GENERALIZED REAL NUMBERS IN CONSTRUCTIVE MATHEMATICS

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Abstract

Two extensions of the real number system, one given by uppercuts the other by lowercuts, are developed within a constructive framework. The first includes distances to arbitrary subsets, the second includes norms of arbitrary bounded linear operators. The intuitive meaning of comparing such quantities to ordinary real numbers is preserved. Difficulties with encompassing both kinds of numbers in a single system are considered.

1 Introduction

In constructive mathematics we often encounter entities ξ that can be compared with any real number r, in that the statements $r < \xi$ and $\xi < r$ make sense, but are not real numbers in that we cannot calculate rational approximations to them. Typically these entities are infima or suprema of sets of real numbers. For example,

1. The **distance** from a point x to a subset Y in a metric space. It makes sense to say that $d(x, Y) < 2$, or $d(x, Y) \geq 1$ even if we cannot calculate $d(x, Y)$ arbitrarily precisely. The first means that there exists y in Y such that $d(x, y) < 2$. The second means that $d(x, y) \geq 1$ for every y in Y. It also makes sense to say $d(x, Y) \leq d(x, Y')$. That means that for each y' in Y', and $\varepsilon > 0$, there exists y in Y such that $d(x, y) < d(x, y') + \varepsilon$.

2. The **norm** of a bounded linear operator T on a normed linear space. To say $\|T\| \leq 2$ means that $\|Tx\| \leq 2$ for each x in the unit ball, that is, T is bounded by 2. To say $\|T\| > 1$ means that there exists x in the unit ball with $\|Tx\| > 1$. To say $\|T\| \leq \|T'\|$, even if neither norm exists, means that for each $\varepsilon > 0$ and x in the unit ball, there exists y in the unit ball such that $\|Tx\| < \|T'y\| + \varepsilon$.

We would like to consider these entities as generalized real numbers, or simply numbers. Distances are infima of sets of real numbers. Norms of operators, like lengths of curves, are suprema.

A distance $d(x, Y)$ may be the infimum of an arbitrary of set of nonnegative real numbers

$$d(x, Y) = \inf\{d(x, y) : y \in Y\}.$$

The number $d(x, Y)$ is naturally described by the set of rational numbers (or real numbers) that are greater than it

$$U = \{q \in \mathbf{Q} : q > d(x, y) \text{ for some } y \in Y\}.$$

This is more informative than the set of rational numbers that are at most $d(x, Y)$

$$\bar{L} = \{q \in \mathbf{Q} : q \leq d(x, y) \text{ for all } y \in Y\}$$

or the set of rational numbers that are less than $d(x, Y)$

$$L = \bigcap_{r>0} (\bar{L} - r)$$

because $\bar{L}$ is the (logical) complement of U, but U cannot be reconstructed from $\bar{L}$.

A norm is a supremum of a set of real numbers

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$$

and is naturally represented by the set of rational number that are less than it

$$L = \{q \in \mathbf{Q} : q < \|Tx\| \text{ for some } x \text{ such that } \|x\| = 1\}.$$

The calculus of distances was developed to a certain extent in [3] where it greatly simplified the exposition. The calculus of norms is equally useful in dealing with operators on Hilbert spaces. The purpose of this paper is to provide a theoretical context for both kinds of numbers.

As our primary interest is in real numbers, we care only how these new numbers compare with real numbers (or rational numbers). That is, two numbers will be considered equal if they compare with each rational number in the same way. So a number ξ determines, and is determined by, the two sets of rational numbers

$$L = \{q \in \mathbf{Q} : q < \xi\} \text{ and } U = \{q \in \mathbf{Q} : q > \xi\}.$$

That is not to say that we can decide, for each rational number q, whether $q < \xi$. It simply says that we know what the statement $q < \xi$ means.

The set L is called a (rational) **lowercut**, by which we mean that is has the two properties:

- L is open,
- $x \in L$ whenever $x < y \in L$.

The first property reflects the strictness of the inequality $q < \xi$, the second its transitivity. An **uppercut** is defined dually, replacing $x < y$ by $x > y$. Thus any generalized real number ξ may be identified with a disjoint pair (L, U) where L is a lowercut and U is an uppercut.

Each real lowercut determines a rational lowercut by restriction; each rational lowercut L determines a real lowercut by setting $r < \xi$ if $r < q$ for some $q \in L$. For the weak inequality, set $r \leq \xi$ if $r - \varepsilon < \xi$ for all $\varepsilon > 0$.

Extending the real numbers to a larger system that coincides with the real numbers in the presence of the law of excluded middle is not a new idea. In [6], two closely related systems of generalized real numbers, $\mathbf{R}^{\mathrm{e}}$ and $\mathbf{R}^{\mathrm{be}}$ are defined, based on Troelstra's paper [5]. Earlier, Staples [4] discussed a notion of Dedekind reals which is equivalent to $\mathbf{R}^{\mathrm{be}}$. These systems use a restricted class of lowercuts, the strongly monotonic ones. We will argue that they are not suitable for the applications we are interested in. Bishop [1, Exercises 7,8 page 109] formulated the notion of a fickle number to represent the supremum of an arbitrary bounded increasing sequence of real numbers. Fickle numbers do not meet the criterion that they be equal if they compare the same way with each rational number.

2 Uppercuts and lowercuts

In this section we develop some generalities about lowercuts (and hence, dually, about uppercuts). The **complement** S^c of an open set S in a metric space X may be defined either as $\{x \in X : x \neq s \text{ for each } s \text{ in } S\}$, or as $\{x \in X : x \notin S\}$. These two definitions need not be equivalent if S is not open, in which case the latter is called the *logical complement*. The complement of an open set is closed (but not necessarily vice versa). The complement of a real lowercut L is the closed set consisting of all upper bounds of L.

The **interior** S^o of an arbitrary set in a metric space X consists of those points x in X that are centers of open balls contained in S—it is the union of all open sets contained in S. The interior of the complement, S^{co}, called the **metric complement** in [1], consists of those points that are bounded away from S. Note that if L is a lowercut, then $\overline{L}^o = L$, where $\overline{L}$ is the closure of L; that is, L is a *regular* open set.

There is a duality in the set of subsets of the real numbers taking a subset S to the subset $-S = \{-s : s \in S\}$. Lowercuts and uppercuts are interchanged under this duality, so results proved for lowercuts are also true for uppercuts, most of the time. Define the sum of two sets of real numbers to be $A+B = \{a+b : a \in A \text{ and } b \in B\}$. The duality preserves this addition.

The generalized reals of Troelstra are defined to be rational lowercuts L that are **strongly monotonic**: if $x < y$, and y cannot fail to be in L, then $x \in L$. Strongly monotonic uppercuts are defined dually. It is easy to see that a lowercut L is strongly monotonic if and only if $L = U^{co}$ for some uppercut U. In fact we can take $U = L^{co}$. The strong monotonicity condition allows us to describe a generalized real as a lowercut L, or as an uppercut U, or, symmetrically, as the pair (L, U) where each element of the pair is the metric complement of the other.

The set $\mathcal{L}$ of lowercuts (either real or rational) is partially ordered by inclusion. Dually, the set $\mathcal{U}$ of uppercuts is partially ordered by setting $U \leq U'$ if $U' \subset U$ (note the reversal of inclusion for uppercuts). There is also a natural strict inequality on $\mathcal{L}$, defined by setting $L < L'$ if $L + q \subset L'$ for some positive rational number q. The partially ordered set $\mathcal{L}$ is a complete lattice: the supremum of a set of lowercuts is their union, the infimum is the interior of their intersection. The lowercuts of the form $(-\infty, r)$, with r real, form a sublattice of $\mathcal{L}$ that is isomorphic to the lattice of real numbers. Note that $(-\infty, r)$ is strongly monotonic.

A lowercut L is **bounded** if it is nonempty and there exists $r \in \mathbf{R}$ such that $r > x$ for each $x \in L$. That is, both L and L^c are nonempty.

Theorem 1 *Let $\mathcal{L}$ be the set of lowercuts.*

- *$\mathcal{L}$ is an additive monoid with $(-\infty, 0)$ as the zero element,*
- *If $L \in \mathcal{L}$ is bounded, then $L + L' \subset L + L''$ if and only if $L' \subset L''$. In particular, L is cancellable,*
- *An element of $\mathcal{L}$ has an inverse if and only if it is of the form $(-\infty, r)$ for some $r \in \mathbf{R}$.*

The dual result holds for uppercuts.

Proof. Because L is a lowercut, $(-\infty, 0) + L \subset L$. As L is open, if $x \in L$, then $x + r \in L$ for some $r > 0$, so $x = (-r) + (x + r) \in (-\infty, 0) + L$. Thus $(-\infty, 0) + L = L$.

Suppose L is bounded, so $x_0 \in L \subset (-\infty, x_1)$, and $L + L' \subset L + L''$. We want to show that $L' \subset L''$. Given $x' \in L'$, and $\varepsilon > 0$, we will show that there exists $y'' \in L''$ such that $x' < y'' + 2\varepsilon$. That will do it because L' is open (so $x' - 2\varepsilon$ is an arbitrary element of L').

Choose a positive integer n such that $n\varepsilon > x_1 - x_0$ and induct on n. As $L + L' \subset L + L''$, there exist $y \in L$ and $y'' \in L''$ such that $x_0 + x' = y + y''$. Either $y > x_0 + \varepsilon$, in which case we can replace x_0 by y and we are through by induction on n, or $y < x_0 + 2\varepsilon$, in which case $x' < y'' + 2\varepsilon$.

The inverse of $(-\infty, r)$ is $(-\infty, -r)$. Indeed the elements $(-\infty, r)$ form a copy of the additive group $\mathbf{R}$ within $\mathcal{L}$. Conversely, suppose $L + L' = (-\infty, 0)$. Then for each $\varepsilon > 0$, there exists $x \in L$ and $x' \in L'$ such that $x + x' = -\varepsilon$. As $y + x' < 0$ for each y in L, we have $(-\infty, x) \subset L \subset (-\infty, -x') = (-\infty, x + \varepsilon)$. As $\mathbf{R}$ is complete, there exists $r \in \mathbf{R}$ within ε of each element of $(x, x + \varepsilon)$ for such x. Thus $(-\infty, r - \varepsilon) \subset L \subset (-\infty, r + \varepsilon)$ for each $\varepsilon > 0$, whence $L = (-\infty, r)$. ∎

Neither the sum nor the union of two strongly monotonic lowercuts need be strongly monotonic.

Theorem 2 *Let P_1 and P_2 be propositions,*

$$U_i = (1, \infty) \cup \{x \in (0, \infty) : P_i\}$$

and $L_i = U_i^{co}$. Then the L_i are strongly monotonic lowercuts, and the following are equivalent.

1. *$L_1 \cup L_2$ is strongly monotonic,*

2. *$\neg(P_1 \wedge P_2) \Longrightarrow \neg P_1 \vee \neg P_2$.*

Moreover, if $L_1 + L_2$ is strongly monotone, then so is $L_1 \cup L_2$.

Proof. Clearly L_i is a strongly monotonic lowercut. For each $x \in (0,1)$ the following equivalences hold.

$$\begin{aligned} x \in L_i &\Longleftrightarrow \neg P_i \\ x \in L_1 \cup L_2 &\Longleftrightarrow \neg P_1 \vee \neg P_2 \\ x \in (L_1 \cup L_2)^{co} &\Longleftrightarrow \neg\neg P_1 \wedge \neg\neg P_2 \\ x \in (L_1 \cup L_2)^{coco} &\Longleftrightarrow \neg(\neg\neg P_1 \wedge \neg\neg P_2) \\ x \in L_1 + L_2 &\Longleftrightarrow \neg P_1 \vee \neg P_2 \\ x \in (L_1 + L_2)^{coco} &\Longleftrightarrow \neg(\neg\neg P_1 \wedge \neg\neg P_2) \end{aligned}$$

Note that $\neg(\neg\neg P_1 \wedge \neg\neg P_2)$ is equivalent to $\neg(P_1 \wedge P_2)$, and that if two lower cuts contain the same nonzero elements, then they are equal.

Now $L_1 \cup L_2 \subset (L_1 \cup L_2)^{coco}$ so (1) is equivalent to $(L_1 \cup L_2)^{coco} \subset L_1 \cup L_2$, and it suffices to test that inclusion on elements $x \in (0,1)$. Thus (1) is equivalent to (2).

If $L_1 + L_2$ is strongly monotone, then $L_1 + L_2 = (L_1 + L_2)^{coco}$. Testing that equality on any $x \in (0,1)$ results in (2). ■

Condition (2) of Theorem 2 is De Morgan's law, which is not a constructive tautology. In fact, if P_i is the proposition that a certain binary sequence contains a 1, then (2) is Bishop's omniscience principle LLPO [2]. So Theorem 2 implies that neither the union nor the sum of two strongly monotonic lowercuts need be strongly monotonic.

3 Troelstra's classical reals

Troelstra's generalized reals are strongly monotonic rational lowercuts L.

- The **classical reals**, $\mathbf{R}^{\mathrm{e}}$, are **weakly bounded**—that is, it is impossible for L to be empty, and it is impossible for L^c to be empty,

- The **extended reals**, $\mathbf{R}^{\mathrm{be}}$, are bounded classical reals,

- The **Dedekind reals**, $\mathbf{R}^{\mathrm{d}}$, are **located** extended reals—given rational numbers $x < y$, either $x \in L$ or $y \notin L$. These are the ordinary real numbers (compare [1, Exercises 7,8 page 58]).

If $\bar{r} = \{q \in \mathbf{Q} : q < r\}$ denotes the lowercut corresponding to the rational number r, then

$$L = \{r \in \mathbf{Q} : \bar{r} < L\}$$

so the classical reals are determined by their relationship to the rational numbers. Because of Theorem 2, when we add or take the supremum of two classical reals L and L', we must form $(L + L')^{cco}$ or $(L \cup L')^{cco}$.

The classical reals, $\mathbf{R}^{\mathrm{e}}$, do not represent distances and norms well. The distance $d(x, Y)$ is naturally represented by the uppercut

$$\{q \in Q : d(x, y) < q \text{ for some } y \in Y\}.$$

This uppercut need not be strongly monotonic. If we represent $d(x, Y)$ by an element of $\mathbf{R}^{\mathrm{e}}$, hence by a lowercut, we lose information. Consider $x = 0$ and the two nonempty subsets

$$Y = \{y \in \mathbf{Q} : y > 2, \text{ or } y > 1 \text{ and } P\}$$

and

$$Y' = \{y \in \mathbf{Q} : y > 2, \text{ or } y > 1 \text{ and } \neg\neg P\}$$

where P is an arbitrary proposition. The uppercuts describing $d(0, Y)$ and $d(0, Y')$ are $U = Y$ and $U' = Y'$. They have the same complement,

$$\{q \in \mathbf{Q} : q \leq 1, \text{ or } q \leq 2 \text{ and } \neg P\}$$

so we cannot distinguish between $d(0, Y)$ and $d(0, Y')$ if we represent them by lowercuts. Note that the interior of that complement is

$$L = \{q \in \mathbf{Q} : q < 1, \text{ or } q < 2 \text{ and } \neg P\}$$

which is a strongly monotonic lowercut. The lowercut L is, in fact, the infimum of both Y and Y' in $\mathbf{R}^{\mathrm{e}}$.

This is not a harmless loss of information. If $d(0, Y) < 2$, then we should be able to produce $y \in Y$ such that $d(0, y) < 2$. We could do that were

$d(0,Y)$ a real number, and if the symbolism doesn't entail that, then it will be misleading at best, useless at worst. However, if the lowercut, L, above is taken to represent $d(0,Y)$, then the statement $d(0,Y) < 2$ is equivalent to $\neg\neg P$, which does not enable us to construct $y \in Y$ with $d(0,y) < 2$—for that we would need P.

Norms, on the other hand, naturally correspond to lowercuts,

$$L = \{q \in \mathbf{Q} : q < \|Tx\| \text{ for some } x \text{ such that } \|x\| = 1\},$$

which need not be strongly monotonic, so need not be in $\mathbf{R}^{\mathrm{e}}$. Consider the subspace

$$K = \{a_1e_1 + a_2e_2 : (a_2 = 0) \vee P\}$$

of a two-dimensional Hilbert space with orthonormal basis e_1, e_2, where P is an arbitrary proposition. Let T be the restriction to K of the linear functional that takes e_1 to 0 and e_2 to 1. If $q \in (0,1] \cap \mathbf{Q}$, then $q \in L$ if and only if P. So L is strongly monotonic only if $\neg\neg P$ implies P.

4 Cuts

We are interested in objects ξ for which it makes sense to write $r < \xi$ and $r > \xi$ for r a real number. Moreover we want ξ to be determined by the sets $L_\xi = \{r \in \mathbf{R} : r < \xi\}$ and $U_\xi = \{r \in \mathbf{R} : r < \xi\}$. So the natural candidate for ξ is the pair (L_ξ, U_ξ) itself. These pairs give us a context for any kind of generalized real number, but in practice it turns out that we still need to focus on either lowercuts or uppercuts.

A **cut** ξ in the real numbers $\mathbf{R}$ is an ordered pair (L_ξ, U_ξ) of disjoint sets, where L_ξ is a lowercut and U_ξ is an uppercut. So the set $\mathcal{C}$ of cuts is a subset of $\mathcal{L} \times \mathcal{U}$, and inherits a strict inequality, and the structure of a commutative monoid and a complete lattice, from that product. Write $r < \xi$ if $r \in L_\xi$, and $r > \xi$ if $r \in U_\xi$. A cut ξ is **bounded above** if U_ξ is nonempty, **bounded below** if L_ξ is nonempty, and **bounded** if both U_ξ and L_ξ are nonempty.

One reason to look at cuts is simply that they provide a way to make statements about lowercuts and uppercuts simultaneously. As long as those statements were dual we needed no such device, but it will be convenient when we consider multiplication of weakly positive cuts.

Each real number r gives rise to a cut by setting $L_r = \{s \in \mathbf{R} : s < r\}$ and $U_r = \{s \in \mathbf{R} : s > r\}$. We identify the real number r with that cut.

Note that $U_r = L_r^{co}$ and $L_r = U_r^{co}$. A cut is **located** if for each pair $r < s$ of real numbers, either $r < \xi$ or $s > \xi$. This is the same as saying that $L_\xi \cup U_\xi$ is dense in $\mathbf{R}$. The sets $\overline{L}_\xi$ and $\overline{U}_\xi$ are closed and there is at most one element in their intersection. A (real) cut is located exactly when this intersection is nonempty. A bounded located cut is a real number. The cut ∞ is defined by setting $L_\infty = \mathbf{R}$, and the cut $-\infty$ by setting $U_{-\infty} = \mathbf{R}$.

Each lowercut $L \in \mathcal{L}$ determines a cut (L, L^{co}). Recall that the strict inequality on $\mathcal{L}$ is defined by setting $L < L'$ if there exists $\varepsilon > 0$ such that $L + \varepsilon \subset L'$. If $L' = (-\infty, r)$ represents the real number r, this says that

$$L < r \text{ if and only if there is } \varepsilon > 0 \text{ such that } \ell + \varepsilon < r \text{ for all } \ell \in L.$$

It's not difficult to verify that this condition is equivalent to $r \in L^{co}$. So embedding $\mathcal{L}$ in the set of cuts by taking L to (L, L^{co}) is consistent with the meaning of $L < r$ in $\mathcal{L}$. Similarly, we can embed the set of uppercuts $\mathcal{U}$ into the set of cuts by taking U to (U^{co}, U). Then $\mathbf{R}^{\text{be}}$ consists of the bounded cuts in $\mathcal{L} \cap \mathcal{U}$, and $\mathbf{R}^{\text{e}}$ consists of the weakly bounded cuts in $\mathcal{L} \cap \mathcal{U}$.

Cuts constitute a true extension of the reals, even from a classical point of view. Not only are ∞ and $-\infty$ among the cuts, but so are intervals $[a, b]$ corresponding to the cuts $((-\infty, a), (b, \infty))$. The latter are unintended consequences of the definition which seem to be harmless. The possibly unbounded cuts, on the other hand, are useful in describing the distance from a point to a possibly empty set, and the norm of a possibly unbounded operator.

The partially ordered sets $\mathcal{L}$ and $\mathcal{U}$ are order embedded in $\mathcal{C}$, but are not sublattices or submonoids because of Theorem 2. Of course we could embed $\mathcal{L}$ in $\mathcal{C}$ by taking L to $(L, \emptyset)$, instead of to (L, L^{co}). This is a lattice and monoid embedding, but does not take the natural copy of $\mathbf{R}$ in $\mathcal{L}$ to the natural copy of $\mathbf{R}$ in $\mathcal{C}$. In particular, $(L, \emptyset)$ is not bounded by any real number in $\mathcal{C}$ even if L is bounded by some real number in $\mathcal{L}$.

In the presence of the law of excluded middle, a bounded cut (L, U) may be identified with the nonempty closed interval $(L \cup U)^c$. Without the law of excluded middle there is the possibility of a Specker sequence: a bounded increasing sequence of rational numbers that is eventually bounded away from any given real number. If we let ξ be the (bounded) cut which is the supremum of such a sequence, then $\mathbf{R} = L \cup U$, so we can't retrieve L and U from $(L \cup U)^c = \emptyset$. Even classically, $(L \cup U)^c$ does not distinguish between the unbounded cuts $L = \mathbf{R}$ and $U = \mathbf{R}$.

The duality on the set of subsets of $\mathbf{R}$ gives a duality on the lattice of cuts taking ξ to $-\xi$, where $L_{-\xi} = -U_\xi$ and $U_{-\xi} = -L_\xi$. This duality preserves sums. Note that $-\xi$ is generally not an additive inverse for ξ.

We can characterize $\mathcal{L}$ and $\mathcal{U}$ in terms of $\mathbf{R}$ and $\mathcal{C}$.

Theorem 3 *Let ξ be a cut. Then the following are equivalent.*

1. *ξ is the infimum in $\mathcal{C}$ of the set U_ξ,*

2. *ξ is the infimum in $\mathcal{C}$ of some set of real numbers,*

3. *$L_\xi = U_\xi^{co}$, that is, $\xi \in \mathcal{U}$.*

Proof. Clearly (1) implies (2). Suppose $\xi = \inf S$, so $U_\xi = \{r \in \mathbf{R} : r > s$ some $s \in S\}$ and $\overline{L}_\xi = \{r \in \mathbf{R} : r \leq s$ for all $s \in S\}$. If $r > \xi$ is impossible, then $r > s$ is impossible for each $s \in S$ so $r \leq s$ for all $s \in S$. Thus $\overline{L}_\xi = U_\xi^c$, so (3) holds. Finally, if $\eta \leq s$ for each s in U_ξ, then $U_s \subset U_\eta$ for each s in U_ξ so $U_\xi \subset U_\eta$. If (3) holds, it follows that $\overline{L}_\eta \subset \overline{L}_\xi$ so $L_\eta \subset L_\xi$. ■

5 Multiplication

There are occasions when we want to multiply cuts. For bounded operators T and T' on a Hilbert space we would like the inequality $\|TT'\| \leq \|T\| \, \|T'\|$ to make sense. Multiplying a cut ξ by a positive real number r is simply pointwise multiplication

$$L_{r\xi} = rL_\xi \text{ and } U_{r\xi} = rU_\xi.$$

For a nonnegative real number r, set

$$U_{r\xi} = U_r U_\xi = \bigcup_{s>r} sU_\xi \text{ and } L_{r\xi} = \left(\bigcap_{s>r} sL_\xi \right)^o,$$

that is, $r\xi = \inf_{s>r} s\xi$.

Mostly we are interested in cuts ξ that are **weakly positive**, that is, $0 \leq \xi$. For lowercuts L that means that L contains the negative reals, for uppercuts U it means that U is contained in the positive reals. Weakly positive cuts can be thought of as cuts in the positive reals. The correspondence between a cut ξ^* in the positive reals, and a weakly positive cut ξ in the reals, is given by:

- $U^* = U$,
- $L^* = L \cap (0, \infty)$,
- $L = (-\infty, 0) \cup \{x : x < y \text{ for some } y \in L^*\}$.

A weakly positive uppercut may be identified with an uppercut in the set of positive reals, a weakly positive lowercut with a lowercut in the set of positive reals.

To multiply weakly positive cuts, it is convenient to think of them as cuts in the positive reals. Then we can simply define

$$U^*_{\xi\eta} \equiv U^*_\xi U^*_\eta \text{ and } L^*_{\xi\eta} \equiv L^*_\xi L^*_\eta.$$

On the other hand, addition for lowercuts in the positive reals is slightly complicated by the possibility that L^* may be empty whereas L always contains $(-\infty, 0)$. So

$$L^*_{\xi+\eta} = L^*_\xi \cup L^*_\eta \cup (L^*_\xi + L^*_\eta).$$

Note that the weakly positive cuts form a commutative monoid under multiplication with $1 \cdot \xi = \xi$ for all ξ, and $0 \cdot \xi = 0$ for all bounded ξ. This multiplication distributes across addition.

Theorem 4 *If ξ_1, ξ_2, and ξ_3 are weakly positive cuts, then $\xi_1(\xi_2 + \xi_3) = \xi_1\xi_2 + \xi_1\xi_3$.*

Proof. The lowercut $L_1(L_2 + L_3)$, viewed as a cut in the positive reals, is

$$L^*_1(L^*_2 \cup L^*_3 \cup (L^*_2 + L^*_3)) = L^*_1 L^*_2 \cup L^*_1 L^*_3 \cup L^*_1(L^*_2 + L^*_3)$$

so the issue is whether

$$L^*_1(L^*_2 + L^*_3) \supset L^*_1 L^*_2 + L^*_1 L^*_3.$$

Let $x_1 x_2 + x'_1 x_3$ be an element of the right hand side, and choose $x''_1 > \sup(x_1, x'_1)$ in L^*_1. So

$$x_1 x_2 + x'_1 x_3 = x''_1 \left(\frac{x_1}{x''_1} x_2 + \frac{x'_1}{x''_1} x_3 \right).$$

As $x_1/x_1'' \leq 1$ and $x_1'/x_1'' \leq 1$, the right hand side is in $L_1^*(L_2^* + L_3^*)$.

For the uppercuts, we must show

$$U_1(U_2 + U_3) \supset U_1U_2 + U_1U_3.$$

Again, consider $x_1x_2 + x_1'x_3$, but this time choose $x_1'' < \inf(x_1, x_1')$ in U_1 and proceed as before. ■

Because of Theorems 1 and 4, the bounded weakly positive cuts can be embedded naturally in a lattice-ordered real algebra $\mathcal{R}$. The elements of $\mathcal{R}$ are formal differences $\xi - \eta$ of bounded weakly positive cuts, with $\xi_1 - \eta_1 \leq \xi_2 - \eta_2$ defined to mean $\xi_1 + \eta_2 \leq \xi_2 + \eta_1$. The lattice operations may be described by

$$(\xi_1 - \eta_1) \vee (\xi_2 - \eta_2) = ((\xi_1 + \eta_2) \vee (\xi_2 + \eta_1)) - (\eta_1 + \eta_2)$$
$$(\xi_1 - \eta_1) \wedge (\xi_2 - \eta_2) = ((\xi_1 + \eta_2) \wedge (\xi_2 + \eta_1)) - (\eta_1 + \eta_2).$$

The second condition of Theorem 1 plays a key role in verifying these equations, and in showing that the partial order is antisymmetric. The bounded cuts are embedded in $\mathcal{R}$ by taking β to $(\beta + r) - r$ where r is any positive real number such that $\beta + r \geq 0$.

Multiplication in $\mathcal{R}$ is defined by

$$(\xi_1 - \eta_1)(\xi_2 - \eta_2) = \xi_1\xi_2 + \eta_1\eta_2 - (\xi_1\eta_2 + \xi_2\eta_1).$$

Here we have to verify that if $\xi_1 \geq \eta_1$ and $\xi_2 \geq \eta_2$, then $\xi_1\xi_2 + \eta_1\eta_2 \geq \xi_1\eta_2 + \xi_2\eta_1$. This multiplication does not agree with the standard multiplication of interval arithmetic, which is not associative. For example, multiplying the interval $[1, 2]$ by the real number -1 gives the interval $[-2, -1]$ in interval arithmetic, while in $\mathcal{R}$ you get the element $-[1, 2]$.

If ξ is a weakly positive cut, and n is a positive integer, then there exists a unique weakly positive cut η such that $\eta^n = \xi$. Indeed

$$L_\eta^* = \{x \geq 0 : x^n \in L_\xi^*\} \text{ and } U_\eta^* = \{x \geq 0 : x^n \in U_\xi^*\}.$$

In particular, we may talk about $\sqrt{\xi}$ for $\xi \geq 0$.

The lower components of the elements of $\mathcal{R}$ form a lattice-ordered real algebra $\mathcal{R}_L$ generated by the bounded weakly positive elements of $\mathcal{L}$, and the upper components form a lattice-ordered real algebra $\mathcal{R}_U$ generated by the nonempty weakly positive elements of $\mathcal{U}$. So if we start with some weakly positive uppercuts, like distances, then we've got a context in which we can add, subtract, and multiply them, and take finite suprema and infima. Note that the algebras $\mathcal{R}_L$ and $\mathcal{R}_U$ are images of $\mathcal{R}$, but are not embedded in $\mathcal{R}$ as subalgebras.

6 Lowercuts and uppercuts versus cuts

We end by considering two situations in which cuts, together with the natural operations on them, do not provide a satisfactory context for generalized real numbers. In each situation we are led to consider the numbers as uppercuts rather than cuts.

Recall that a metric d is an **ultrametric** if

$$d(x,z) \leq d(x,y) \vee d(y,z)$$

for all x, y, z. Ultrametrics come up naturally in the theory of abelian groups. Suppose A is an abelian group and p is a prime such that $\bigcap_n p^n A = 0$. Define

$$|a| = \inf\{p^{-n} : a \in p^n A\}.$$

Then $|a|$ is an uppercut, and the ultrametric inequality is

$$|a+b| \leq |a| \vee |b| .$$

Think of this as an inequality of cuts. If $|a| \leq p^{-n}$ and $|b| \leq p^{-n}$, then $|a+b| \leq p^{-n}$, so the inequality is satisfied by the upper portions. However, the inequality for the lower portions says that if $a+b \notin p^n A$, then either $a \notin p^n A$ or $b \notin p^n A$. Were this true in general, we could derive De Morgan's law $\neg(P \wedge Q) \Rightarrow \neg P \vee \neg Q$ as follows. Define an abelian group A by

$$A = \left\{ (x,y) \in \frac{\mathbf{Z}}{p^2\mathbf{Z}} \oplus \frac{\mathbf{Z}}{p^2\mathbf{Z}} : px \neq 0 \Rightarrow P \text{ and } py \neq 0 \Rightarrow Q \right\}$$

and consider the elements $a = (p,0)$ and $b = (0,p)$ of A. Now

$$a+b \in pA \Leftrightarrow (p,p) = (px,py) \text{ for some } (x,y) \text{ in } A \Leftrightarrow P \wedge Q$$

while $a \in pA$ if and only if P, and $b \in pA$ if and only if Q, so if $a \notin p^n A$ or $b \notin p^n A$, then $\neg P \vee \neg Q$.

Thus, if we want the ultrametric inequality to hold, we must consider the supremum $|a| \vee |b|$ to take place in $\mathcal{U}$, not in $\mathcal{C}$.

Next consider the Hausdorff metric

$$d(A, B) = \sup_{\substack{a \in A \\ b \in B}} \{d(a, B), d(b, A)\}$$

on bounded nonempty closed subsets of a metric space. This is a supremum of uppercuts, which are infima of real numbers, so we might expect the right setting to be cuts (although the ultrametric example suggests the contrary). The triangle inequality is

$$d(A, C) \leq d(A, B) + d(B, C).$$

On the upper portions, this is true: we must show that if $d(A, B) < r$ and $d(B, C) < s$, then $d(A, C) < r + s$. By hypothesis, there exist $r' < r$ and $s' < s$ such that for each a there exists b with $d(a, b) < r'$, and for each b there exists c with $d(b, c) < s'$. Hence for each a there exists c with $d(a, c) < r' + s' < r + s$. Similarly, for each c there exists a with $d(a, c) < r' + s' < r + s$.

On the lower portions, however, the inequality can fail. Suppose the space consists of three points a, b, c each a distance 1 from the other two. Let $A = \{a, b\}$ and $C = \{b, c\}$ and

$$B = \{b\} \cup \{a : P\} \cup \{c : \neg P\}.$$

Then $d(A, C) = 1$. But $0 < d(A, B)$ if and only if $\neg P$, and $0 < d(B, C)$ if and only if $\neg\neg P$. If the triangle inequality holds, then $1/2$ is in the lower portion of $d(A, B) + d(B, C)$, so 0 must be in the lower portion of $d(A, B)$ or of $d(B, C)$. So the triangle inequality entails $\neg P \vee \neg\neg P$, the weak law of excluded middle (which is equivalent to De Morgan's law).

Thus, for the Hausdorff metric to satisfy the triangle inequality, we must interpret $d(A, B)$ as an uppercut. The moral seems to be that once you are dealing with uppercuts, you deal with them from then on. This probably holds for lowercuts also, but I don't know a good example of the phenomenon. Note that the metric on bounded operators on a Hilbert space, given by the norm $\|T\|$ which is a lowercut, satisfies the triangle inequality when we consider it as an inequality of cuts. Even so, the triangle inequality is sharper on the upper portions, but still true, if the sum is taken to be the sum in $\mathcal{L}$ rather than in $\mathcal{C}$.

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