_i$, then $u_0 \subset u_i$ for some $i > 0$ (**irredundance**).

We can strengthen the irredundance condition to a somewhat intriguing distributive law.

Theorem 5. *If U is a uniserial basis of a module A, and $u_1, \ldots, u_m, v_1, \ldots, v_n \in U$, then*

$$\sum_{i=1}^{m} u_i \cap \sum_{j=1}^{n} v_j = \sum_{i,j} u_i \cap v_j.$$

Proof. Note that irredundance is equivalent to the case $n = 1$. The three conditions in the definition of a uniserial basis are preserved upon passing to $A/\sum_{i,j} u_i \cap v_j$, and replacing the elements of U by their nonzero images in $A/\sum_{i,j} u_i \cap v_j$, so we may assume that $\sum_{i,j} u_i \cap v_j = 0$. Let $K = \sum_{i=1}^{m} u_i$ and $L = \sum_{i=2}^{n} v_j$. We want to show that $K \cap (v_1 + L) = 0$. It suffices, by induction on the length of $\bigoplus_{j=1}^{n} v_j$, to show that

$$K \cap (v_1 + L) \subset \operatorname{rad} v_1 + L. \tag{$*$}$$

As

$$\frac{K \cap (v_1 + L)}{K \cap (\operatorname{rad} v_1 + L)} \subset \frac{v_1 + L}{\operatorname{rad} v_1 + L},$$

the left hand side has length at most 1. If the length is 0, then $(*)$ holds. If the length is 1, then the inclusion is equality so

$$v_1 \subset K + \operatorname{rad} v_1 + L.$$

By irredundance, $v_1 \subset u_i$ for some i, or $v_1 \subset \operatorname{rad} v_1$, or $v_1 \subset v_j$ for some $j > 1$. In each of these cases we can discard v_1 and are done by induction on n. ■

Corollary 6. *Let U be a uniserial basis of a module A, and w a finitely generated submodule of A. Then there exist unique pairwise incomparable elements $u_1, \ldots, u_m$ of U such that*

- $w \subset u_1 + \cdots + u_m$
- *if $V \subset U$ and $w \subset \sum_{v \in V} v$, then for each i there exists $v \in V$ such that $u_i \subset v$.*

Proof. As U generates A, and w is finitely generated, w is contained in some finite sum of elements of U. Choose $u_1, \ldots, u_m$, minimizing the sum of the lengths, such that $w \subset u_1 + \cdots + u_m$. Clearly the u_i are pairwise incomparable. As w is finitely generated, we may assume that V is finite in the second condition. From Theorem 5 we get

$$w \subset \sum_{i=1}^{m} u_i \cap \sum_{v \in V} v = \sum_{i,v} u_i \cap v$$

so if the uniserial module u_i were not contained in any v, we could either eliminate u_i or replace it by a proper submodule, decreasing the sum of the lengths. ■

The set $u_1, \ldots, u_m$ in Corollary 6 is called the **support** of w. It follows that the set of finite sums of elements of U forms a distributive lattice, as does the set of arbitrary sums. Note that a similar phenomenon occurs with the set of all sums of summands in a given direct sum decomposition of a module

The idea of a uniserial basis generalizes the notion of a T-basis [4] (or p-basis [19]) from the theory of abelian p-groups. A p-**basis** for an abelian p-group A is a subset X of A so that $px \in X$ whenever $x \in X$ and $px \neq 0$, and such that each element of A has a unique representation as a sum of elements of X (possibly empty) with coefficients in $\{1, \ldots, p-1\}$.

Theorem 7. *Let A be a p-group. If X is a p-basis of A, then $\{\langle x \rangle : x \in X\}$ is a uniserial basis of A. Conversely, if U is a uniserial basis of A, then there exists a p-basis X of A such that $U = \{\langle x \rangle : x \in X\}$.*

Proof. Let X be a p-basis for A. Then $\{\langle x \rangle : x \in X\}$ clearly consists of nonzero uniserial subgroups, closed under nonzero subgroups and generating A. It remains to show irredundance. Suppose that $\langle x_0 \rangle \subset \sum_{i=1}^{m} \langle x_i \rangle$. Then x_0 can be written as a linearly combination of nonzero elements $p^{s_i} x_i$, with coefficients in $\{1, \ldots, p-1\}$, and $i \in \{1, \ldots, m\}$. By uniqueness of such representations, $x_0 = p^{s_i} x_i$ for some i, that is, $\langle x_0 \rangle \subset \langle x_i \rangle$.

To establish the converse, choose one generator for each subgroup of order p in U. Then choose one generator for each subgroup of order p^2 in U such that p times that generator is equal to the chosen generator of its subgroup of order p, and so on. We will show that the totality X of these generators form a p-basis of A.

Clearly each element of A can be written as a sum of elements of X with coefficients in $\{1, \ldots, p-1\}$. Suppose we do this in two ways. Choose $x \in X$ to be an element of maximum order among those that appear in either representation. If the coefficients of x in the two representations were different, then their difference would be prime to p, so $\langle x \rangle \subset \langle y \rangle$ for some $y \neq x$ in X by irredundance. By the choice of x, this says $\langle x \rangle = \langle y \rangle$, which implies $x = y$ by the construction of X. So the coefficient of x must be the same in each representation. By induction, all the coefficients are the same. ■

Notice that by thinking of p-bases as uniserial bases, we can suppress reference to a prime p and talk about uniserial bases in a torsion group. Clearly a uniserial basis in a torsion group consists of a p-basis in each primary component.

The external and internal characterizations of simply presented modules are the same, as in the classical case of p-groups.

Theorem 8. *A $\mathcal{U}$-module is simply presented if and only if it admits a uniserial basis.*

Proof. Suppose u is a $\mathcal{U}$-forest. We must show that $S(u)$ admits a uniserial basis. By Theorem 4 the maps $u_x \to S(u)$ are monic. Let U be the set of images in $S(u)$ of the nonzero uniserials u_x, which we will identify with the u_x themselves. Because $u_{\pi x}$ maps isomorphically onto $\operatorname{rad} u_x$, it is clear that nonzero submodules of modules in U are in U. Because $S(u)$ is the colimit of the u_x's, it is generated by U.

To show irredundance, suppose $u_{x_0} \subset u_{x_1} + \cdots + u_{x_n}$. Let $u_x \to S(u)/u_{\pi x_0}$ be the natural map $u_x \to u_x/u_{\pi x_0}$ if x is a descendant of x_0, and the zero map otherwise. These maps are compatible with the maps $u_{\pi x} \to u_x$, so they define a map $\varphi : S(u) \to S(u)/u_{\pi x_0}$. As $\varphi(u_{x_0}) = u_{x_0}/u_{\pi x_0} \neq 0$ it follows that $\varphi(u_{x_i}) \neq 0$ for some $i > 0$. So x_i is a descendant of x_0, whence $u_{x_0} \subset u_{x_i}$.

Conversely, suppose U is a uniserial basis for the $\mathcal{U}$-module A. Then U, together with the zero submodule of A, has the structure of a $\mathcal{U}$-forest by Property (1) of a uniserial basis. We will show that $S(U) \cong A$. The inclusion maps from elements of U to A induce a map $S(U) \to A$ which is onto by Property (2) of a uniserial basis. It remains to show that the kernel K of the map is zero. Because of AB-5, it suffices to show that $K \cap (u_{x_1} + \cdots + u_{x_n}) = 0$ in $S(U)$. We may assume, by induction, that $K \cap (\operatorname{rad} u_{x_1} + u_{x_2} + \cdots + u_{x_n}) = 0$. If $T = K \cap (u_{x_1} + \cdots + u_{x_n})$ is not zero, then

$$T + \operatorname{rad} u_{x_1} + u_{x_2} + \cdots + u_{x_n} = u_{x_1} + \cdots + u_{x_n}$$

because $(u_{x_1} + \cdots + u_{x_n})/(\operatorname{rad} u_{x_1} + u_{x_2} + \cdots + u_{x_n})$ is simple, so

$$T + u_{x_2} + \cdots + u_{x_n} = u_{x_1} + \cdots + u_{x_n}$$

because radical submodules are nongenerators. This equation, viewed in A, says that $u_{x_1} \subset u_{x_2} + \cdots + u_{x_n}$, so, by irredundance, x_1 is an ancestor of some x_i. Thus $u_{x_1} \subset u_{x_2} + \cdots + u_{x_n}$ in $S(U)$, and we are done by induction on n. ■

4. Height

The **height** of an element of a torsion forest is ∞ if it has an infinite chain of descendants. Otherwise the height is an ordinal, defined inductively by

$$\operatorname{ht} y = \sup\{\operatorname{ht} x + 1 : y = \pi x\}$$

The uniserial submodules of any module A form a torsion forest by defining πx to be the radical of x. The **height** of a uniserial submodule is its height in that forest.

Recall that a $\mathcal{U}$-module is called a **tag module** if $\mathcal{U}$ is a tag class.

Let A be a module. For each ordinal α, define $\rho^\alpha A$ to be the submodule generated by the uniserial submodules of A of height at least α. If A is a tag module, then $\rho^\alpha A$ is the iterated radical of A. Before showing this we characterize the uniserial submodules of $\rho^\alpha A$.

Theorem 9. *Let A be a tag module. If u is a uniserial submodule of A, then $u \subset \rho^\alpha A$ if and only if* $\operatorname{ht} u \geq \alpha$.

Proof. The "if" part follows directly from the definition of $\rho^\alpha A$. The "only if" part is proved by induction on α. If $u \subset p^\alpha A$, then $u \subset \sum_{i=1}^n u_i$ with $\operatorname{ht} u_i \geq \alpha$ by AB-5. The induction step is easy at limits and 0, so suppose $\alpha = \beta + 1$. We can find uniserials v_i, all of height at least β, so that $u_i \subset \operatorname{rad} v_i$. Hence

$$u \subset \sum_{i=1}^n \operatorname{rad} v_i = \operatorname{rad} \sum_{i=1}^n v_i = \operatorname{rad} \bigoplus_{j=1}^m w_j = \bigoplus_{j=1}^m \operatorname{rad} w_j$$

where w_j has height at least β, by induction. Replacing the w_j by submodules, we may assume that u projects onto $\operatorname{rad} w_j$ for each j, and, because u is uniserial, that u projects isomorphically onto $\operatorname{rad} w_1$. Let π_i denote the projection of $\bigoplus_{j=1}^m \operatorname{rad} w_j$ on w_i, and let $\theta : \operatorname{rad} w_1 \to u$ be the inverse π_1 restricted to u. For $j = 2, \ldots, m$ we have onto maps $f_j = \pi_j \theta : \operatorname{rad} w_1 \to \operatorname{rad} w_j$. Consideration of the diagram

$$\begin{array}{ccc} \dfrac{\operatorname{rad} w_1}{\ker f_j} & \subset & \dfrac{w_1}{\ker f_j} \\ \updownarrow & & \\ \operatorname{rad} w_j & \subset & w_j \end{array}$$

shows that the f_j extend to maps from w_1 into w_j. Consider the complementary summand w_1' of $\bigoplus_{j=2}^m w_j$ in $\bigoplus_{j=1}^m w_j$ given by $w_1' = (1 + \sum_{j=2}^n f_j) w_1$. We have

$$\operatorname{rad} w_1' = (1 + \sum_{j=2}^n f_j) \operatorname{rad} w_1 = (1 + \sum_{j=2}^n \pi_j \theta) \operatorname{rad} w_1 = (1 + (1 - \pi_1)\theta) \operatorname{rad} w_1 = \theta \operatorname{rad} w_1 = u$$

so w_1' is a uniserial submodule of A that contains u properly and has height at least β (by induction). ■

So height can be defined in terms of the submodules $\rho^\alpha A$: a uniserial submodule u has ordinal height α if and only if it is in $\rho^\alpha A$ but not in $\rho^{\alpha+1} A$, and it has height ∞ if it is in $\rho^\alpha A$ for all ordinals α. As a consequence, if B is a finite length submodule of A, then there are only finitely many heights realized by uniserial submodules of B.

Corollary 10. *If A is a tag module, then ρA is the radical of A, and $\rho^{\alpha} A$ is the iteration of ρ on A for each ordinal α.*

Proof. For the first part, clearly ρA is contained in the radical of A, so it suffices to show that $A/\rho A$ is semi-simple. But A is generated by uniserials, and ρA contains all of their radicals, so $A/\rho A$ is generated by simples.

For the second part, Theorem 9 shows that $\rho^{\alpha} A = \bigcap_{\beta<\alpha} \rho^{\beta} A$ for α a limit ordinal, as $\bigcap_{\beta<\alpha} \rho^{\beta} A$ is a tag module. So it suffices to show that $\rho(\rho^{\alpha} A) = \rho^{\alpha+1} A$. But a uniserial submodule u of A has height at least $\alpha + 1$, if and only if u is properly contained in a uniserial module of A of height at least α, that is, has height at least 1 in $\rho^{\alpha} A$. ■

If A is a submodule of B, then we define $A(\alpha)$ to be

$$A(\alpha) \equiv A \cap \rho^{\alpha} B.$$

A **partial uniserial basis** of a module is a uniserial basis U for a submodule A such that $A(\alpha) = \sum_{u \in U} u(\alpha)$ for each ordinal α. If we wish to construct a uniserial basis of a tag module, we must make sure that the subset we construct at each stage, when closed under nonzero submodules, is a partial uniserial basis.

Theorem 11. *Let U be a subset of a uniserial basis of a tag module. If U is closed under nonzero submodules, then U is a partial uniserial basis.*

Proof. Let U be a subset of the uniserial basis V. Clearly U is a uniserial basis for the submodule $\sum_{u \in U} u$. Let α be an ordinal. We will show that any uniserial $w' \subset \sum_{u \in U} u$ of height at least α is contained in $\sum_{u \in U} u(\alpha)$. If $\alpha = \beta + 1$, then $w' = \operatorname{rad} w$ where $w \subset \sum_{v \in V} v(\beta)$, by induction on α (applied to V). So

$$w' = \operatorname{rad} w \subset \operatorname{rad} \sum_{v \in V} v(\beta) = \sum_{v \in V} \operatorname{rad} v(\beta) = \sum_{v \in V} v(\alpha).$$

As V is a uniserial basis, Theorem 5 says that

$$w' \subset \sum_{u \in U} u \cap \sum_{v \in V} v(\alpha) = \sum_{\substack{u \in U \\ v \in V}} u \cap v(\alpha) \subset \sum_{u \in U} u(\alpha).$$

If α is a limit, then $w' \subset \sum_{u \in U} u(\beta)$ for each $\beta < \alpha$. Let $v_1, \ldots, v_m$ in V be the support of w' (Corollary 6). Then for each i, and $\beta < \alpha$, there exists u in U such that $v_i \subset u(\beta)$. So $v_i \in U$ and $v_i = v_i(\alpha)$. ■

We will show that any partial uniserial basis in a countably generated tag module extends to a uniserial basis, so each countably generated tag module is simply presented. This generalizes the corresponding theorem for p-groups given by Hunter and Walker in [6]. The proof breaks up into three lemmas.

Lemma 12. *Let A be a tag module, U a finite partial uniserial basis for A with span K, and v a nonzero uniserial submodule of A. Then $U \cup \{v\}$ is a partial uniserial basis if*

- $\operatorname{rad} v = 0$ *or* $\operatorname{rad} v \in U$,
- *if* $\alpha = \operatorname{ht} v$, *then v is not contained in* $K + A(\alpha+1)$, *where we set* $A(\infty+1) = 0$.

Proof. To show that $U \cup \{v\}$ is a uniserial basis for $K + v$, it suffices to show that $U \cup \{v\}$ is irredundant. As v is not contained in K, it suffices to consider

$$u_0 \subset u_1 + u_2 + \cdots + u_m + v$$

with u_i in U. If

$$u_0 \subset u_1 + u_2 + \cdots + u_m + \operatorname{rad} v$$

then we are done by the irredundance of U. Otherwise $u_0 \supset v$ modulo $u_1 + u_2 + \cdots + u_m + \operatorname{rad} v$, so $v \subset K$, contrary to assumption. So $U \cup \{v\}$ is irredundant.

Let $L = K + v$. To show that $U \cup \{v\}$ is a partial p-basis for A, it suffices to show that $L(\gamma) \subset K(\gamma) + v(\gamma)$ for each ordinal γ. If $\gamma \leq \alpha$, then $v \subset A(\gamma)$, so $L(\gamma) = K(\gamma) + v$ by the modular law. So suppose $\gamma > \alpha$. As $v(\alpha+1) = \operatorname{rad} v$, the only question is whether $L(\alpha+1) \subset K$.

As $L(\alpha+1) \cap K(\alpha) = K(\alpha+1)$ we have

$$\frac{L(\alpha+1)}{K(\alpha+1)} \subset \frac{L(\alpha)}{K(\alpha)}.$$

But the latter is simple, being the image of v, so either $L(\alpha+1) + K(\alpha) = L(\alpha)$, which cannot be because v is not contained in $K + A(\alpha+1)$, or $L(\alpha+1) = K(\alpha+1) \subset K$, which is what we wanted to show. ∎

Lemma 13. *Let A be a tag module and U a finite partial uniserial basis for A. Suppose $u \in U$ has height β. If $\alpha < \beta$, or $\alpha = \beta = \infty$, then there exists a uniserial submodule $v \subset A(\alpha)$ such that* $\operatorname{rad} v = u$ *and $U \cup \{v\}$ is a partial uniserial basis.*

Proof. If $\alpha < \beta$, and β is not a limit, then we may assume $\alpha + 1 = \beta$. Because U is finite, if β is a limit, then we may increase α until there is no element of U with height in the half open interval $[\alpha, \beta)$.

Let $K = \sum_{w \in U} w$. From the definition of height, there is a uniserial $v \subset A(\alpha)$ such that $\operatorname{rad} v = u$. By increasing α we may assume that $\alpha = \operatorname{ht} v$. If v is not contained in $K + A(\alpha+1)$, then $U \cup \{v\}$ is a partial uniserial basis by Lemma 12.So we may assume that $v \subset K + A(\alpha+1)$.

Intersecting with $A(\alpha)$ and applying the modular law, $v \subset K(\alpha) + A(\alpha+1)$, so $K(\alpha) \neq K(\alpha+1)$. Thus there is an element of U of height α, so we may assume $\alpha + 1 = \beta$. Intersecting $u \subset \operatorname{rad} K(\alpha) + A(\beta+1)$ with K and using the modular law, $u \subset \operatorname{rad} K(\alpha) + K(\beta+1)$ so $u \subset \operatorname{rad} K(\alpha)$ by the irredundance of U. Thus $u \subset \sum_{w\in U} \operatorname{rad} w(\alpha)$, so $u \subset \operatorname{rad} w(\alpha)$ for some w in U, and we can pick v in U. ■

Lemma 14. *Let A be a tag module, U a finite partial uniserial basis for A, and w a uniserial submodule of A. Then there exists a finite partial uniserial basis U' containing U such that w is in the span of U'.*

Proof. By the previous lemma, we may assume that each node of U of height $\alpha+1$ has a child in U of height α. Let $K = \sum_{u\in U} u$ denote the span of U. If $w \subset K$, set $U' = U$, so we may assume that w is not contained in K. By induction on the length of w, we may assume that $\operatorname{rad} w \subset K$, say $\operatorname{rad} w \subset t_1 + \cdots + t_m$ with $t_i \in U$ of height at least that of $\operatorname{rad} w$. Because there are only finitely many heights of uniserial submodules of $K + w$ (see the remark before Corollary 10), we may also assume that w has maximum height among those uniserials w' such that $K + w' = K + w$.

Let $\alpha = \operatorname{ht} w$. The maximality condition on the height of w is equivalent to the condition that w not be contained in $K + A(\alpha+1)$. For if $w \subset K + A(\alpha+1)$, then the simple module $(K+w)/K$ is a submodule of $(K + A(\alpha+1))/K$, so there is a uniserial $w' \subset A(\alpha+1)$ such that $K + w' = K + w$.

If w is simple, set $U' = U \cup \{w\}$. This is a finite partial uniserial basis by Lemma 12. So we may assume that $\operatorname{rad} w \neq 0$.

If $\alpha < \infty$, then $\operatorname{ht} t_i \geq \alpha + 1$. If $\operatorname{ht} t_i = \alpha + 1$, then t_i has a child $t_i' \in U$ of height α. If $\operatorname{ht} t_i > \alpha + 1$, then, by the previous lemma, we can enlarge U to include a child t_i' of t_i of height greater than α without changing the fact that w is not contained in $K + A(\alpha+1)$.

So $\operatorname{rad} w \subset \operatorname{rad}(t_1' + \cdots + t_m') = \operatorname{rad} t_1' + \cdots + \operatorname{rad} t_m'$ whence there is a uniserial $w' \subset t_1' + \cdots + t_m'$ so that $\operatorname{rad} w' = \operatorname{rad} w$. So there is an isomorphism $f : w \to w'$ that is the identity on $\operatorname{rad} w$. The image of w under $1 - f$ is a simple submodule w'' of $w' + w \subset A(\alpha)$ such that $w' + w'' = w' + w$, so $K + w'' = K + w$. So we can set $U' = U \cup \{w''\}$ as before. ■

Theorem 15. *Let A be a countably generated tag module, and U a partial uniserial basis for A. Then U can be extended to a uniserial basis for A.*

Proof. This is a straightforward application of the preceding lemma. ■

Corollary 16. *Every countably generated tag module is simply presented.*

5. Ulm's theorem

Let s be a simple module in $\mathcal{U}$ and α an ordinal. Consider the submodule of $\rho^\alpha A$ generated by submodules u in $\mathcal{U}$ with $u/\operatorname{rad} u \cong s$ and $\operatorname{rad} u \subset \rho^{\alpha+2}A$, modulo $\rho^{\alpha+1}A$, that is

$$\operatorname{Ulm}_A(\alpha, s) = \frac{\sum\{u \subset \rho^\alpha A : u \in \mathcal{U},\ u/\operatorname{rad} u \cong s, \text{ and } \operatorname{rad} u \subset \rho^{\alpha+2}A\}}{p^{\alpha+1}A}.$$

This is a submodule of $\rho^\alpha A/\rho^{a+1}A$, hence is semisimple—in fact, it is clearly a direct sum of copies of s. We also let

$$\operatorname{Ulm}_A(\infty, s) = \sum\{u \in \rho^\infty A : u \in \mathcal{U} \text{ and } u \cong s\}.$$

The α^{th} s-**Ulm invariant** $U_A(\alpha, s)$ is the number of copies of s in that direct sum—the rank of $\operatorname{Ulm}_A(\alpha, s)$.

Alternatively, let $A[s]$ denote the s-**socle** of A—the sum of the copies of s in A. Then the α^{th} s-Ulm invariant of A is equal to the rank of $\rho^\alpha A[s]/\rho^{\alpha+1}A[s]$. In fact, $\rho^\alpha A[s]/\rho^{\alpha+1}A[s]$ is isomorphic to $\operatorname{Ulm}_A(\alpha, s)$ under the map induced by the inclusion

$$\rho^\alpha A[s] \subset \sum\{u \subset \rho^\alpha A : u \in \mathcal{U},\ u/\operatorname{rad} u \cong s, \text{ and } \operatorname{rad} u \subset \rho^{\alpha+2}A\}.$$

The only problem is showing that the map is onto $\operatorname{Ulm}_A(\alpha, s)$. This is a standard result in abelian group theory, going back to the Kaplansky-Mackey proof of Ulm's theorem. Suppose u is a uniserial submodule of $\rho^\alpha A$ with $u/\operatorname{rad} u \cong s$, and $\operatorname{rad} u \subset \rho^{\alpha+2}A$. Then there exists a uniserial $v \subset \rho^{\alpha+1}A$ such that $\operatorname{rad} u = \operatorname{rad} v$. There is an isomorphism $f : u \to v$ that is the identity on $\operatorname{rad} u$. The map $1_u - f$ taking u into $u + v$ is zero on $\operatorname{rad} u$, hence induces a map from s into $u + v$. Let s' be its image. Clearly $s' + v = u + v$, so, as $v \subset \rho^{\alpha+1}A$, the submodule u is in the image, modulo $\rho^{\alpha+1}A$, of $\rho^\alpha A[s]$.

Consider the example of representations of the quiver

$$\cdots \to 3 \to 2 \to 1$$

by vector spaces over k. There is one simple module for each positive integer n, namely the representation A with $A_n = k$ and $A_m = 0$ for $m \neq n$. The Ulm invariants are then cardinal numbers indexed by an ordinal and a positive integer.

We will restrict our attention to standard $\mathcal{U}$-forests u, where the labels $\sigma_x(u)$ on the nodes are simple modules in $\overline{\mathcal{U}}$. The proof of Ulm's theorem for simply presented tag modules follows the ideas in [13, Proposition 2]. We define the notion of a stripping function $u \to u'$ between $\mathcal{U}$-forests. Such a function simply eliminates some of the edges joining the nodes in u, without decreasing the height of any node. We show that if $u \to u'$ is a stripping function, then $S(u) \cong S(u')$. We observe how to read the Ulm invariants of $S(u)$ from u. Finally, we show that if $S(u)$ and $S(v)$ have the same Ulm invariants, then there exist stripping functions $u \to u' \leftarrow w \to v' \leftarrow v$.

Let u and u' be $\mathcal{U}$-forests. A function $f : u \to u'$ is a **stripping function** if f is a height-preserving and label-preserving bijection such that $f(\pi x) = \pi f(x)$ whenever $\pi f(x) \neq 0$. Note that $f(\pi x)$ may be nonzero when $\pi f(x)$ is zero—exactly the situation where an edge in u is eliminated. Clearly the composition of two stripping functions is a stripping function. A $\mathcal{U}$-forest u is said to be **fully stripped** if each node of height $\alpha + 1$ or ∞ has exactly one child.

Theorem 17. *If u is a $\mathcal{U}$-forest, then there is a stripping function from u to a fully stripped $\mathcal{U}$-forest.*

Proof. For each node of u of height $\alpha + 1$ choose a child of height α, and for each element of height ∞, choose a child of height ∞. Let C be the set of chosen children. Define π' on u by $\pi' x = \pi x$ if πx is defined and has height a limit ordinal, or if x is in C. Otherwise $\pi' x$ is undefined. Let u' be the resulting $\mathcal{U}$-forest, and f the identity map from the nodes of u to the nodes of u'. It is readily seen that f is a stripping function and that u' is fully stripped. ■

The point of a stripping function is that it is a combinatorial operation on a $\mathcal{U}$-forest u that leaves the isomorphism class of the module $S(u)$ unchanged.

Theorem 18. *If $u \to u'$ is a stripping function, then $S(u) \cong S(u')$.*

Proof. We may consider u and u' to consist of the same nodes and labels, but u' lacks some of the edges of u. For x in u, let $e(x)$ denote the smallest nonnegative integer n such that $\pi^n x = 0$, and $e'(x)$ the corresponding integer when we view x as an element of u'. So $e'(x) \leq e(x)$.

By induction on $e(x)$, we can define a function $\theta : u \to u$ such that

- $\operatorname{ht} \theta x \geq \operatorname{ht} x$,
- $\theta x = x$ if $e'(x) = e(x)$,
- If $e(x) > 1$, then $e'(\theta x) > 1$ and $\pi \theta x = \begin{cases} \theta \pi x & \text{if } e'(x) > 1, \\ \pi x & \text{if } e'(x) = 1, \end{cases}$.

It follows, by induction on $e(x)$, that $\pi^{e'(x)}x = \pi^{e'(x)}\theta x$. This node is nonzero unless $\theta x = x$, so $e(\theta x) = e(x)$ and $u_x = u_{\theta x}$ for each x. It follows, by induction on $e'(x)$, that if $e'(x) < e(x)$, then $e'(\theta x) > e'(x)$. So if $n \geq e(x) - e'(x)$, then $\theta^n x = \theta^{n+1}x$. Moreover, each node is contained in a finite set of nodes that is invariant under π and θ.

We will define monic maps $u'_x \to S(u)$. If $e'(x) = e(x)$, then $u'_x = u_x$ and we simply take the natural map $u_x \to S(u)$. So we may assume that $e'(x) < e(x)$, whence $e(x) > 1$. Let $i = e'(x)$ and $y = \pi^i x \neq 0$. Then $\pi^i\theta x = \pi^i x$ and $\pi^j\theta x = \theta\pi^j x \neq \pi^j x$ for $j < i$. Noting that $u_{\theta x}$ is u_x, consider the composite map

$$u_x \xrightarrow{1,-1} u_x \oplus u_{\theta x} \to S(u)$$

The kernel of this map is u_y, so we get a monic map $u'_x \to S(u)$ from the short exact sequence

$$0 \to u_y \to u_x \to u'_x \to 0$$

where the map $u_x \to u'_x$ is the unique epic in $\overline{\mathcal{U}}$.

If $e'(x) > 1$, then the diagram

$$\begin{array}{ccc} u'_x & \to & S(u) \\ \uparrow & \nearrow & \\ u'_{\pi x} & & \end{array}$$

commutes, so there is an induced map $S(u') \to S(u)$.

To show that this last map is an isomorphism, it suffices to show that the induced map $S(v') \to S(v)$ is an isomorphism for v the restriction of u to a finite set of nodes closed under π and θ. As $S(v')$ and $S(v)$ have the same finite length, equal to the number of nonzero nodes in v, it suffices to show that $S(v') \to S(v)$ is onto. For this, it is enough to show that each v_x is contained in the image of $S(v')$. If $\theta x = x$, then v_x is the image of v'_x, so we may assume, by induction, that $v_{\theta x}$ is contained in the image of $S(v')$. The image of v'_x in $S(v)$ is the image of the composite map

$$v_x \xrightarrow{1,-1} v_x \oplus v_{\theta x} \to S(v)$$

so it, together with (the image of) $v_{\theta x}$, generates a submodule containing v_x. ∎

As in the case of abelian p-groups [13, Proposition 2], we can read off the Ulm invariants of the simply presented module $S(u)$ from the $\mathcal{U}$-forest u.

Theorem 19. *Let u be a standard $\mathcal{U}$-forest based on a forest $F(u)$. For each simple s in $\overline{\mathcal{U}}$, let*

$$F_s(u) = \{x \in F(u) : \sigma_x(u) = s\}.$$

If we let $\infty + 1 = \infty$*, then the Ulm invariant* $U_{S(u)}(\alpha, s)$ *is equal to the cardinality of*

$$F_{s,\alpha}(u) = \{x \in F_s(u) : \operatorname{ht} x = \alpha, \text{ and } \operatorname{ht} \pi x > \alpha + 1 \text{ or } \pi x = 0\}$$

plus the sum, for each node of height $\alpha+1$ *whose children are in* $F_s(u)$*, of the number, less* 1*, of its children of height* α *(there is always at least one such child).*

Proof. If $u \to v$ is a stripping function, then clearly these cardinalities are the same for u and v, so we may assume that u is fully stripped. By Theorem 11

$$\rho^\alpha S(u) = \sum_{\operatorname{ht} x \geq \alpha} u_x,$$

so there is a natural isomorphism

$$U_{S(u)}(\alpha, s) \cong \bigoplus_{x \in F_{s,\alpha}(u)} \frac{u_x}{\operatorname{rad} u_x}.$$

■

We will use this as the *definition* of the **Ulm invariants of a $\mathcal{U}$-forest** u.

It remains to show the combinatorial part of Ulm's theorem: if two $\mathcal{U}$-forests u and v have the same Ulm invariants, then there exist stripping functions

$$\begin{array}{ccccccccc} u & & & & w & & & & v \\ & \searrow & & \swarrow & & \searrow & & \swarrow & \\ & & u' & & & & v' & & \end{array}$$

This is called the **W-theorem**. We will indicate why the proof for unlabeled forests goes through intact for labeled forests.

One considers **T-functions** between fully stripped $\mathcal{U}$-forests u and v. These are one-to-one height and label preserving correspondences $f : u \to v$ with the property that $f\pi x = \pi f x$ unless $x \in F_{s,\alpha}(u)$ and $fx \in F_{s,\alpha}(v)$ for $s = \sigma_x(u)$ and $\alpha = \operatorname{ht} x$. A T-function is uniquely specified by giving one-to-one correspondences $F_{s,\alpha}(u) \simeq F_{s,\alpha}(v)$, so if u and v have the same Ulm invariants, then there is a T-function between them. Clearly any stripping function is a T-function, as is the inverse of a stripping function.

The W-theorem is proved in the unlabeled case [13],[5] by constructing the intermediate forests and stripping functions from a T-function $f : u \to v$. The nodes on all five forests may be considered the same, with the nodes of u and v identified via f. In the labeled case, this identification respects the labels because we required f to be label-preserving. The same construction works in the labeled case because the parent, if any, of a node in an intermediate forest is either its parent in u or its parent in v, so those forests will also be standardized $\mathcal{U}$-forests.

6. Zippin's theorem

Zippin's theorem states the conditions under which there exists a countable p-group with prescribed Ulm invariants. More generally, the term is used to describe necessary and sufficient conditions for the existence of something with prescribed Ulm invariants. For countable $\mathcal{U}$-forests, Zippin's theorem is not too hard to formulate and prove.

Let $\mathcal{U}$ be a tag class and S the set of simple modules in $\overline{\mathcal{U}}$. For $s, s' \in S$, and $n \in \omega$, we write $s' = s + n$, or $s = s' - n$, if $s' \cong w/\rho w$ and $s \cong \rho^n w/\rho^{n+1} w$ for some $w \in \mathcal{U}$. So if u is a standardized $\mathcal{U}$-forest, and $\pi^i x \neq 0$, then $\sigma_x(u) = \sigma_{\pi^i x}(u) + i$.

Theorem 20. *Let $\mathcal{U}$ be a tag class and S the set of simple modules in $\overline{\mathcal{U}}$. Let λ be a countable ordinal. Let f a function from $(\lambda \cup \{\infty\}) \times S$ to $\omega \cup \{\aleph_0\}$. Then there is a standardized $\mathcal{U}$-forest u with Ulm invariants $U_u(\alpha, s) = f(\alpha, s)$ if and only if f satisfies the conditions:*

- *If $f(\alpha, s) > 0$, then $s + n$ exists for all $n \leq \alpha$.*
- *If α is a limit ordinal, $\beta < \alpha$, and $f(\alpha + m, s) > 0$, then $f(\gamma, s + m + 1) > 0$ for some $\gamma \in [\beta, \alpha)$.*

We can also write the latter condition as

$$f(\alpha + m, s) \leq \inf_{\beta < \alpha} \sum_{\gamma \in [\beta, \alpha)} f(\gamma, s + m + 1)$$

for α a limit ordinal.

Proof. Let u be a standardized $\mathcal{U}$-forest, and $f(\alpha, s) = U_u(\alpha, s)$. If $f(\alpha, s) > 0$ and $n \leq \alpha$, then there is $x \in u$ with $\operatorname{ht} x = \alpha$ and $\sigma_x = s$. Write $x = \pi^n y$. Then $\sigma_y = s + n$, verifying the first condition. If $f(\alpha + m, s) > 0$, then there is $x \in u$ with $\operatorname{ht} x = \alpha + m$ and $\sigma_x = s$. Write $x = \pi^m y$ with $\operatorname{ht} y = \alpha$, and $y = \pi z$ with $\operatorname{ht} z = \gamma \geq \beta$. Then z represents an element of $U_u(\gamma, s + m + 1)$, verifying the second condition.

Conversely, suppose f satisfies the conditions of the theorem. We have to build a $\mathcal{U}$-forest u. The set S is partitioned into countable sets in which any two elements s and s' are related by $s = s' \pm n$ for some $n \in \omega$. Because of the nature of the conditions of the theorem, we may assume that all elements of S are so related—in particular, that S is countable.

For each $s \in S$ such that $s + n$ exists for all n, construct $f(\infty, s)$ infinite branchless trees with roots labeled s. Consider triples (α, s, m) in $\lambda \times S \times \omega$, such that $s - m$

exists, and $s+n$ exists for all $n \leq \alpha$. For each such triple let $N_{(\alpha,s,m)}$ be a set of $f(\alpha+m, s-m)$ nodes. These will be the remaining nodes of our forest u. Note that u is countable. If $x \in N_{(\alpha,s,m)}$ we write $\operatorname{ht} x = \alpha$ and $\sigma_x = s$, and say that x has **index** m.

We construct the partial map π on the remaining nodes of u in two stages. As the sets $N_{(\alpha,s,m+1)}$ and $N_{(\alpha+1,s-1,m)}$ have the same number of elements, namely $f(\alpha+m+1, s-m-1)$, we can define $\pi : N_{(\alpha,s,m+1)} \to N_{(\alpha+1,s-1,m)}$ to be any one-to-one correspondence. So now π is defined on all nodes of index different from 0, and π lowers the index by 1 and raises the height by 1. Because of the latter, and because there is no branching, we have introduced no Ulm invariants as yet. Note that the cardinality of $N(\alpha, s, 0)$ is $f(\alpha, s)$. On the nodes $x \in N(\alpha, s, 0)$, either π will remain undefined, or πx will be a node of limit height. So we will get a fully stripped forest with the right Ulm invariants. What remains is to arrange for the actual height of an element of $N_{(\alpha,s,m)}$ to be equal to α in the forest u.

Let $(x_1, \alpha_1), (x_2, \alpha_2), \ldots$ be an enumeration of all pairs (x, α) with x a node of limit height, and $\alpha < \operatorname{ht} x$. Initially, all the nodes of index 0 are unmarked. At step i, if $x_i \in N_{(\beta,s,m)}$, choose an unmarked $y \in N_{(\gamma,s+1,0)}$ for some $\gamma \in [\alpha_i, \beta)$, set $\pi y = x_i$ and mark y. Also mark $\pi^k x_i$, for k the largest nonnegative integer such that $\pi^k x_i$ is defined, so we will end up with a *torsion* forest. Note that $\pi^k x_i \in N_{(\delta,s-k,0)}$ for some $\delta \geq \beta + k$. At each step only a finite number of nodes of index 0 have been marked, so we can always proceed. ■

Zippin's theorem in the general case seems a little tricky even to formulate, and is left as an open question.

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