Subrings of zero-dimensional rings

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It is an honor to write an article in a volume dedicated to the work of Robert Gilmer.

1 Introduction

When Sarah Glaz, Bill Heinzer and the junior author of this article approached Robert with the idea of editing a book dedicated to his work, we asked him to give us a list of his work and to comment on it to the extent he felt comfortable. As usual, he was extremely thorough in his response. When the authors of this article began to consider what topic we wanted to write about, we were impressed by Robert's comment that he was particularly pleased with his series of papers with Bill on the embeddability of a ring in a zero-dimensional ring. So we decided to write about that. Our task was complicated by the fact that Robert had already written several excellent expository papers on the subject [8, 9, 10].

The problem is easy to state.

Problem 1. Find necessary and sufficient conditions on a ring R for it to be embeddable in a ring of (Krull) dimension zero.

Note that it is not required that the total quotient ring of R be zero dimensional, only that R can be embedded in some zero-dimensional ring. Although interesting on its face, the problem appears quite innocuous. Indeed, when Robert and Bill began to look at it seriously, it had already been solved by Arapović [3, Theorem 7]. More specifically, Arapović proved the following result.

Theorem 1 (Arapović). *A ring R is embeddable in a zero-dimensional ring if and only if R has a family of primary ideals $\{Q_\lambda\}_{\lambda\in\Lambda}$, such that:*

A1. $\bigcap_{\lambda\in\Lambda} Q_\lambda = 0$*, and*

A2. For each $a \in R$, there is $n \in \mathbf{N}$ such that for all $\lambda \in \Lambda$, if $a \in \sqrt{Q_\lambda}$, then $a^n \in Q_\lambda$.

In [16] Bill says, "This result of Arapovic is definitive, but there [remain] a number of questions concerning the existence of zero-dimensional extensions."

Condition A1, as a stand-alone condition on R, simply says that the intersection of *all* primary ideals in R is equal to 0, while A2 is a uniformity condition on the primary ideals of the given family $\{Q_\lambda\}_{\lambda\in\Lambda}$. Moreover, as conditions on the family of ideals Q_λ, condition A1 is inherited by superfamilies and A2 by subfamilies, so there is some sort of balance going on. Notice that each of these conditions makes sense for any family of ideals, not just for a family of primary ideals.

Condition A2 on a family of ideals can be characterized as saying that intersection commutes with radical for any (countable) subfamily.

Theorem 2. *A family $\{Q_\lambda\}_{\lambda\in\Lambda}$ of ideals in a ring R satisfies A2 if and only if for each (countable) subset $\Gamma \subset \Lambda$*

$$\sqrt{\bigcap_{\lambda\in\Gamma} Q_\lambda} = \bigcap_{\lambda\in\Gamma} \sqrt{Q_\lambda}.$$

Proof. If $\{Q_\lambda\}_{\lambda\in\Lambda}$ satisfies A2 and $x \in \bigcap_{\lambda\in\Gamma}\sqrt{Q_\lambda}$, then there exists n such that $x^n \in Q_\lambda$ for each $\lambda \in \Gamma$. Hence $x \in \sqrt{\bigcap_{\lambda\in\Gamma} Q_\lambda}$. The other inclusion always holds. Conversely, given $x \in R$, let $\Gamma = \left\{\lambda \in \Lambda : x \in \sqrt{Q_\lambda}\right\}$. Then the displayed equation says that there is n such that $x^n \in Q_\lambda$ for all $\lambda \in \Gamma$.

Finally, we note that if the displayed equation fails for some subset $\Gamma \subset \Lambda$, then it fails for a countable subset of Γ. Indeed, if $x \notin \sqrt{\bigcap_{\lambda\in\Gamma} Q_\lambda}$, then for each n there exists $\lambda_n \in \Gamma$ such that $x^n \notin Q_{\lambda_n}$, so $x \notin \sqrt{\bigcap_{n=1}^{\infty} Q_{\lambda_n}}$.

2 Zero-dimensional rings

We will need an arithmetic characterization of zero-dimensional rings. Such characterizations take the form of properties satisfied by every element of the ring. The following lemma gives three of the more interesting properties.

Lemma 1. *Let x be an element of a ring R. The following three conditions are equivalent:*

1. *There is an idempotent $e \in R$ such that xe is a unit in Re and $x(1-e)$ is nilpotent.*
2. *There exists n such that $Rx^n = Rx^{n+1}$.*
3. *$Rx + \bigcup_{n=1}^{\infty}(0 : x^n) = R$.*

The idempotent in (1) is unique.

Proof. If x satisfies (1), then $x = xe + x(1-e)$. So $x^n = x^n e$, because $x(1-e)$ is nilpotent, and $rxe = e$ for some $r \in R$, because xe is a unit in Re. Thus $rx^{n+1} = rxx^n e = x^n e = x^n$. If x satisfies (2), then $x^n = rx^{n+1}$, so $x^n(1-rx) = 0$, which says that $1 - rx \in (0 : x^n)$ whence $1 \in Rx + \bigcup_{n=1}^{\infty}(0 : x^n)$. If x satisfies (3), then $rx + a = 1$ where $ax^n = 0$. So $rx^{n+1} = x^n$. Let $e = r^n x^n$. Then $e^2 = r^n\left(r^n x^{2n}\right) = r^n x^n = e$. Moreover, $rxe = r^n rx^{n+1} = r^n x^n = e$ and $(x(1-e))^n = x^n(1-e) = x^n(1-r^n x^n) = x^n - r^n x^{2n} = 0$.

To see that the idempotent in (1) is unique, suppose $e, f \in R$ are idempotents such that xe is a unit in Re and $x(1-f)$ is nilpotent. Then $e = rxe = rxfe + rx(1-f)e$. As $x(1-f)$ is nilpotent, $e = r^n x^n fe$ whence $e \leq f$. By symmetry, the idempotent in (1) is unique.

Notice that each of the three conditions of Lemma 1 asserts that a certain set of natural numbers n is nonempty, and the proof of Lemma 1 shows that these three sets are the same.

Theorem 3. *A ring R is zero dimensional if and only if the conditions of Lemma 1 are satisfied for every $x \in R$.*

Proof. We first prove the contrapositive of the statement that if R is zero dimensional, then condition (3) of the lemma holds for all $x \in R$. If $xR + \bigcup_{n=1}^{\infty}(0 : x^n)$ is a proper ideal, then it is contained in a prime ideal Q of R. Look at the multiplicative system $S = \{x^n y : n = 0, 1, 2, \ldots \text{ and } y \in R \setminus Q\}$. Now $0 \notin S$ because $y \notin Q$ and $Q \supseteq \bigcup_{n=1}^{\infty}(0 : x^n)$. Thus there is a prime ideal P of R that misses S. Since $S \supseteq R \setminus Q$, we have $P \subseteq Q$. Moreover, $x \in Q \setminus P$ so $P \neq Q$ whence P is not maximal.

To finish, we prove the contrapositive of the statement that condition (3) for all x implies that R is zero dimensional. Suppose there exist distinct prime ideals $P \supset Q$ in R and let $x \in P \setminus Q$. Then $\bigcup_{n=1}^{\infty}(0 : x^n) \subset Q$ and $Rx \subset P$, so $Rx + \bigcup_{n=1}^{\infty}(0 : x^n) \subset P$.

Condition (1) seems to be the most perspicuous. The outstanding arithmetic property of zero-dimensional rings is that they have lots of idempotents while the difference between zero-dimensional rings and von Neumann regular rings is that the former have nilpotent elements. Condition (1) is reminiscent of the decomposition of a linear transformation into its semisimple and nilpotent parts. Condition (2) is the descending chain condition on the powers of Rx. A variant of (1) and (2), which Gilmer and Heinzer use, is that some power of Rx is an idempotent ideal (see Lemma 2).

Arapović [1, Theorem 6] showed that R is zero dimensional if and only if R is a total quotient ring and for every $x \in R$ there exists an idempotent e_x such that $x + (1 - e_x)$ is invertible and $x(1 - e_x)$ is nilpotent. Clearly this is the same e_x as in (1). It's not too difficult to show that $e_{xy} = e_x e_y$.

Corollary 1. *The intersection of an arbitrary nonempty set of zero-dimensional subrings of a ring R is zero dimensional. More specifically, if $x \in R$ and $\mathcal{S}_x$ is the set of subrings containing x and satisfying condition (1) of Lemma 1, then $\mathcal{S}_x$ is closed under nonempty intersection.*

Proof. Each $S \in \mathcal{S}_x$ contains an idempotent e such that xe is invertible in the ring Se, and $x(1-e)$ is nilpotent. Clearly xe is also invertible in the ring Re, so if $\mathcal{S}_x$ is nonempty, then $R \in \mathcal{S}_x$. Now the uniqueness of e in condition (1) implies that $e \in \bigcap \mathcal{S}_x$. Moreover, if $S \in \mathcal{S}_x$, then there is an element $s \in S$ such that $sxe = e$ so $se \in S$ is the inverse of xe in the ring Re, that is, the inverse t of xe in the ring Re lies in Se. So $t \in \bigcap \mathcal{S}_x$. But any subring containing x, e, and t is in $\mathcal{S}_x$, so $\mathcal{S}_x$ is closed under nonempty intersection.

The first part of Corollary 1 was proved by Arapović [2, Theorem 7] for R zero-dimensional and by Gilmer and Heinzer [13] in general. Arapović didn't really need the zero-dimensional hypothesis for his proof, as you might expect given that the general theorem is true. The element t in the proof of Corollary 1 is the "pointwise inverse" of x, see [20, Lemma 2] and [15, Lemma 4.3.9], used by Gilmer and Heinzer. It is characterized by the two equations $txt = x$ and $xtx = t$. These equations make sense in any semigroup, not necessarily commutative, see Clifford and Preston [4, §1.9].

Condition (2) of Lemma 1 with $n = 1$ is the defining condition for a von Neumann regular ring. What kinds of rings do you get if you impose condition (2) for, say, $n = 2$?

The ideal $I = \bigcup_{n=1}^{\infty}(0 : x^n)$ in condition (3) is the kernel of the localization map $R \to R_S$ where S consists of the powers of x. So condition (3) says that the localization map $R \to R_S$ is onto for any multiplicatively closed set S. Note also that x is regular in R/I, and that if $\varphi : R \to R'$ takes x to a regular element, then $\varphi(I) = 0$. So condition (3) is equivalent to the condition that every regular homomorphic image of x is invertible. It follows that a ring has dimension zero if and only if every homomorphic image is a total quotient ring.

In [14, Proposition 2.4], Robert and Bill show that for products of local rings, the localization map is onto at any maximal ideal. Of course that's also true for zero-dimensional rings by condition (3). What other rings have that property?

It's not hard to see that condition (3) is equivalent to the condition $xR + \left(\sqrt{0} : x\right) = R$. Indeed

$$xR + \bigcup_{n=1}^{\infty}(0 : x^n) \subset xR + \left(\sqrt{0} : x\right) \subset \sqrt{xR + \bigcup_{n=1}^{\infty}(0 : x^n)}.$$

Both containments above can be strict. The quotient ring of R modulo $xR + \left(\sqrt{0} : x\right)$ is called the **upper boundary** $R^{\{x\}}$ of x in R by Coquand, Lombardi, and Roy in [5]. This allows an elegant inductive definition of (Krull) dimension: $\dim R \leq n$ if $\dim R^{\{x\}} \leq n-1$ for all $x \in R$, the dimension of the trivial ring being set equal to -1.

Arapović's theorem implies, of course, that condition A2 must hold for some family of primary ideals in any zero-dimensional ring. It is an interesting fact that A2 holds for the family of *all* ideals in a zero-dimensional ring, hence also for any family of ideals. Moreover this property characterizes zero-dimensional rings.

Theorem 4. *The following conditions on a ring R are equivalent.*

1. *The ring R is zero dimensional,*
2. *Condition A2 holds for the family of all ideals of R,*
3. *Condition A2 holds for the family of all primary ideals of R.*

Proof. Suppose that R is zero dimensional. For $x \in R$, let n be such that $x^n R = x^{n+1} R$, as in Theorem 3, condition (3). If I is any ideal, and $x \in \sqrt{I}$, then $x^m \in I$ for some m, hence $x^n \in I$ also. So A2 holds for the family of all ideals of R.

Conversely, suppose A2 holds for the family of all primary ideals of R. To show that R is zero dimensional, it suffices, by Theorem 2, to show that if $\dim R > 0$, then there is a sequence of primary ideals Q_n such that $\sqrt{\bigcap_{n=1}^{\infty} Q_n} \neq \bigcap_{n=1}^{\infty} \sqrt{Q_n}$.

Let P' be a prime ideal of R that is not maximal. The primary ideals of R/P' are in one-to-one correspondence with the primary ideals of R that contain P', and this correspondence respects intersections and radicals, so we may assume that R is an integral domain.

Let x be a nonzero nonunit of R and let P be a minimal prime of the principal ideal Rx. For each positive integer n, let

$$Q_n = R_P x^n \cap R = \{r \in R : sr \in Rx^n \text{ for some } s \in R \backslash P\}.$$

Each Q_n is P-primary, so $\bigcap_{n=1}^{\infty} \sqrt{Q_n} = P$. To show that $x \notin \sqrt{\bigcap_{n=1}^{\infty} Q_n}$, it suffices to show that $x^{n-1} \notin Q_n$. Suppose, by way of contradiction, that $x^{n-1} \in Q_n$. Then $sx^{n-1} \in Rx^n$ for some $s \in R \backslash P$, so $s \in Rx \subset P$ because R is an integral domain, a contradiction.

The authors are indebted to the referee for the proof that the last condition in Theorem 4 is equivalent to the other two.

Why is the intersection of all primary ideals in a zero-dimensional ring equal to zero? At the end of [7], Robert says that it would be interesting to have a characterization of the intersection of all primary ideals in a ring R. For the nonce we will call that intersection the *Gilmer radical* of R and denote it by $G(R)$. So A1 for primary ideals is the condition that the Gilmer radical be zero.

The following theorem shows that $G(R)$ is indeed a radical, that is, $G(R/G(R)) = 0$. It also characterizes the condition $G(R) = 0$ in terms of zero-dimensional rings.

Theorem 5. *The following ideals of R are equal.*

1. *The intersection of all primary ideals of R,*
2. *The intersection of the kernels of all maps of R into zero-dimensional local rings,*
3. *The intersection of the kernels of all maps of R into zero-dimensional rings.*

Proof. To show that (2) is contained in (1), we note that if Q is primary, and $P = \sqrt{Q}$, then R_P/Q_P is a zero-dimensional local ring and the kernel of the natural map from R to R_P/Q_P is Q because if $r/1 = q/s$, then $s'sr \in Q$ so $r \in Q$ because $s's \notin P$. To see that (1) is contained in (2), note that the kernel of a map from R into a zero-dimensional local ring is primary because zero is a primary ideal in a zero-dimensional local ring. It remains to show that (2) is contained in (3). The key observation for that is that any ring is a subring of the product of its localizations at each of its maximal ideals, that is, the natural map $R \to \prod R_M$ is one-to-one.

So $G(R) = 0$ if and only if R is a subring of a product of zero-dimensional (local) rings, as Arapović proves in [3, Theorem 13]. In particular, A1 and A2 hold for the family of all primary ideals of a zero-dimensional ring, hence for the corresponding family of primary ideals that are contractions of these in any subring. This establishes the necessity of Arapović's two conditions.

If I is an ideal of a zero-dimensional ring R, then R/I is also zero dimensional, so I is an intersection of primary ideals. Note that the nilradical of a ring is the intersection of the kernels of maps into fields, and the Jacobson radical is the intersection of the kernels of maps onto fields.

3 Products of zero-dimensional rings

If direct products of zero-dimensional rings were zero dimensional, then $G(R) = 0$ would be a necessary and sufficient condition for the embeddability of R in a zero-dimensional ring. However, there are direct products of zero-dimensional rings that are not zero dimensional. In particular, the following ring was described by Robert [9] as "a good test case for several questions Heinzer and I have considered". First note that if a ring R is zero dimensional, then its Jacobson radical $J(R)$ is equal to its nilradical $N(R)$, and that the Jacobson radical of a direct product of rings is equal to the direct product of their Jacobson radicals.

Example 1. Let $R = \prod_{n=1}^{\infty}(\mathbf{Z}/p^n\mathbf{Z})$. Then

$$J(R) = J\left(\prod_{n=1}^{\infty} \frac{\mathbf{Z}}{p^n\mathbf{Z}}\right) = \prod_{n=1}^{\infty} J\left(\frac{\mathbf{Z}}{p^n\mathbf{Z}}\right) = \prod_{n=1}^{\infty} \frac{p\mathbf{Z}}{p^n\mathbf{Z}},$$

so the element $p \cdot 1$ of R belongs to $J(R)$. But $p \cdot 1$ does not belong to $N(R)$, so R is not zero dimensional.

We will see below that the dimension of this ring is actually infinite. Maroscia [19] gave necessary and sufficient conditions for a direct product of zero-dimensional rings to be zero dimensional. As noted above, if a ring has dimension zero, then its Jacobson radical is equal to its nilradical. This condition on radicals is not sufficient for a ring to be zero dimensional as the ring of integers shows, but for a direct product of rings of dimension zero, it is exactly what is needed.

Theorem 6 (Maroscia). *Let $\{R_\lambda\}_{\lambda \in \Lambda}$ be a family of zero-dimensional rings. The following conditions are equivalent:*

1. *The ring $S = \prod R_\lambda$ is zero dimensional.*
2. *$J(S) = N(S)$.*
3. *$N(S) = \prod N(R_\lambda)$.*

Proof. It is clear that (1) implies (2) because the $J = N$ in any zero-dimensional ring. The implication from (2) to (3) is true because the J commutes with products and the R_λ have dimension zero. Now suppose (3) holds. Then

$$\frac{S}{N(S)} = \frac{\prod R_\lambda}{\prod N(R_\lambda)} \simeq \prod \frac{R_\lambda}{N(R_\lambda)}.$$

For each $\lambda \in \Lambda$, the ring $R_\lambda / N(R_\lambda)$ is zero dimensional and reduced, that is, it is a von Neumann regular ring. As products of von Neumann regular rings are von Neumann regular, $\prod(R_\lambda / N(R_\lambda))$ is von Neumann regular. Since the dimension of $S/N(S)$ is zero, the dimension of S is zero.

In [12, Theorem 3.4], Gilmer and Heinzer added a fourth condition to these three: $\dim(S) < \infty$. Thus, the dimension of a direct product of zero-dimensional rings is either zero or infinite; there is no in between. This significant contribution shows that the ring in the example above is infinite dimensional. The proof is much more involved than that of the equivalence of conditions (1)–(3) and we shall omit it here in favor of an alternative proof in Section 7 using an arithmetic characterization of Krull dimension due to Lombardi [18].

Gilmer and Heinzer introduced two other equivalent conditions in [11]. One is of particular interest to us because, for our purposes, it would be enough to embed an arbitrary direct product of zero-dimensional rings in a zero-dimensional ring. But, the following theorem of Gilmer and Heinzer shows that approach won't work.

Theorem 7. *If $S = \prod_{\lambda \in \Lambda} R_\lambda$ is a product of zero-dimensional rings, then the following conditions are equivalent.*

1. *The ring S can be embedded in a zero-dimensional ring;*

2. *The set* $\Delta_m = \{\lambda \in \Lambda : x^m \neq 0$ *for some* $x \in N(R_\lambda)\}$ *is finite for some positive integer* m*;*
3. *The ring* S *is zero dimensional.*

Proof. Clearly (3) implies (1). To see that (2) implies (3), choose m such that Δ_m is finite and let $\Gamma = \Lambda \setminus \Delta_m$. Then

$$S = T \times \prod_{\lambda \in \Delta_n} R_\lambda,$$

where $T = \prod_{\lambda \in \Gamma} R_\lambda$. If $x \in \prod_{\lambda \in \Gamma} N(R_\lambda)$, then $x^m = 0$ by definition of Γ so $x \in N(T)$. Since the inclusion $N(T) \subset \prod_{\lambda \in \Gamma} N(R_\lambda)$ always holds, we conclude that $N(T) = \prod_{\lambda \in \Gamma} N(R_\lambda)$, so by the result of Maroscia, $\dim(T) = 0$. As Δ_n is finite, and R_λ is zero dimensional, $\dim(S) = 0$ as well.

The implication from (1) to (2), or rather its contrapositive, is handled easily by a simple criterion that Robert introduced in [7], namely that if S can be embedded in a zero-dimensional ring, and $x \in S$, then the ascending chain of ideals $0 : x^k$ stabilizes (see Theorem 10). If (2) fails, then we can find distinct $\lambda_1, \lambda_2, \lambda_3, \ldots$ in Λ, elements $x_i \in N(R_{\lambda_i})$, and positive integers $m_1 < m_2 < m_3 < \cdots$ so that $x_i^{m_i} \neq 0$ and $x_i^{m_{i+1}} = 0$. Let the λ_i-th coordinate of $x \in S$ be x_i, for $i = 1, 2, \ldots$, and the rest of the components be zero (or whatever). Then the element of S whose coordinate is 1 in R_{λ_i} and 0 elsewhere is in $0 : x^{m_{i+1}}$ but not in $0 : x^{m_i}$.

All this taken together gives the following lovely result about when a direct product of zero-dimensional rings has dimension zero.

Theorem 8 (Maroscia, Gilmer, and Heinzer). *Let* $\{R_\lambda\}_{\lambda \in \Lambda}$ *be a family of zero-dimensional rings and let* $S = \prod_{\lambda \in \Lambda} R_\lambda$*. The following conditions are equivalent.*

1. *The ring* S *is zero dimensional;*
2. *The ring* S *is finite dimensional;*
3. $J(S) = N(S)$*;*
4. $N(S) = \prod_{\lambda \in \Lambda} N(R_\lambda)$*;*
5. *The ring* S *is a subring of a zero-dimensional ring;*
6. *The set* $\{\lambda \in \Lambda : x^m \neq 0$ *for some* $x \in N(R_\lambda)\}$ *is finite for some positive integer* m*.*

4 Sufficiency

We return to the embedding problem. We have seen that if $G(R) = 0$, then R can be embedded in a direct product S of zero-dimensional rings, but that S need not be zero dimensional. Nor can that deficiency be remedied merely by enlarging S. What if we look at rings between S and R?

Gilmer and Heinzer [13, Theorem 3.1] proved a very pretty generalization of the arithmetic characterization of a zero-dimensional ring, which distills and clarifies two related constructions of Arapović in [2, Theorem 7] and [3, Theorem 7]:

> If R is a subring of S, then R is contained in a zero-dimensional subring of S if and only if for each $x \in R$, some power of xS is an idempotent ideal of S.

The condition here is the relative version of condition (1) of Lemma 1 and says that S, rather than R itself, has enough idempotents to decompose each element of R into a unit and a nilpotent coordinate. That's all you need for Arapović's two constructions. Gilmer and Heinzer rely on the unique "pointwise inverse" of an element x that generates an idempotent ideal: this is the inverse of x within the idempotent ideal viewed as a ring [20, Lemma 2], [15, Lemma 4.3.9]. The required zero-dimensional subring of S is generated over R by the pointwise inverses of the elements x^m where m is the smallest positive integer (or any positive integer) such that $x^m S$ is idempotent.

To see how this result relates to Arapović's constructions, we start with a simple lemma connecting the two approaches.

Lemma 2. *Let S be a ring and $x \in S$. Then the following three conditions on an idempotent e of S are equivalent.*

- *$x(1-e)$ is nilpotent and $x+(1-e)$ is invertible (Arapović),*
- *$x(1-e)$ is nilpotent and xe is invertible in Se,*
- *Some power of Sx is equal to Se (Gilmer-Heinzer).*

At most one such idempotent e exists.

Proof. Note that $x+(1-e)$ is invertible if and only if $xe = (x+(1-e))\,e$ is invertible in Se and $(1-e)+x(1-e) = (x+(1-e))(1-e)$ is invertible in $S(1-e)$. But if $x(1-e)$ is nilpotent, then $(1-e)+x(1-e)$ is invertible in $S(1-e)$. So the first two conditions are equivalent. As $x = xe + x(1-e)$, the Gilmer-Heinzer condition says that $(x(1-e))^n = 0$ and $Sx^n = Se$ for some n. The second condition says exactly the same thing because xe is invertible in Se if and only if $x^n e$ is.

The uniqueness of e follows immediately from the Gilmer-Heinzer condition and the fact that an idempotent principal ideal has a unique idempotent generator.

Arapović considers two cases $R \subset S$ in which every element $x \in R$ admits such an idempotent in the larger ring S. The first [2, Theorem 7] is where $\dim S = 0$, in which case every element of S admits such an idempotent. The second [3, Theorem 7] is where $S = \prod T(R/Q_\lambda)$ where the Q_λ are primary ideals satisfying A1 and A2. Here $T(R/Q_\lambda)$ is the total quotient ring of R/Q_λ. Because of A1, the ring R can be considered a subring of S, and because of A2, for each $x \in R$ there exists an idempotent e of S satisfying the conditions of Lemma 2. This idempotent is constructed using the n in the definition of A2 and the fact that each element of $T(R/Q_\lambda)$ is either nilpotent or invertible.

In both of these cases, Arapović constructed a minimal zero-dimensional extension ring of R within S, pretty much as follows.

Theorem 9. *Let $R \subset S$ be rings such that for each element $x \in R$ there is an idempotent $e \in S$ satisfying the conditions of Lemma 2. Then there is a unique smallest zero-dimensional subring of S containing R.*

Proof. Let R' be the ring generated by R and the idempotents $e_x \in S$ for $x \in R$. Any zero-dimensional subring of S containing R must contain R' because of the uniqueness of the idempotents e_x. Let E be the boolean algebra of idempotents generated by the idempotents e_x for $x \in R$. Then each element $a \in R'$ can be written as $a = r_1 f_1 + \cdots + r_n f_n$ where the $f_i \in E$ are orthogonal. Let $e = e_{r_1} f_1 + \cdots + e_{r_n} f_n$. Then

$$a(1-e) = r_1(1-e_{r_1}) f_1 + \cdots + r_n(1-e_{r_n}) f_n$$

is nilpotent because the elements $r_i(1-e_{r_i})$ are nilpotent. Moreover, ae is invertible in Se because $r_i e_{r_i}$ is invertible in Se_{r_i}. If a is regular in R', then $e = 1$ so $f_i \leq e_{r_i}$ for each i. But $r_i e_{r_i}$ is invertible in Se_{r_1}, so $r_i f_i$ is invertible in Sf_i, whence a is invertible in S. Thus regular elements of R' are invertible in S.

The total quotient ring T of R' within S, is the desired subring. It is contained in any zero-dimensional subring of S containing R because such a subring must contain R' and be a total quotient ring. It is zero dimensional because if $x = a/b$ is in the total quotient ring of R' within S, then the e constructed above for $a \in R'$ has the property that ae is invertible in Se and $a(1-e)$ is nilpotent. so xe is invertible in Te and $x(1-e)$ is nilpotent.

In [13], Gilmer and Heinzer showed that the intersection of any nonempty family of zero-dimensional subrings of a commutative ring S is zero dimensional. This follows, rather impredicatively, from Theorem 9 upon taking R to be the intersection.

5 The new criterion

We turn to the construction of rings which are not embeddable in a zero-dimensional ring. For that purpose, we use a simple consequence of Theorem 3. This observation is due to Robert [7].

Theorem 10. *If the ring R is embeddable in a zero-dimensional ring S, then for each element $x \in R$, there exists a positive integer m such that x^m and x^{m+1} have the same annihilator in R.*

Proof. Let $x \in R$. By Theorem 3, choose a positive integer n such that $Sx^n = Sx^{n+1}$. Then $\mathrm{Ann}_S x^n = \mathrm{Ann}_S x^{n+1}$, where Ann_S denotes the annihilator in S. Consequently, $\mathrm{Ann}_R(x^n) = R \cap \mathrm{Ann}_S x^n = R \cap \mathrm{Ann}_S x^{n+1} = \mathrm{Ann}_R x^{n+1}$.

We will refer to the condition of Theorem 10 as "the new criterion", from the title of [7]. The general idea has a history in the study of subrings of a class of rings: you take a condition on ideals and restrict it to annihilator ideals (annulets). For example, if you want to characterize subrings of Noetherian rings, an obvious property to consider is the ascending chain condition on annulets (acc$\perp$) because annulets are contracted from any extension. This condition does not characterize subrings of Noetherian rings because of Jeanne Kerr's example [17] of a commutative Goldie ring R (acc$\perp$ and finite Goldie dimension) such that $R[X]$ does not have acc$\perp$, so acc$\perp$ is not inherited by polynomial rings but being embeddable in a Noetherian ring is (see also Moshe Roitman [21]).

Note that the new criterion for (fixed) $m = 1$ simply says that R is reduced, which is the exact condition necessary for embedding R in a von Neumann regular ring, that is, a ring S such that $Sx = Sx^2$ for all x. So it is not that far fetched that the new criterion would also be a *sufficient* condition for embeddability in a zero-dimensional ring. What can you deduce from the condition that x^2 and x^3 have the same annihilator for all x in R?

6 Valuation rings

Following Fuchs and Salce [6], we call a ring R a *valuation ring* if the (principal) ideals of R form a chain under inclusion. These are also called *uniserial rings* or *chained rings* (Robert's preference). Arapović [3, Theorem 8] gave a class of examples of rings where the intersection of all primary ideals is not zero. He then gave a generic example of a valuation ring with that property. Here is such an example:

Let E be the set of weakly positive elements of the group $\mathbf{Z} \oplus \mathbf{Z}$ under the lexicographic order, that is, the elements (a, b) such that either $a > 0$, or $a = 0$ and $b \geq 0$:

$$(0,0) < (0,1) < \cdots < (1,-2) < (1,-1) < (1,0) < (1,1) < \cdots.$$

Consider polynomials in X with coefficients in a fixed field k and exponents in E. Then allow denominators with nonzero constant terms. The result is a valuation domain R. The claim is that the ideal I generated by $X^{(1,1)}$ is not an intersection of primary ideals. Pass to the valuation ring R/I. Now $X^{(1,0)}X^{(0,1)} = 0$, and $X^{(0,1)}$ is not nilpotent, so $X^{(1,0)}$ is in every primary ideal of R/I.

It's an easy observation that in a valuation ring the prime ideals are exactly the complements of saturated submonoids that don't contain 0 (because in a valuation ring, a nonempty subset that's closed under multiplication by ring elements is an ideal—you don't need to require closure under addition). For valuation domains, that's the well-known result that prime ideals correspond to convex subgroups of the divisibility group (a totally ordered abelian group). So the total quotient ring of a valuation ring is obtained by localizing at the prime ideal which is the complement of the set of regular elements.

The following lemma was observed by Robert in his proof of [10, Result 1.8].

Lemma 3. *The intersection of any set (chain) of primary ideals in a valuation ring is primary.*

Proof. Let $Q = \bigcap_i Q_i$, where Q_i is primary, and let $P = \bigcap_i \sqrt{Q_i}$. Note that P is a prime ideal containing Q. If P' is any prime ideal containing Q, then, because we are in a valuation ring, either $P' = Q$ or $P' \supset Q_i$ for some i. In particular, either Q is prime, and we're done, or P is the smallest prime ideal containing Q, whence $P = \sqrt{Q}$. Moreover, either $P = Q$ or $P \supset Q_i$ for some i, so we may assume that $P = \sqrt{Q_i}$ for some i and thus we may assume that $P = \sqrt{Q_i}$ for all i. Now if $st \in Q$ and $s \notin P$, then $t \in Q_i$ for all i, hence $t \in Q$. So Q is P-primary.

In particular, there is a smallest primary ideal in any valuation ring. Another way to get this minimal primary ideal is to let P be the minimal prime ideal and $Q = \{x \in R : xs = 0 \text{ for some } s \notin P\}$.

It is not true that the intersection of a chain of primary ideals in a general ring need be primary. Let k be a field and R be $k[x,y]/(xy)$, the generic ring with a zero divisor. Then M^n is primary and $\bigcap M^n = 0$ but 0 is not primary. Of course we could make this example local. Note also that the family of primary ideals M^n does not satisfy Arapović's condition A2, but the pair of prime ideals (x) and (y) does, as does $(x)^n$.

The equivalence of conditions (1), (2), and (4) in the next theorem is Robert's [10, Result 1.8].

Theorem 11. *The following conditions are equivalent for a valuation ring R:*

1. *The minimal primary ideal is zero,*
2. *Zero is an intersection of primary ideals (Arapović's condition A1 holds),*
3. *The new criterion of Theorem 10 holds,*
4. *R is a subring of a zero-dimensional ring.*

Proof. Clearly (1) and (2) are equivalent and both imply (4). We know that (4) implies (3), that is, the new criterion is a necessary condition for a ring to be a subring of a zero-dimensional ring. To see that (3) implies (1), let a_1 be an element of the minimal primary ideal Q. There exists $s \notin \sqrt{Q}$ such that $sa_1 = 0$. Moreover, as R is a valuation ring, we can write $a_1 = sa_2$, so $s^2a_2 = 0$. Continuing, we write $a_n = sa_{n+1}$. Now $0 : s^n$ stabilizes, and $s^na_n = 0$, so we must have $s^na_{n+1} = 0$ for some n. But $s^na_{n+1} = a_1$.

Here is Robert's [7, Theorem 4.3].

Theorem 12. *Suppose that R is a valuation ring with total quotient ring T. If $\dim T > 0$, then R does not satisfy the new criterion of Theorem 10, so R cannot be embedded in a zero-dimensional ring.*

Proof. We'll prove the contrapositive. If R satisfies the new criterion, then, by the preceding theorem, zero is a primary ideal of R, hence a primary ideal of T. Thus all zero divisors of T are nilpotent, so T, being a total quotient ring, has dimension zero.

Rephrased, this result says that if a valuation ring R can be embedded in a zero-dimensional ring, then each zero divisor of R must be nilpotent.

7 An arithmetic approach to Krull dimension

Lombardi [18] introduced a characterization of Krull dimension that does not refer to prime ideals. Let the polynomial $P_m(X_1, \ldots, X_m, Y_1, \ldots, Y_m)$ be defined as

$$Y_1 Y_2 \cdots Y_m + X_1 Y_1 Y_2 \cdots Y_m + X_2 Y_2 Y_3 \cdots Y_m + \cdots + X_m Y_m$$

There are $m+1$ monomials here of degrees $m, m+1, m, m-1, \ldots, 2$. We say that a sequence $x_1, \ldots, x_m$ in R is **pseudoregular** if for all elements $a_1, \ldots, a_m \in R$ and positive integers e, we have

$$P_m(a_1 x_1, \ldots, a_m x_m, x_1^e, \ldots, x_m^e) \neq 0.$$

Lombardi showed that the Krull dimension of R is at least m if and only if there exists a pseudoregular sequence of length m in R. Note that for $m = 1$ this says that the Krull dimension is greater than zero if and only if there exists an element x such that $x^n \notin Rx^{n+1}$ for all n, which is the denial of the characterization of dimension zero (see Theorem 3) from which Robert's new criterion was derived.

We will use this idea to show that the ring $R = \prod \mathbf{Z}_{p^n}$ is infinite dimensional. We must show how to construct pseudoregular sequences in R of arbitrary length. For $r \in (0,1) \cap \mathbf{Q}$, define $x_r \in R$ by

$$x_r(n) = \begin{cases} p^{n^r} \text{if} n^r \in \mathbf{N} \\ 0 \text{ otherwise.} \end{cases}$$

Suppose $r_1, \ldots, r_k \in (0,1) \cap \mathbf{Q}$. Then $x_{r_1}(n) \cdots x_{r_k}(n) \neq 0$ for infinitely many n. Indeed, if d is a common denominator of $r_1, \ldots, r_k$, and n is of the form m^d, then the product is nonzero and the exponent of p in the product is equal to

$$n^{r_1} + n^{r_2} + \cdots + n^{r_k}.$$

If $s \in (0,1) \cap \mathbf{Q}$ is greater than $\max(r_1, \ldots, r_m)$, then

$$\lim_{n\to\infty} (n^s - n^{r_1} - n^{r_2} - \cdots - n^{r_k}) = \infty$$

because $n^{r_i}/n^s \to 0$. Of course this expression is only relevant for us when all the powers are integers—but that happens for infinitely many n.

Now suppose $r_1 < r_2 < \cdots < r_m$. We claim that $x_{r_1}, \ldots, x_{r_m}$ is a pseudoregular sequence in R. Let $a_1, \ldots, a_m \in R$ and $e \in \mathbf{N}$. We want to show that

$$P_m\left(a_1 x_{r_1}, \ldots, a_m x_{r_m}, x_{r_1}^e, \ldots, x_{r_m}^e\right) \neq 0.$$

Look at the terms of the polynomial

$$P_m\left(x_{r_1}, \ldots, x_{r_m}, x_{r_1}^e, \ldots, x_{r_m}^e\right).$$

The exponent of p in $x_{r_1}^e(n) \cdots x_{r_m}^e(n)$, when the latter is nonzero, is

$$n^{er_1} + n^{er_2} + \cdots + n^{er_m}.$$

The exponent of p in $x_{r_1} x_{r_1}^e(n) \cdots x_{r_m}^e(n)$ is

$$n^{(e+1)r_1} + n^{er_2} + \cdots + n^{er_m}$$

and the difference of these exponents, $n^{(e+1)r_1} - n^{er_1}$ goes to infinity. The exponent of p in $x_{r_2} x_{r_2}^e x_{r_2}^e \cdots x_{r_m}^e$ is

$$n^{(e+1)r_2} + n^{er_3} + \cdots + n^{er_m}$$

and if we subtract the first exponent from this we get $n^{(e+1)r_2} - n^{er_1} - n^{er_2}$ which goes to infinity. So, eventually, the exponent of p in $x_{r_1}^e(n) \cdots x_{r_m}^e(n)$ becomes smaller than all the exponents of the other terms. This means that the order of the n-th coordinate of $x_{r_1}^e \cdots x_{r_m}^e$ is greater than the order of the n-th coordinates of the other terms. This situation does not change if we replace the unexponentiated terms x_{r_i} by $a_i x_{r_i}$ because they do not occur in the first term. As the order of the first term is bigger than the orders of the other terms at some n, the sum of all the terms cannot be 0.

This construction easily extends to any product of rings $\prod R_i$ where $N(\prod R_i) \neq \prod N(R_i)$.

8 Inheritance by polynomial rings

In [7], Robert commented that, "In practice Theorem 3.1 [Arapović's criterion] has limited ease of application." (He and Bill did use it to prove that $\prod \mathbf{Z}_{p^n}$ was not a subring of a zero-dimensional ring in [11]—Bill points this out in [16]—but this is now done more easily with the new criterion.) Although this seems generally to be true, it is possible to use Arapović's criterion to prove that embeddability in a zero-dimensional ring is inherited by polynomial rings. In fact, Arapović proves this [3, Theorem 12] for an arbitrary set of indeterminants. Interestingly, Arapović does not use his criterion here! Instead he uses [1, Proposition 8] which says that the total quotient ring of a polynomial ring over a zero-dimensional ring is zero dimensional.

Theorem 13. *If R is a subring of a zero-dimensional ring, then so is $R[X]$.*

Proof. We may think of X as standing for an arbitrary set of indeterminants. First note that if Q is a primary ideal of R, then $Q[X]$ is a primary ideal of $R[X]$. So if we have an Arapović family of primary ideals Q_λ in R, we get a family of primary ideals $Q_\lambda[X]$ in $R[X]$ whose intersection is zero. Let P_λ be the radical of Q_λ. Suppose $f[X] \in R[X]$. There exists n such that if a is a coefficient of f that is in P_λ, then $a^n \in Q_\lambda$. Let I be the ideal generated by the d coefficients of $f[X]$. Then $m = dn - 1$ has the property that if $I \subset P_\lambda$, then $I^m \subset Q_\lambda$. So if $f[X] \in P_\lambda[X]$, then $I^m \subset Q_\lambda$, so $f[X]^m \in Q_\lambda[X]$.

Can one show that the new criterion is inherited by polynomial rings? That would have to be the case if the new criterion were sufficient for embedding in a zero-dimensional ring, which it undoubtedly is not. A somewhat related question is: Does the new criterion imply that if I is a finitely generated ideal, then the ideals $\mathrm{Ann} I^n$ stabilize? In fact it does, and we will end our paper by proving this not very deep fact.

Theorem 14. *Let I be a finitely generated ideal in a ring R that satisfies the new criterion. Then there exists n such that $\mathrm{Ann} I^n = \mathrm{Ann} I^{n+1}$.*

Proof. Let $I = (x_1, \ldots, x_t)$ and choose n_i such that $\mathrm{Ann} x_i^{n_i} = \mathrm{Ann} x_i^{n_i+1}$. The claim is that we can take n to be $n_1 + \cdots + n_t$. Any standard generator of $I^{n_1+\cdots+n_t}$ is of the form $x_1^{e_1} \cdots x_t^{e_t}$ where $e_1 + \cdots + e_t = n_1 + \cdots + n_t$. So $e_i \geq n_i$ for some i. Therefore $\mathrm{Ann} x_1^{e_1} \cdots x_t^{e_t} = \mathrm{Ann} x_1^{e_1} \cdots x_i^{e_i+1} \cdots x_t^{e_t}$. But $x_1^{e_1} \cdots x_i^{e_i+1} \cdots x_t^{e_t} \in I^{n_1+\cdots+n_t+1}$, so anything that kills $I^{n_1+\cdots+n_t+1}$ must kill every generator of $I^{n_1+\cdots+n_t}$. (The argument also works for $n = n_1 + \cdots + n_t - t + 1$.)

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