Pointwise differentiability

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Abstract

We study what can be done with pointwise properties as opposed to uniform properties on compact intervals.

1 Introduction

Bishop defines continuity and differentiability of functions in terms of uniform behavior on compact intervals. So "continuity" is really uniform continuity on compact intervals, and "differentiability" entails uniform convergence of the difference quotient on compact intervals. Bishop says that "the concept of a pointwise continuous function is not relevant." This is a sentiment with which I have a lot of sympathy. Nevertheless, the purpose of this paper is to see what can be done with the pointwise definitions.

One feature of the pointwise approach is the prominent role played by countable choice. If, like me, you prefer to reject countable choice, then you seem to be forced to use Bishop's approach. If, indeed, Bishop's approach is the right one, then this is an argument for rejecting countable choice, for by so doing you are forced to do the right thing.

2 The mean value theorem

A function f, defined on a nontrivial interval $[a, b]$, has a **derivative** m at a point x in $[a, b]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ so that for all y in $[a, b]$, if $|x - y| \leq \delta$, then $|f(x) - f(y) - m(x - y)| \leq \varepsilon |x - y|$. This is the pointwise version of the Bishop definition.

If $x \neq y$ are real numbers, define the difference quotient by

$$\Delta_f(x,y) = \frac{f(x) - f(y)}{x - y}.$$

If I is the interval $[x, y]$, we also write $\Delta_f(I)$ for $\Delta_f(x, y)$. Note that $\Delta_f(x, y)$ is symmetric in x and y.

An immediate consequence of these definitions is

Lemma 1 *Let f be defined on $[a, b]$ and differentiable at x. If I_n is a sequence of subintervals of $[a, b]$ containing x, whose lengths converge to 0, then $\Delta_f(I_n) \to f'(x)$.*

Proof. Suppose $a \leq c \leq x \leq d \leq b$. It suffices to show that if $d - c > 0$ is small, then so is $\Delta_f(c, d) - f'(x)$. If $d - c < \delta$, then

$$|f(x) - f(c) - f'(x)(x - c)| \leq \varepsilon(x - c)$$

and $|f(d) - f(x) - f'(x)(d - x)| \leq \varepsilon(d - x)$, so

$$|f(d) - f(c) - f'(x)(d - c)| \leq \varepsilon(d - c),$$

so $|\Delta_f(c, d) - f'(x)| \leq \varepsilon$. ∎

A sort of converse to this lemma is that if $\Delta_f(x, y) \to d$ as $y \to x$, and if f is strongly extensional at x, then $f'(x) = d$.

With this lemma in hand, we can prove a mean value theorem (using countable choice) that suffices for the usual applications. In the theorem, the indicated supremum need not exist, but it is clear what it means for it to exceed a real number.

Theorem 2 *Let f be differentiable on the nontrivial interval $[a, b]$. Then*

$$\sup_{x \in [a,b]} f'(x) \geq \Delta_f(a, b).$$

Proof. The conclusion means that, for each $\varepsilon > 0$, there exists x in $[a, b]$ such that $f'(x) \geq \Delta_f(a, b) - \varepsilon$. We will construct a sequence of compact intervals $[a, b] = I_0, I_1, I_2, \ldots$ such that I_n is one of the two pieces obtained by bisecting I_{n-1}, and $\Delta_f(I_n) > \Delta_f(I_{n-1}) - \varepsilon/2^n$. It follows from the lemma that $f'(x) \geq \Delta_f(a, b) - \varepsilon$ for $x \in \bigcap I_n$.

Set $I_0 = [a, b]$. If I_{n-1} has been constructed, let L and R be its left and right halves. Then $\Delta_f(L) + \Delta_f(R) = 2\Delta_f(I_{n-1})$, so either $\Delta_f(L)$ or $\Delta_f(R)$ exceeds $\Delta_f(I_{n-1}) - \varepsilon/2^n$. Therefore we can choose I_n as desired. ∎

Corollary 3 *Let f be differentiable on the nontrivial interval $[a, b]$. If $f'(x) \geq 0$ for each x in $[a, b]$, then $f(b) \geq f(a)$. If, in addition $f'(x) > 0$ for some $x \in [a, b]$, then $f(b) > f(a)$.*

Kushner [2, Theorem 1, page 202] shows that if $f'(x) > 0$ for $x \in (a, b)$, then $f(b) > f(a)$ on $[a, b]$. (Keep in mind that, for Kushner, all functions are pointwise continuous by Tseĭtin's theorem. Actually, he appeals to the nondiscontinuity theorem, rather than to Tseĭtin's theorem.) The key argument is his proof of [2, Lemma 1, page 200], that if f is increasing in a neighborhood of each point of (a, b), then f is increasing on $[a, b]$. This argument is essentially the same as that given in the proof of Theorem 2 above.

Strictly increasing pointwise continuous functions satisfy the intermediate value theorem (no choice necessary).

Theorem 4 *Let f be a strictly increasing, pointwise continuous function on the nontrivial interval $[a, b]$. If $f(a) \leq c \leq f(b)$, then there exists x in $[a, b]$ such that $f(x) = c$.*

Proof. Let $a = x_0, x_1, \ldots, x_n = b$ be equally spaced points in $[a, b]$. As f is strictly increasing, there exists i such that $f(x_i) \leq c \leq f(x_{i+2})$, and there may be up to three such i, but not four. The set of such intervals $[x_i, x_{i+2}]$, as n varies, has the finite intersection property, and contains arbitrarily small intervals. Thus, by the completeness of $[a, b]$, the intersection of this set contains a unique number x. The conclusion follows from the continuity of f at x. ■

3 Example

Let $q_0 < q_1 < q_2 < \cdots$ be a sequence of rational numbers in $[0, 1]$ that is eventually bounded away from any real number. Define $f(x)$, for $q_n \leq x \leq q_{n+1}$, to be positive, continuous, and piecewise linear with $f(q_n) = f(q_n + 1) = 0$, $f((q_n + q_{n+1})/2) \geq n$ and

$$\int_{q_n}^{q_{n+1}} f(x)dx = \frac{1}{2^{n+1}}.$$

The function $F(x) = \int_0^x f(t)dt$ exists for each x, and $F'(x) = f(x)$. The function f is pointwise continuous but not uniformly continuous on $[0, 1]$ (it is unbounded), and the function F has a continuous pointwise derivative on $[0, 1]$, but does not have a derivative in the sense of Bishop.

In what sense does that integral exist? The supremum of $\int_0^x g(t)dt$ exists, where g ranges over uniformly continuous nonnegative functions that are dominated by f. It exists in the sense of Bishop-Cheng: There is a sequence f_n of uniformly continuous functions with compact support such that $\sum_n \int_0^x |f_n(x)|\, dx$ converges, and $f(x) = \sum_n f_n(x)$ whenever $\sum_n |f_n(x)|$ converges (which it always does). (See [1, page 222].) Indeed, f_n is the restriction of f to $[q_0, q_n]$.

References

[1] Bishop, Errett and Douglas Bridges, *Constructive analysis*, Springer-Verlag 1985

[2] Kushner, Boris A., Lectures on constructive mathematical analysis, Amer. Math. Soc. 1984 (Moscow 1973)