Pre-abelian clan categories

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Abstract

Categories of representations of clans without special loops, and with a linear ordering at each vertex, are studied with an eye toward identifying those that have kernels and cokernels. A complete characterization is given for simple graphs whose vertices have degree at most two.

1 Representations of clans

I'm not going to give the definition of an arbitrary clan [1], but only a very restricted version which will cover the cases I want to look at here. A (linear ordinary) **clan** consists of

- A finite graph, possibly with multiple edges and loops,
- At each vertex v an enumeration $e(v,1),\ldots,e(v,d)$ of the edges incident to v in which each incident loop appears twice and the other edges appear once. The integer d is the **degree** of the vertex.

We say that an edge e **joins** (v,i) **with** (w,j) if $e = e(v,i) = e(w,j)$ and $(v,i) \neq (w,j)$.

In contrast to the general notion of a clan, no field is mentioned because we don't allow "special loops." Representations of clans decompose canonically into representations of their components, so we may assume that the graph is connected. As in [1], we will assume that there are no vertices of degree 0, which for a connected graph simply says that it has an edge.

If k is a field, then a k-**representation** M of a clan associates a finite-dimensional vector space $M(v)$ over k to each vertex v of the clan, together with a filtration

$$0 = M(v)_0 \subset M(v)_1 \subset \cdots \subset M(v)_{d(v)} = M(v)$$

of $M(v)$ where $d(v)$ is the degree of v. Moreover, if the edge e joins (v, i) with (w, j), then M associates with e an isomorphism M_e between $M(v)_i/M(v)_{i-1}$ and $M(w)_j/M(w)_{j-1}$.

A **map** f between representations consists of a linear transformation between the vector spaces at each vertex that

- respects the filtrations, that is, $f(M(v)_i) \subset M'(v)_i$,
- respects the isomorphisms associated with the edges, that is, if the edge e joins (v, i) with (w, j), then the diagram

$$\begin{tikzcd}
M(v)_i/M(v)_{i-1} \arrow[r, leftrightarrow, "M_e"] \arrow[d] & M(w)_j/M(w)_{j-1} \arrow[d] \\
M'(v)_i/M'(v)_{i-1} \arrow[r, leftrightarrow, "M'_e"] & M'(w)_j/M'(w)_{j-1}
\end{tikzcd}$$

  commutes, where the vertical arrows are induced by f.

Let V assign a vector space V_e to each edge of a clan C. We can construct a representation M_V of C by setting

$$M_V(v)_i = \bigoplus_{i' \leq i} V_{e(v,i')}.$$

Every representation of C is isomorphic to one constructed in this manner. If $V_e = U_e \oplus W_e$ for each edge e, then $M_V = M_U \oplus M_W$. So M_V is indecomposable if and only if $V_e = k$, for some edge e, and V is zero on the other edges. Thus the indecomposable representations of C are in one-to-one correspondence with the edges of C. This also follows from the general theory in [1].

What are the maps between indecomposables I_e and $I_{e'}$? For any edge e the dimension of $I_e(v)_i$ is the number of $i' \leq i$ such that $e = (v, i')$. This is at most 1 if e is not a loop, and at most 2 if e is a loop. If e is not a loop, then $\dim \operatorname{Hom}(I_e, I_e) = 1$. If e is a loop, then $\dim \operatorname{Hom}(I_e, I_e) = 2$. If

$e \neq e'$, then for any v and i either $I_e(v)_i/I_e(v)_{i-1} = 0$ or $I_{e'}(v)_i/I_{e'}(v)_{i-1} = 0$, so any putative map $f : I_e \to I_{e'}$ respecting the filtrations respects the edge isomorphisms for trivial reasons. Thus if $e \neq e'$, then $\dim \operatorname{Hom}(I_e, I_{e'})$ is the number of triples (v, i, i') such that $e = e(v, i)$, $e' = e(v, i')$, and $i' < i$.

The **dual clan** C^* is obtained by reversing the order at each vertex of C: the graph is unchanged and we set $e^*(v, i) = e(v, d(v) - i + 1)$.

Theorem 1 *The category of representations of C^* is dual to the category of representations of C.*

Proof. The duality takes a representation M of C to a representation M^* of C^* defined as follows:

- $M^*(v) = M(v)^*$, the space of linear functionals on $M(v)$,
- $M^*(v)_i = M(v)^{\perp}_{d(v)-i} = \{\varphi \in M(v)^* : \varphi(M(v)_{d(v)-i}) = 0\}$
- If e joins (v, i) with (w, j) in C, hence joins $(v, d(v) - i + 1)$ with $(w, d(w) - j + 1)$ in C^*, then M^*_e is the isomorphism induced by M_e between

  $$M^*(v)_{d(v)-i+1}/M^*(v)_{d(v)-i} = M(v)^{\perp}_{i-1}/M(v)^{\perp}_i \cong (M(v)_i/M(v)_{i-1})^*$$

  and

  $$M^*(w)_{d(w)-j+1}/M^*(w)_{d(w)-j} = M(w)^{\perp}_{j-1}/M(w)^{\perp}_j \cong (M(w)_j/M(w)_{j-1})^*.$$

Clearly

$$0 = M^*(v)_0 \subset M^*(v)_1 \subset \cdots \subset M^*(v)_{d(v)} = M^*(v)$$

because of the index reversal together with taking annihilators. A map from M to N consists of maps from $M(v)$ to $N(v)$ for each vertex v, and these correspond to maps from $N(v)^*$ to $M(v)^*$. ∎

2 A category where idempotents don't split

This paper was provoked by a remark in [1] that it is not surprising that idempotents split in clan categories because "a representation of a clan is a cross between a representation of a poset and a representation of a quiver." The same reasoning suggests that clan categories are **pre-abelian**, that is, *all* maps have kernels and cokernels, not just idempotents. That's not always the case, so the question arises as to what clans have pre-abelian categories. Before looking at a clan category that is not pre-abelian, we consider a class of categories where even idempotents need not have kernels.

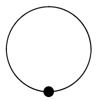
Theorem 2 *Let $\mathcal{C}$ be the category of finite-rank free modules over a commutative ring R. For $a \in R$, define $f_a : R \to R$ by $f_a(x) = ax$. Then f_a has a kernel in $\mathcal{C}$ if and only if a is regular or $a = 0$.*

Proof. If a is regular, then the inclusion $0 \to R$ is a kernel of f_a, and if $a = 0$, then the identity map $R \to R$ is a kernel of f_a. Conversely, suppose $g : R^n \to R$ is a kernel of f_a. If $n = 0$, then $R^n = 0$, so f_a is monic—that is, a is regular. If $n > 0$, then $g(r_1, \ldots, r_n) = \sum r_i b_i$, and $ab_i = 0$ for all i. Consider the map $h : R \longrightarrow R^n$ given by $h(1) = (a, \ldots, a)$. Then $gh = 0$ so $h = 0$ as g is monic. Therefore $a = 0$. ■

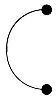
In particular, if a is a nontrivial idempotent in R, then f_a is an idempotent in $\mathcal{C}$ without a kernel.

3 The smallest clan

Consider the clan C with one vertex and one edge:



I want to change this picture to one that reflects the notation introduced in Section 1. Each vertex v is expanded into an ascending vertical column of vertices $(v, 1), \ldots, (v, d)$, each edge joining a unique pair of the expanded vertices. Thus the clan C would be drawn as



Theorem 3 *The category of k-representations of the clan C above is equivalent to the category of finite-rank free modules over the ring $R = k[d]/(d^2)$.*

Proof. A representation M of the clan C consists of a finite-dimensional vector space V, a subspace S of V, and an isomorphism M_a between V/S and $S/0 = S$. The isomorphism M_e gives V the structure of a module over $R = k[d]/(d^2)$ with $\ker d = \operatorname{im} d$.

Such R-modules are exactly the free R-modules: Indeed, free R-modules obviously have the property that $\ker d = \operatorname{im} d$. Conversely, suppose A is an R-module with $\ker d = \operatorname{im} d$. Let da_i be a vector space basis for dA and set $B = \sum Ra_i$. We first show that this sum is direct: if $\sum r_i a_i = 0$, then $\sum r_i da_i = 0$ so $r_i = s_i d$ with $s_i \in k$ because the da_i are independent over k. So $\sum s_i da_i = 0$ whence $s_i = 0$ and thus $r_i = 0$. Suppose $a \in A$. Then $db = da$ for some $b \in B$, so $d(b - a) = 0$ whence $b - a \in dA = dB$ so $a \in B$.

Conversely, if A is a free R-module, then setting $V = A$ and $S = dA$, and letting M_e be the isomorphism of A/dA and dA induced by d, gives a representation of the clan C.

A map from a representation M to a representation M' is a linear transformation $f : V \to V'$ such that $f(S) \subset S'$ and the diagram

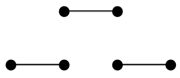
$$\begin{tikzcd}
\frac{V}{S} \arrow[r, leftrightarrow, "M_a"] \arrow[d] & S \arrow[d] \\
\frac{V'}{S'} \arrow[r, leftrightarrow, "M'_a"] & S'
\end{tikzcd}$$

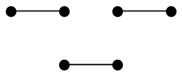
where the vertical maps are induced by f, commutes. For $V = A$, this is exactly the condition that $df = fd$, that is, that f is a map of R-modules. ■

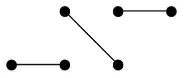
The element $d \in R$ is a nonzero zero-divisor, hence, by Theorem 2, induces a map from R to R that does not have a kernel in the category of free R-modules.

4 Simple degree-2 clans

We now restrict ourselves to clans whose graphs are **simple** (no loops or multiple edges), and particularly to those whose vertices have degree at most two. Here are the diagrams of three such clans:



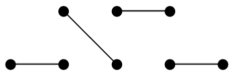




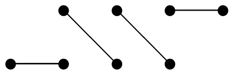
For simple clans, $\dim \operatorname{Hom}(A, B) \leq 1$ if A and B are indecomposable representations. In fact, $\dim \operatorname{Hom}(I_{e'}, I_e) = 1$ exactly when $e = e'$ or $e = e(v, i)$ and $e' = e(v, j)$ for some vertex v and $i < j$. Simple degree-2 clans have the additional property that if f and g are maps between irreducible representations, and $fg \neq 0$, then either f or g is an isomorphism.

In a representation of any one of the three clans above, we can ignore the spaces on the end vertices because they are forced to be isomorphic to certain quotients. In the middle, denote the spaces on the top vertices by V and W with subspaces S and T at the bottom vertices. For the first clan we have an isomorphism between V/S and W/T, in the second an isomorphism between S and T, and in the third an isomorphism between V/S and T. Representations of the third clan may be identified with representations of the quiver $\bullet \longrightarrow \bullet$, hence form an abelian category.

The second clan is clearly dual to the first. The third is self dual because if we turn it upside down (duality) and then reverse the vertices (just change the picture) we are back where we started. We leave it to the reader to show directly that the category of the second clan is pre-abelian—kernels are easy, cokernels a little harder. This result will be a consequence of the general theory developed in the following sections. We will also see that



is pre-abelian while



is not.

5 A more abstract setting

With the model of simple degree-2 clans in mind, the remainder of the paper is devoted to the study an arbitrary additive k-category $\mathcal{C}$ such that:

- $\mathcal{C}$ has a finite number of indecomposables $I_1, \ldots, I_n$ up to isomorphism,
- $\dim \operatorname{Hom}(I_i, I_j) \leq 1$ for $i, j \in N = \{1, \ldots, n\}$.

- every object in $\mathcal{C}$ is a finite direct sum of indecomposables,
- the product of any two maps between indecomposables of $\mathcal{C}$ is zero unless one of the maps is an isomorphism.

Let $\mathcal{I}$ be the full subcategory $\{I_1, \ldots, I_n\}$ of $\mathcal{C}$. We say that $i \in N$ is a **predecessor** of $j \in N$, or that j is a **successor** of i, if $i \neq j$ and $\operatorname{Hom}(I_i, I_j) \neq 0$. We write this as $i \to j$. In this way we turn N into a directed graph (digraph). For the five clans of the preceding section, these digraphs are $1 \leftarrow 2 \to 3$, $1 \to 2 \leftarrow 3$, $1 \leftarrow 2 \leftarrow 3$, $1 \leftarrow 2 \leftarrow 3 \to 4$, and $1 \leftarrow 2 \leftarrow 3 \leftarrow 4$.

For $i \in N$, an object in $\mathcal{C}$ is said to be i-**homogeneous** if it is a direct sum of indecomposables isomorphic to I_i. The full subcategory $\mathcal{C}_i$ of i-homogeneous objects is abelian: in fact $\mathcal{C}_i$ is equivalent to the category of finite-dimensional vector spaces over k. When operating totally within $\mathcal{C}_i$, we may proceed as if the objects were vector spaces.

Each object A in $\mathcal{C}$ can be decomposed as $A = \bigoplus_{i \in N} A_i$ where A_i is i-homogeneous. The summand A_i is not unique unless $A_j = 0$ whenever $i \to j$, but it is unique up to isomorphism. The subset

$$\operatorname{spt} A = \{i \in N : A_i \neq 0\}$$

is independent of the particular decomposition of A.

Theorem 4 *Every map in $\mathcal{I}$ has a kernel in $\mathcal{C}$ if and only if the digraph N has no paths of length 3. In that case, if $j \to k$, then $\bigoplus_{i \to j} I_i$ is the kernel of any nonzero map $I_j \to I_k$.*

Proof. Let $f : I_j \to I_k$ be a nonzero map and $\varphi : C = \bigoplus_i C_i \to I_j$. If C_i has a summand isomorphic to $I_i \oplus I_i$, or if $C_i \neq 0$ and $\varphi_i = 0$, then the zero map $I_i \to I_j$ factors in two ways through φ. If $\varphi_j \neq 0$, then $f\varphi \neq 0$. If $i \to j$ and $\varphi_j = 0$, and a nonzero map from I_i to I_j factors through φ, then $C_i \neq 0$. So if φ is a kernel of f, then C is isomorphic to $\bigoplus_{i \to j} I_i$.

If C is isomorphic to $\bigoplus_{i \to j} I_i$ and $\ell \to i \to j$, then the zero map from I_ℓ to I_j factors in two ways through φ. So if φ is a kernel of f, then no predecessor of j can have a predecessor. Hence if every map in $\mathcal{I}$ has a kernel in $\mathcal{C}$, then there are no paths of length 3 in N.

Finally, suppose there are no paths of length 3 in N. To show that $\varphi : \bigoplus_{i \to j} I_i \to I_j$ is a kernel of f, it suffices to show that a map $\theta : I_\ell \to I_j$, such that $f\theta = 0$, factors uniquely through φ. Because no predecessor of j can have a predecessor, if $\lambda : I_\ell \to \bigoplus_{i\to j} I_i$ is nonzero, then $\ell \to j$ and $\varphi\lambda \neq 0$. So if $\theta = 0$, then θ factors uniquely through φ. If $\theta \neq 0$, and $f\theta = 0$, then $\ell \to j$ so θ factors uniquely through φ. ■

As the path condition on the digraph N is self dual, it is also equivalent to the condition that every map in $\mathcal{I}$ has a *cokernel* in $\mathcal{C}$. We want to show that the path condition implies that every map in $\mathcal{C}$ has a kernel, hence that every map in $\mathcal{C}$ has a cokernel, so $\mathcal{C}$ is pre-abelian. This will give us a simple criterion for the category $\mathcal{C}$ to be pre-abelian.

6 Covers and kernels

Let M be a subset of N. A map $g : K \to A$ is an M-**kernel** of the map $f : A \to B$ if

1. $fg = 0$,
2. $\operatorname{spt} K \subset M$, and
3. if $g' : K' \to A$ where $fg' = 0$ and $\operatorname{spt} K' \subset M$, then $g' = gh$ for a unique $h : K' \to K$.

It follows easily that M-kernels are unique up to isomorphism (if they exist). Note that it suffices to check Condition 3 for K' indecomposable.

Define $\lambda(i)$ for $i \in N$ to be the length of a maximal chain of successors starting at i. Thus $\lambda(i) = 0$ if i has no successors, $\lambda(i) = \infty$ if there is a path from i to a circuit, and $\lambda(i) = \sup_{i\to j}(1 + \lambda(j))$.

We will show that M-kernels exist when $M = \{i\}$ and when $M = N_m = \{i \in N : \lambda(i) \leq m\}$ for $m = 0$, 1, and 2. Proving that N_2-kernels exist will complete the proof that $\mathcal{C}$ is pre-abelian exactly when there are no paths of length 3 in N.

For $A \in \mathcal{C}$ and $i \in N$, we say that $\varphi : C_i(A) \to A$ is an i-**cover** if

1. $C_i(A)$ is i-homogeneous, and
2. any map from an i-homogeneous object J to A factors uniquely through φ.

It suffices to verify Condition 2 for $J = I_i$. Note that an i-cover of A is the same as an i-kernel of the map $A \to 0$.

Theorem 5 *For each $A \in \mathcal{C}$ and $i \in N$, there is an i-cover of A.*

Proof. It suffices to take A indecomposable, say $A = I_j$. If $\mathrm{Hom}(I_i, I_j) = 0$, then let $C_i(I_j) = 0$. Otherwise let $C_i(I_j) = I_i$ and $\varphi : C_i(I_j) \to I_j$ any nonzero map. Then φ induces a nonzero linear transformation $\mathrm{Hom}(I_i, I_i) \to \mathrm{Hom}(I_i, I_j)$, which must be an isomorphism because both spaces have dimension 1. ■

Note that $C_i(A) = A_i \oplus \bigoplus_{i \to j} C_i(A_j)$. If we choose a particular cover for each object, we get a functor C_i because any map $A \to B$ composes to give a map $C_i(A) \to B$ which factors uniquely through $C_i(B)$.

Theorem 6 *For each $i \in N$, every map in $\mathcal{C}$ has an i-kernel.*

Proof. If $A \to B$ is a map in $\mathcal{C}$, let K_i be the kernel in $\mathcal{C}_i$ of the induced map $C_i(A) \to C_i(B)$. The composite map $K_i \to C_i(A) \to A$ is easily seen to be an i-kernel of $A \to B$. ■

That takes care of M-kernels for $M = \{i\}$. Just as easy are N_0-kernels.

Theorem 7 *Every map in $\mathcal{C}$ has an N_0-kernel.*

Proof. Any map $\varphi : A \to B$ in $\mathcal{C}$ restricts to maps $\varphi_k : A_k \to B_k$ for each $k \in N_0$. Let K_k be the kernel in $\mathcal{C}_k$ of φ_k. The induced map $\bigoplus_{k \in N_0} K_k \to A$ is an N_0-kernel of φ. ■

Note that the N_0-kernel, unlike the i-kernel, is always a summand of A. The next theorem establishes that N_1-kernels exist, and paves the way to showing that N_2-kernels exist.

Lemma 8 *Let $\varphi : A \to B$ be a map in $\mathcal{C}$ and $j \in N_1$. Let K_j be the j-kernel of φ and K_k the kernel in $\mathcal{C}_k$ of the restriction of φ to A_k. Then*

1. $K_j \cap \bigoplus_{j \to k} C_j(A_k) = \bigoplus_{j \to k} K_j \cap C_j(A_k)$,
2. $C_j(K_k) = K_j \cap C_j(A_k)$ *and*
3. *if K'_j is a complement of $K_j \cap \bigoplus_{j \to k} C_j(K_k)$ in K_j, then $K'_j \to A$ is a summand (has a left inverse).*

Moreover, we can choose A_j so that K'_j is a summand of it.

Proof. In $\mathcal{C}_j$ we have the diagram

$$\begin{tikzcd} C_j(A) \arrow[d] & = & A_j \oplus \bigoplus_{j \to k} C_j(A_k) \\ C_j(B) & = & B_j \oplus \bigoplus_{j \to k} C_j(B_k) \end{tikzcd}$$

As $k \in N_0$, the vertical map takes $C_j(A_k)$ to $C_j(B_k)$ because $\varphi(A_k) \subset B_k$. The kernel of the vertical map is K_j, so that establishes 1. To see 2, note that $C_j(K_k)$ is the kernel in $\mathcal{C}_j$ of $C_j(A_k) \to C_j(B_k)$ as the sequence $K_k \to A_k \to B_k$ is split exact. For 3, as $K'_j \cap \bigoplus_{j \to k} C_j(K_k) = 0$, the map $K'_j \to A_j$ induced by $K_j \to A$ and projection onto A_j has zero kernel. Let g be the left inverse of the map $K'_j \to A_j$, and define the map $\theta : A \to K'_j$ to be g on A_j and zero on A_ℓ for $\ell \neq j$. Then θ is a left inverse of $K'_j \to A$.

The final claim follows from a fact about vector spaces: If $K \subset Q \oplus R$, then there exists $f : Q \to R$ so that if $(q, r) \in K$, then $(q, f(q)) \in K$. The function f gives another decomposition $Q \oplus R = Q' \oplus R$, where $Q' = \{(q, f(q)) : q \in Q\}$, so that $K \cap (Q' \oplus R) = K \cap Q' \oplus K \cap R$. ■

Theorem 9 *Let $\varphi : A \to B$ be a map in $\mathcal{C}$, and decompose $A = \bigoplus_{i \in N} A_i$ in accordance with the last line of Lemma 8. Then the summand $K_{N_1} = \bigoplus_{k \in N_0} K_k \oplus \bigoplus_{j \in N_1 \setminus N_0} K'_j$ of A is an N_1-kernel of φ.*

Proof. Let $j \in N_1$ and $\theta : I_j \to A$ be such that $\varphi\theta = 0$. If $j \in N_0$, then θ maps I_j into the N_0-kernel $\bigoplus_{k \in N_0} K_k \subset K_{N_1}$. If $j \in N_1 \setminus N_0$, then θ lifts to a map into the j-kernel $K_j = K'_j \oplus K_j \cap \bigoplus_{j \to k} C_j(K_k)$ of φ, hence maps into K_{N_1}. ■

Corollary 10 *Let $\varphi : A \to B$. Let $K'_k = K_k$ for $k \in N_0$, so $K_{N_1} = \bigoplus_{j \in N_1} K'_j$. For $i \in N_2 \setminus N_1$, define K'_i so that $K_i = K'_i \oplus \bigoplus_{i \to j} C_i(K'_j)$. Then*

$$K_{N_2} = \bigoplus_{i \in N_2} K'_i = K_{N_1} \oplus \bigoplus_{i \in N_2 \setminus N_1} K'_i$$

is an N_2-kernel of φ.

Proof. Note that while K_{N_1} is a summand of A, the external direct sum $\bigoplus_{i \in N_2 \setminus N_1} K_i'$ maps into A via $K_i' \to K_i \to C_i(A) \to A$, and this latter map need not have any component in A_i. To show that K_{N_2} is an N_2-kernel, let $\theta : I_i \to A$ with $\varphi\theta = 0$. If $i \in N_1$, then I_i can't map into $\bigoplus_{i \in N_2 \setminus N_1} K_i'$, so $\theta : I_i \to A$ lifts uniquely to a map into K_{N_2}. If $i \in N_2 \setminus N_1$, then θ lifts uniquely to map into $K_i = K_i' \oplus \bigoplus_{i \to j} C_i(K_j')$ hence to $K_i' \oplus \bigoplus_{i \to j} K_j' \subset K_{N_2}$. The composite map into K_{N_2} is unique because any map of I_i into K_{N_2} goes into $K_i' \oplus \bigoplus_{i \to j} K_j'$. ■

If there are no paths of length 3 in N, then N_2-kernels are kernels. Hence Corollary 10, together with Theorem 4, gives us:

Theorem 11 *The category $\mathcal{C}$ is pre-abelian if and only if there are no paths of length* 3 *in* N.

7 Abelian categories

We can also answer the question as to when $\mathcal{C}$ is abelian.

Theorem 12 *The category $\mathcal{C}$ is abelian if and only if N has no paths of length 3 and every edge of N is in a path of length 2.*

Proof. Suppose $\mathcal{C}$ is abelian. As $\mathcal{C}$ is pre-abelian, N can have no paths of length 3. Suppose $i \to j$ with i having no predecessors and j no successors. Then the kernel and cokernel of any nonzero map $I_i \to I_j$ would be zero, whence $\mathcal{C}$ would not be abelian.

Conversely, suppose $\mathcal{C}$ is pre-abelian and every edge of N is in a path of length 2. We will show that if the kernel and cokernel of $\varphi : A \to B$ are zero, then φ is an isomorphism.

We first show that if $i \to j \to k$, then the map $\lambda = \pi_{B_j}\varphi\iota_{A_j} : A_j \to B_j$ is an isomorphism. Suppose $\theta : I_j \to A_j$ and $\lambda\theta = 0$. Let $\xi : I_i \to I_j$ be nonzero. Then $\varphi\iota_{A_j}\theta\xi = 0$ because any map from I_i through A_j must go into B_j. As the kernel of φ is zero, the map $\theta\xi$ is zero whence the map θ is also zero. Thus the kernel in $\mathcal{C}_j$ of λ is zero. Similarly the cokernel in $\mathcal{C}_j$ of λ is zero, so λ is an isomorphism.

Now $\varphi : A_j \oplus C \to B_j \oplus D$, where $C = \bigoplus_{\ell \neq j} A_\ell$ and $D = \bigoplus_{\ell \neq j} B_\ell$, and the induced map $\lambda = \pi_{B_j}\varphi\iota_{A_j} : A_j \to B_j$ is an isomorphism. So there exist complementary summands C' of A_j and B_j' of D so that φ maps A_j

isomorphically onto B_j' and $\varphi(C') \subset D$. Indeed, C' is C mapping into $A = A_j \oplus C$ by $\iota_C' = \iota_C - \iota_A \lambda^{-1} \pi_{B_j} \varphi \iota_C$, and B_j' is B_j mapping into B by $\iota_B' = \varphi \iota_{A_i} \lambda^{-1}$.

Passing to C' and D, we may assume that A_j and B_j are zero. Inducting, we may assume that each element of $\operatorname{spt} A \cup \operatorname{spt} B$ either has no predecessor or no successor. Thus any map between an indecomposable summand of A and an indecomposable summand of B is either zero or an isomorphism. It follows that $\varphi : A \to B$ is an isomorphism. ■

8 Prescribed digraphs

If N is any digraph without loops or multiple edges, then there is a category $\mathcal{C}$, of the type described in Section 5, with N as its digraph. Indeed we can take the objects of $\mathcal{C}$ to be finite sequences of vertices of N and the maps to be matrices (a_{ij}) over the field whose rows and columns are labeled, via a function ν, by vertices of N. The restriction on the matrices is that $a_{ij} = 0$ unless $\nu(i) = \nu(j)$ or $\nu(j) \to \nu(i)$. Multiplication of matrices is given by

$$c_{ik} = \sum_{\nu(i)=\nu(j) \text{ or } \nu(j)=\nu(k)} a_{ij} b_{jk}.$$

If N is isomorphic to the digraph of a Section 5 category $\mathcal{C}$, then this construction produces a category equivalent to $\mathcal{C}$.

Not every digraph without loops or multiple edges is the digraph of a simple degree-2 clan category. Indeed, the latter are characterized as those digraphs whose underlying graphs are simple, connected, and have vertices of degree at most two (lines and circles).

References

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