LINEAR INDEPENDENCE WITHOUT CHOICE

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Abstract

The notions of linear and metric independence are investigated in relation to the property: if U is a set of $n+1$ independent vectors, and X is a set of n independent vectors, then adjoining some vector in U to X results in a set of $n+1$ independent vectors. It is shown that this property holds in any normed linear space. A related property—that finite-dimensional subspaces are proximinal—is established for strictly convex normed spaces over the real or complex numbers. It follows that metric independence and linear independence are equivalent in such spaces. Proofs are carried out in the context of intuitionistic logic without the axiom of countable choice.

1 Introduction

A commutative ring with identity is **local** if whenever $a+b$ is a unit, either a or b is a unit. A **Heyting field** is a commutative local ring such that if a is not a unit, then $a=0$. Any local ring has a natural inequality, $a \neq b$, defined to mean that $a-b$ is a unit. Because the ring is local, if $a+b \neq 0$, then $a \neq 0$ or $b \neq 0$, that is, the inequality is an **apartness**. In a Heyting field, this inequality is **tight**: if a is not different from b, then $a=b$. This does *not* mean that a Heyting field is **discrete**: that is, either $a \neq b$ or $a=b$.

A **Heyting vector space** is a module over a Heyting field, with an inequality such that the algebraic operations are strongly extensional—so that, for example, if $x + y \neq x' + y'$, then $x \neq x'$ or $y \neq y'$. In particular, if $x + y \neq 0$, then $x \neq 0$ or $y \neq 0$, and if $ax \neq 0$, then $a \neq 0$ and $x \neq 0$.

The real numbers and the complex numbers are Heyting fields, as are other valued fields such as the p-adic numbers (see [5, page 287]). The reader of this paper may simply keep the real or complex numbers in mind. A normed vector space (see [1, page 244]) over a valued field is a Heyting vector space if we define $x \neq 0$ to mean $\|x\| \neq 0$. As we will be dealing exclusively with Heyting fields and Heyting vector spaces, we will suppress the qualifier "Heyting."

Vectors $x_1, \ldots, x_n$ are **linearly independent** if $\sum a_i x_i \neq 0$ whenever some $a_i \neq 0$. Heyting called such a family "free" to distinguish this property from its contrapositive, weak linear independence: if $\sum a_i x_i = 0$, then $a_i = 0$ for all i. For normed vector spaces there is an even stronger form of independence: $x_1, \ldots, x_n$ are **metrically independent** if there exists $\delta > 0$ so that $\|\sum a_i x_i\| \geq \delta$ whenever $\sum |a_i| \geq 1$—or if, equivalently, the coordinate projections on the span of $x_1, \ldots, x_n$ are uniformly continuous. It is easily seen that metric independence implies linear independence.

Let Y be a subspace of a vector space, and x a vector. We say that x is in the **complement** of Y, and write $x \in Y^c$, if $x \neq y$ for each y in Y. Note that if $x \in Y^c$, then $ax + y \neq 0$ whenever $a \neq 0$ or $y \neq 0$. It is readily seen [5, Lemma XII.4.1] that $x_1, \ldots, x_n$ are linearly (metrically) independent if and only if x_i is in the complement of (bounded away from) the span of $x_1, \ldots, x_{i-1}$ for $i = 1, \ldots, n$.

An abstract vector space is **finite-dimensional** if it is spanned by a finite linearly independent family of vectors. For a normed space to be finite-dimensional, we require that it be spanned by a finite *metrically* independent family (see [1]). It is a question of what category we are operating in: vector spaces and strongly extensional linear transformations, or normed vector spaces and bounded linear transformations.

Heyting [4, Theorem 1, page 56] proved the following extension property for finite-dimensional vector spaces.

EXT. Let $u_1, \ldots, u_{n+1}$ and $x_1, \ldots, x_n$ be two families of linearly independent vectors. Then there exists i such that $x_1, \ldots, x_n, u_i$ are linearly independent.

Let EXT_m be the property EXT with "linearly independent" replaced by "metrically independent." The motivating problem for this paper was to establish EXT_m in a not necessarily finite-dimensional normed vector space.

Bishop [1, Lemma 7, page 177] showed that if Y is a nonempty, complete, located subset of a metric space, and $x \in Y^c$, then x is bounded away from Y. In fact, he constructed, for any point x, a point y_0 in Y such that if $x \neq y_0$, then $d(x, Y) > 0$. In the proof, Bishop tacitly uses countable choice, possibly even dependent choice. Using Bishop's construction (and, through it, countable choice) one can show that linear independence is the same as metric independence in any normed space over the real or complex numbers. For more on this equivalence, see the discussion following Corollary 7

Bishop's construction suggests two properties that a subset Y might have:

1. Y is **strongly reflective**: for each x there exists y_0 in Y such that if $x \neq y_0$, then x is bounded away from Y.

2. Y is **reflective**: for each x there exists y_0 in Y such that if $x \neq y_0$, then $x \in Y^c$.

The first property makes sense in a metric space, the second in any set with an inequality. Note that if Y is **proximinal**, that is, if for each x there exists a closest point to x in Y, then Y is strongly reflective.

We prove the following results.

- If every finite-dimensional subspace is reflective, then EXT holds (Corollary 3). This follows from a general theorem which is a positive form of the fact that an n-dimensional subspace cannot contain $n+1$ independent vectors (Theorem 2).

- EXT_m holds in any normed vector space (Theorem 4).

- In a strictly convex normed space over the real or complex numbers, every finite-dimensional subspace is strongly reflective. In fact, any complete located subspace is proximinal (Theorem 6). Hilbert spaces and the L^p spaces for $1 < p < \infty$ are strictly convex normed spaces.

2 Systems of linear equations

In order to establish EXT, we are led to analyze systems of equations. The idea is that either the vectors $x_1, \ldots, x_n, u_i$ are independent, or there is a vector in the span of $x_1, \ldots, x_n$ that approximates u_i in some sense. So either EXT holds, or there are $n+1$ vectors in an n-dimensional subspace that are close to independent vectors. To rule out this latter possibility, we would like to show that any such $n+1$ vectors would have to be linearly dependent, that is, that a homogeneous system of linear equations, with more variables than equations, has a nontrivial solution.

This can't quite be done, constructively. A nontrivial solution to the equation $ax + by = 0$, over the real numbers, would establish that either a divides b, or b divides a (see also the example in [4, page 53]). But that property, for arbitrary real numbers a and b, is equivalent to Bishop's omniscience principle LLPO, so does not admit a constructive proof [6, Proposition 1.3]. We can, however, get approximate solutions that are uniformly nontrivial.

Theorem 1 *Let (a_{ij}) be an n-by-$(n+1)$ matrix over a valued field, and δ a positive number. There exist $x_1, \ldots, x_{n+1}$ such that $\sum_{j=1}^{n+1} |x_j| \geq 1$ and $\sum_{i=1}^{n} \left| \sum_{j=1}^{n+1} a_{ij} x_j \right| < \delta$.*

Proof. First consider $x_1 = 1$ and $x_j = 0$ for $j > 1$. Either $\sum_{i=1}^{n} |a_{i1}| < \delta$, and we are done, or $|a_{i1}| > 0$ for some i. So we may assume that $a_{11} \neq 0$. Clear the first column with row operations to get a matrix (a'_{ij}) with $a'_{i1} = 0$ for $i > 1$, and $a'_{1j} = a_{1j}$ for all j. By induction we can find $x_2, \ldots, x_{n+1}$ such that $\sum_{j=2}^{n+1} |x_j| \geq 1$ and

$$\sum_{i=2}^{n} \left| \sum_{j=2}^{n+1} a'_{ij} x_j \right| < \delta.$$

Choose x_1 so that $\sum_{j=1}^{n+1} a_{1j} x_j = 0$. Reversing the row operations yields

$$\sum_{i=1}^{n} \left| \sum_{j=1}^{n+1} a_{ij} x_j \right| < \delta,$$

completing the proof. ■

Theorem 1 says that if k is a valued field, and $\xi_1, \ldots, \xi_{n+1}$ are vectors in k^n, then there are small linear combinations of $\xi_1, \ldots, \xi_{n+1}$ with large coefficients. We would have liked to prove that $\xi_1, \ldots, \xi_{n+1}$ were linearly dependent, but we couldn't. When k is simply a Heyting field, we can't talk in terms of "large" and "small." The following theorem is a purely algebraic version of Theorem 1 (no talk of size) whose setting is an arbitrary Heyting vector space (not necessarily finite-dimensional) where $\xi_1, \ldots, \xi_{n+1}$ are in the span of n vectors. We show that $\xi_1, \ldots, \xi_{n+1}$ are distinct from any linearly independent set $u_1, \ldots, u_{n+1}$—a positive form of the fact that $\xi_1, \ldots, \xi_{n+1}$ cannot be linearly independent. (Heyting does not address this formulation in [4], although the negative statement, within k^n, follows easily from his results on linear independence and the rank of matrices.)

Theorem 2 *Let X be the linear span of $x_1, \ldots, x_n$ in a Heyting vector space. If $u_1, \ldots, u_{n+1}$ are linearly independent, and $\xi_1, \ldots, \xi_{n+1}$ are elements of X, then there exists i such that $\xi_i \neq u_i$.*

Proof. Because $u_1 \neq 0$, either $\xi_1 \neq u_1$, in which case we are done, or else $\xi_1 \neq 0$. Suppose the latter; we will show, by induction, that $\xi_i \neq u_i$ for some i. Write

$$\xi_i = \sum_{j=1}^{n} a_{ij} x_j$$

for $i = 1, \ldots, n+1$. As $\xi_1 \neq 0$, we may assume that $a_{11} \neq 0$. For $i > 1$ let

$$\xi_i' = \xi_i - (a_{i1}/a_{11})\xi_1,$$
$$u_i' = u_i - (a_{i1}/a_{11})u_1.$$

Then ξ_i' is in the span of $x_2, \ldots, x_n$, and $u_2', \ldots, u_{n+1}'$ are linearly independent. By induction, $\xi_i' \neq u_i'$ for some $i > 1$. As

$$\xi_i' - u_i' = (\xi_i - u_i) + \frac{a_{i1}}{a_{11}}(\xi_1 - u_1)$$

it follows that either $\xi_i \neq u_i$ or $\xi_1 \neq u_1$. ■

From Theorem 2 it follows that if finite-dimensional subspaces are reflective, then EXT holds.

Corollary 3 *Let $u_1, \ldots, u_{n+1}$ be linearly independent, and $x_1, \ldots, x_n$ be vectors whose span, X, is reflective. Then there exists i such that $u_i \in X^c$. So if $x_1, \ldots, x_n$ are linearly independent, then $x_1, \ldots, x_n, u_i$ are linearly independent.*

Proof. By reflectivity, for each i there exist ξ_i in X such that if $u_i \neq \xi_i$, then $u_i \in X^c$. Theorem 2 says that $u_i \neq \xi_i$ for some i. ■

We have the analogue of Corollary 3 for metric independence.

Theorem 4 *Let $u_1, \ldots, u_{n+1}$ be metrically independent, and $x_1, \ldots, x_n$ vectors whose linear span X is located. Then there exists i such that $d(u_i, X) > 0$. So if $x_1, \ldots, x_n$ are metrically independent, then $x_1, \ldots, x_n, u_i$ are metrically independent.*

Proof. We may assume that $\|x_j\| \leq 1$. By metric independence, there is $\varepsilon > 0$ such that if $\sum |\lambda_i| \geq 1$, then $\sum \|\lambda_i u_i\| \geq \varepsilon$. Either the desired i exists, or $d(u_i, X) < \varepsilon/2(n+1)$ for all i. We will show that the latter leads to a contradiction.

If $d(u_i, X) < \varepsilon/2(n+1)$ for all i, then there exist a_{ji} such that

$$\left\| u_i - \sum_{j=1}^{n} a_{ji} x_j \right\| < \frac{\varepsilon}{2(n+1)}$$

for $i = 1, \ldots, n+1$. By Theorem 1, there exist $\lambda_1, \ldots, \lambda_{n+1}$ with $\sum |\lambda_i| = 1$ and

$$\sum_{j=1}^{n} \left| \sum_{i=1}^{n+1} a_{ji} \lambda_i \right| < \varepsilon/2.$$

So

$$\left\| \sum_{i=1}^{n+1} \lambda_i u_i - \sum \lambda_i a_{ji} x_j \right\| < \varepsilon/2$$

and

$$\left\| \sum \lambda_i a_{ji} x_j \right\| = \left\| \sum_{j=1}^{n} \left(\sum_{i=1}^{n+1} a_{ji} \lambda_i \right) x_j \right\| < \varepsilon/2.$$

Hence $\sum \|\lambda_i u_i\| < \varepsilon$, a contradiction. ■

Theorem 4 raises the question: When is a finitely generated subspace located? A subspace of a finite-dimensional normed space is located if and only if it is finite dimensional. However, the span of a single vector in an infinite-dimensional Hilbert space can be located without being finite-dimensional: consider the vector $\sum \frac{1}{n} a_n e_n$ where e_n is an orthonormal basis, and a_n is a binary sequence that contains at most one 1.

3 Strictly convex normed spaces

Let V be a normed vector space over a subfield of the complex numbers. Following Bishop [1, Corollary page 256] we say that V is (uniformly) **strictly convex** if for each $\varepsilon > 0$, there exists $r < 1$ so that if u and v are unit vectors, and $\|u - v\| \geq \varepsilon$, then $\left\|\frac{1}{2}(u+v)\right\| \leq r$.

Hilbert spaces are strictly convex because

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2,$$

so if $\|u\| = \|v\| = 1$, and $\|u - v\| \geq \varepsilon$, then

$$\left\|\frac{u+v}{2}\right\| = \sqrt{1 - \frac{1}{4}\|u-v\|^2} \leq \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2} \leq 1 - \frac{\varepsilon^2}{8}.$$

Any complete located subspace S of a strictly convex normed space X is strongly reflective—in fact, S is **proximinal**: each $x \in X$ has a closest point in S. This was proved for finite-dimensional subspaces in [2, 3.1 Theorem]. We shall prove the general result, without using countable choice. First we put strict convexity in a more usable form.

Lemma 5 *Let V be a strictly convex normed space. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $1 \leq \|u_i\| \leq 1 + \delta$ for $i = 1, 2$, and if $\left\|\frac{1}{2}(u_1 + u_2)\right\| \geq 1$, then $\|u_1 - u_2\| \leq \varepsilon$.*

Proof. Choose $r < 1$ so that if u and v are unit vectors, and $\|u - v\| \geq \varepsilon/2$, then $\left\|\frac{1}{2}(u+v)\right\| \leq r$. Choose

$$\delta < \min\left(1 - r, \frac{\varepsilon}{4}\right)$$

and let $u_i' = u_i / \|u_i\|$. To show that $\|u_1 - u_2\| \leq \varepsilon$, assume that $\|u_1 - u_2\| > \varepsilon$. Then

$$\|u_1' - u_2'\| \geq \|u_1 - u_2\| - \|u_1' - u_1\| - \|u_2' - u_2\| > \varepsilon - 2\delta > \varepsilon/2$$

so

$$r \geq \left\|\frac{u_1' + u_2'}{2}\right\| \geq \left\|\frac{u_1 + u_2}{2}\right\| - \delta \geq 1 - \delta > r,$$

a contradiction which shows that $\|u_1 - u_2\| \leq \varepsilon$. ■

Theorem 6 *Let Y be a complete located subspace of a strictly convex normed space, and let x be a point at a distance d from Y. Then there exists a unique y_0 in Y such that $\|x - y_0\| = d$. So Y is strongly reflective.*

Proof. Let $\Delta(\varepsilon)$ be the set of $\delta < 1$ satisfying Lemma 5. For each $\varepsilon > 0$, consider the union, S_ε, of the two sets

$$A_\varepsilon = \{y \in Y : d < \varepsilon/4 \text{ and } \|x - y\| < \varepsilon/2\}$$
$$B_\varepsilon = \{y \in Y : d > 0 \text{ and } \|x - y\| < d(1+\delta) \text{ for some } \delta \in \Delta(\varepsilon/d)\}.$$

The set S_ε is supposed to be a set of ε-approximations to the desired point y_0. Note that each S_ε is nonempty, and that $S_{\varepsilon'} \subset S_\varepsilon$ if $\varepsilon' < \varepsilon$. If we can show that the diameter of S_ε is at most ε, then, by the completeness of Y, we will have determined an element y_0 of Y that is within ε of each element of S_ε.

Let y_1 and y_2 be two elements of S_ε. Clearly $\|y_1 - y_2\| < \varepsilon$ if they are both in A_ε. If one is in A_ε and one in B_ε, then $d < \varepsilon/4$ and $\|y_1 - y_2\| < \varepsilon/2 + (\varepsilon/4)(1+1) = \varepsilon$. If they are both in B_ε, then $d > 0$,

$$1 \leq \left\| \frac{x}{d} - \frac{y_i}{d} \right\| < 1 + \delta$$

and

$$\left\| \frac{\left(\frac{x}{d} - \frac{y_1}{d}\right) + \left(\frac{x}{d} - \frac{y_2}{d}\right)}{2} \right\| = \left\| \frac{x}{d} - \frac{(y_1 + y_2)/2}{d} \right\| \geq 1$$

so, as $\delta \in \Delta(\varepsilon/d)$, it follows from Lemma 5 that

$$\left\| \frac{y_1}{d} - \frac{y_2}{d} \right\| \leq \frac{\varepsilon}{d}$$

The uniqueness follows easily from strict convexity. ∎

Corollary 7 *In a strictly convex normed space over the real or complex numbers, linear independence is the same as metric independence.*

Proof. Note that a finite metrically independent family over the real or complex numbers spans a complete located subspace. We induct on the number of elements in the family $x_1, \ldots, x_n$, so we may assume that $x_1, \ldots, x_{n-1}$

are metrically independent and span a complete located subspace Y. Let d be the distance from x_n to Y, and

$$y_0 = \sum_{i=1}^{n-1} a_i x_i$$

be as in Theorem 6. Then $x_n - y_0 \neq 0$ by independence, so x_n is bounded away from Y, whence $x_1, \ldots, x_n$ are metrically independent. ■

Recall that, using countable choice, one can prove Corollary 7 for *any* normed space over the real or complex numbers. In [3] it is shown that this can be proved for normed spaces over a locally compact valued field using only a weak countable choice principle that can be derived from the law of excluded middle. Henri Lombardi has informed us that Corollary 7 can be proved for normed spaces over the real numbers without invoking any choice principle.

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