Flat dimension, constructivity, and the Hilbert syzygy theorem

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1. CONSTRUCTIVE MATHEMATICS

This paper is written in the spirit of constructive mathematics in the sense of Errett Bishop [3]. For a general introduction to constructive algebra see [11]. An algebraist with no background in constructive mathematics should have no trouble understanding the arguments, but might not see what obstacles are being overcome or sidestepped. From a procedural point of view, the difference is that we do not use the law of excluded middle. From an intuitive point of view, the difference is that our theorems are true under a computational interpretation.

For example, a set is **discrete** if, for any two elements x and y, either $x = y$ or $x \neq y$. Assuming the law of excluded middle, all sets are discrete; but in the absence of that law, there is no reason to believe that all sets are discrete. Moreover, the computational interpretation of a discrete set is one for which you have an algorithm that decides whether or not $x = y$. So, for example, the set of binary sequences is not discrete because (1) we cannot prove it is without using some form of the law of excluded middle, or (2) there is no algorithm that will decide, given an algorithm for a binary sequence, whether or not the sequence consists entirely of zeros. This is not to say that it is a theorem in constructive mathematics that the set of binary sequences is not discrete—on the contrary, any theorem in constructive mathematics is also a theorem in classical mathematics.

As we can reduce questions about the equality of binary sequences to questions about the equality of real numbers, the real numbers constitute another example of a set that is not discrete.

Arguments that require the axiom of choice are not constructively valid; in fact, the axiom of choice implies the law of excluded middle [6]. The failure of the axiom of choice comes up in the present context in that free modules need not be projective.

2. PROJECTIVE DIMENSION

Let R be a ring. Although our interest is in the case when R is commutative, none of our results will require this. However, we will not investigate the relationship between the categories of left R-modules and of right R-modules. The default is that "module" means left module, but this will rarely play any role.

If A is an R-module, with generators $(a_i)_{i\in I}$, then we can map the free R-module $R^{(I)}$, which consists of finite formal sums $r_1 i_1 + \cdots + r_n i_n$, onto A by mapping the element $r_1 i_1 + \cdots + r_n i_n$ to $r_1 a_{i_1} + \cdots + r_n a_{i_n}$. The kernel K of this map is the relation module, or module of syzygies. In the nicest situation, $K = 0$, and A is free on the given generators. Almost as good is when K is a summand, in which case A is projective—if I is not a finite set, then this holds only classically, because the projectivity of free modules requires some sort of choice axiom. With the same proviso, the module K is unique up to a free summand, independent of the choice of generating set. Indeed, given another generating set indexed by I', then Schanuel's trick, again assuming the free modules are projective, shows that

$$K \oplus R^{(I')} \cong K' \oplus R^{(I)}$$

If A is projective, we say that the **projective dimension** of A is zero, and write $\operatorname{pd} A = 0$. If K is projective, we say that $\operatorname{pd} A \leq 1$. If $\operatorname{pd} K \leq 1$, we say that $\operatorname{pd} A \leq 2$, etc. The **global projective dimension** of R is the supremum of the projective dimensions of its modules. M. Auslander [1] showed that it is enough to consider cyclic modules, so the idea of computing global dimension is not so outlandish as it might appear.

Let k be a discrete field. The (inhomogeneous) **Hilbert syzygy theorem** says that the polynomial ring $R = k[x_1, \ldots, x_n]$ has global projective dimension n. The original (homogeneous) Hilbert syzygy theorem, [10, Theorem VII.6.4], says that *graded* modules over R have free resolutions of length n. We are interested here in a constructive treatment of the usual (inhomogeneous) **abstract Hilbert syzygy theorem**, [10, Theorem VII.4.2], that if k is a commutative ring of global dimension r, then $k[x]$ has global dimension $r+1$. Or at least that is our starting point; we will see that a proper constructive treatment requires a modification of this, while still subsuming the concrete theorem.

We have to be careful when defining global dimension for a constructive development. Consider the case $n = 0$ of the concrete theorem. It is too much to ask that every module, or even every finitely generated module, over the rational numbers $\mathbf{Q}$ be projective. If a_n is a binary sequence with at most one 1, let S be the subspace of $V = \mathbf{Q} \oplus \mathbf{Q}$ generated by the countable set $\{(1, n) : a_n = 1\}$. Then $M = V/S$ is a finitely generated discrete $\mathbf{Q}$-module, which locally looks two-dimensional if $a_n = 0$ for all n up to some large value. But if M were projective, then S would be a summand of V, hence finite-dimensional, and this would imply that either $a_n = 1$ for some n, or $a_n = 0$ for all n.

So we have to limit the class of modules that we consider for global dimension. One possibility is the class of *discrete cyclic* modules. This works for fields, but we can't compute the dimension of an arbitrary discrete cyclic module over the integers—we have to know whether or not it is torsion. This latter observation leads us to consider the class of *finitely presented* modules—we can compute the dimension of any finitely presented abelian group. This takes care of the case $n = 0$ because finitely presented vector spaces are finite-dimensional, hence free.

In order to be able to compute the projective dimensions of modules in a certain class, it would obviously be desirable, and probably essential, for the class to be closed under taking modules of relations. For this to hold for finitely presented modules, we need the ring to be *coherent*, that is, finitely generated ideals are finitely presented. While this follows classically from the ascending chain condition, it is an additional hypothesis from a constructive point of view.

For R coherent and Noetherian, and global dimension defined in terms of the finitely presented modules only, we can prove the abstract inhomogeneous theorem. A central role is played by the countably generated, coherent modules. The reason for this is that $R[x]$ is such an R-module, and that global dimension computed for such modules is no greater than for finitely presented modules. The fact that R is Noetherian plays very little part in the argument: it is used to show that $R[x]$ is coherent ([12], [14], [11], [15]), and that coherent $R[x]$-modules are coherent R-modules.

But restricting the class of modules changes the classical meaning of global dimension. In addition, by requiring R to be Noetherian, we have placed a classically significant restriction on the ring. Less significant, but still annoying, is that we need to invoke the countable axiom of choice—a principle used by all constructive schools—because we need countable rank free modules to be projective. A clue to the proper treatment is that a finitely presented module is projective if and only if it is flat ([4, Exercise 2.23.a] and [11]). In [2] this is used to decide whether a finitely presented module is projective (for commutative rings). It follows that if R is coherent, then the projective dimension of a finitely presented module is the same as its flat dimension. Thus what we are really computing here is flat dimension. Moreover, for the concrete inhomogeneous Hilbert syzygy theorem, we might as well be dealing with flat dimension.

3. Flat dimension

What is the definition of flat dimension? If we define it in terms of free resolutions that end up in flat modules, we must worry about dependence on the resolution. The usual proof of independence requires projectivity of the free modules, which requires a choice axiom. In fact this definition turns out to be equivalent to the one we adopt (see Theorem 3, keeping in mind that free modules are flat). We will consider other alternative definitions at the end of the paper.

Here is a rephrasing of the usual definition of flat. We say that A is **flat** if, whenever we are given a map f from a finite-rank free module F into A, and an element θ in the kernel of f, then f factors through a finite-rank free module, $F \to F' \to A$, such that θ goes to zero in F'.

- Vector spaces over discrete fields are flat [11, Exercise III.5.5]. Note, however, that vector spaces over the real numbers need not be flat. In fact, all R-modules are flat if and only if R is von Neumann regular [11, Exercise III.5.6], and if the real numbers were von Neumann regular, they would be discrete: $axa = a$ implies $ax(1-ax) = 0$, and either $ax \neq 0$, whence $a \neq 0$, or $1 - ax \neq 0$, whence $ax = 0$, and $a = axa = 0$. (If $r + s = 1$ in the real numbers, then either $r \neq 0$ or $s \neq 0$. This can be shown by approximating r and s by rational numbers. It is essentially the statement that the real numbers form a local ring.)
- Free modules are flat [11, III.5.2]. This follows rather easily from the definition of equality in the construction of a free module on an arbitrary index set as a set of formal sums [11, p. 54]. The reason it's not completely trivial is that the index set need not be discrete, and so we must be a little careful in defining equality of formal sums. Notice that every module is the homomorphic image of a free module—there are enough free modules—but the same cannot be said for projective modules.
- Summands of flat modules are flat. This follows immediately from the definition.
- Projective modules are flat—they are summands of free modules.
- Finitely presented flat modules are projective [11].

We say that A has **flat dimension** at most n, and write $\operatorname{fd} A \leq n$, if given any complex

$$F_n \to F_{n-1} \to \cdots \to F_0 \to A \tag{1}$$

where the F_i are finite-rank free, and an element θ in F_n that goes to zero in F_{n-1}, then there exists a commutative diagram

$$\begin{array}{ccccccccc} F_n & \to & F_{n-1} & \to & \cdots & \to & F_0 & \to & A \\ \downarrow & & \downarrow & & & & \downarrow & & \| \\ F'_n & \to & F'_{n-1} & \to & \cdots & \to & F'_0 & \to & A \end{array} \tag{2}$$

where the F'_i are finite-rank free, and $\theta \in F_n$ goes to zero in F'_n. Some observations:

- By iteration, we can arrange for any finitely enumerable subset of F_n that goes to zero in F_{n-1} to go to zero in F'_n. (A set $\{x_1, x_2, \ldots, x_m\}$ is finitely enumerable; it may fail to be finite because it may not be discrete.)

- Hence $\mathrm{fd}\, A \leq n$ if and only if whenever $F_{n+1} \to F_n \to \cdots \to F_0 \to A$ is a complex, with the F_i finite-rank free, then we can construct a commutative diagram

$$\begin{array}{ccccccccccc} F_{n+1} & \to & F_n & \to & F_{n-1} & \to & \cdots & \to & F_0 & \to & A \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \| \\ 0 & \to & F'_n & \to & F'_{n-1} & \to & \cdots & \to & F'_0 & \to & A \end{array}$$

  with the F'_i finite-rank free. It's not hard to see that we could replace "finite-rank free" by "finitely generated projective," which has the virtue of being definable purely in terms of the category of modules. Note also that, for $n = 0$, this characterization gives an alternative definition of "flat" which amounts to the characterization in [5, 1.6, Theorem 1(iii)] that every map from a finitely presented module factors through a finite-rank free module.

- If $\mathrm{fd}\, A \leq n$, then $\mathrm{fd}\, A \leq n + 1$. This follows easily from the preceding remark. So if we define $\mathrm{fd}\, A = \{n \in \mathbf{N} : \mathrm{fd}\, A \leq n\}$, then $\mathrm{fd}\, A$ is a set of nonnegative integers that is closed under successor. The natural partial order $S_1 \leq S_2$ on such sets, defined by $S_1 \supseteq S_2$, allows us to work with them as if they were elements of $\mathbf{N} \cup \{\infty\}$, up to a point. Of course we can identify $\mathrm{fd}\, A$ with an element of $\mathbf{N} \cup \{\infty\}$ only if we can compute the exact dimension of A. In any event, we will not need to interpret the symbol "$\mathrm{fd}\, A$" standing alone; it will always occur in the expression "$\mathrm{fd}\, A \leq n$."

- We can simplify the notation by letting $\mathbf{F}$ stand for a complex, indexed by the nonnegative integers. A **map** from a complex $\mathbf{F}$ to a module A is a map $F_0 \to A$ so that $F_1 \to F_0 \to A$ is a complex. This amounts to a map of complexes where A is considered as a complex with $A_i = 0$ for $i \neq 0$.

- We say that $\deg \mathbf{F} \leq n$ if $F_m = 0$ for $m > n$. Then $\mathrm{fd}\, A \leq n$ if and only if every map of a finite-rank free complex into A factors through one of degree at most n.

- We can think of our complexes as indexed by the integers, with $F_n = 0$ for $n < 0$.

- The internal maps of a complex will be denoted generically by ∂.

- If $\mathbf{F}$ is a complex, the **translates** $\mathbf{F}_-$ and $\mathbf{F}_+$ of $\mathbf{F}$ are defined by $(\mathbf{F}_-)_n = F_{n-1}$ and $(\mathbf{F}_+)_n = F_{n+1}$ for $n \geq 0$. Note that $\mathbf{F}_{-+} = \mathbf{F}$, and $(\mathbf{F}_-)_0 = 0$.

- Given a map $f : \mathbf{F} \to \mathbf{F}'$ between two complexes, the **mapping cylinder** $\mathbf{M}(f)$ of f is the complex $\mathbf{F} \oplus \mathbf{F}'_+$ where the map $\partial : F_i \oplus F'_{i+1} \to F_{i-1} \oplus F'_i$ takes (x, y) to $(-\partial x, \partial y + fx)$.

  It is readily checked that $\mathbf{M}(f)$ is a complex, and that the natural map from $\mathbf{F}'_+$ to $\mathbf{M}(f)$ is a map of complexes, so $\mathbf{F}'_+$ is a subcomplex of $\mathbf{M}(f)$ in a natural way. In our applications, F'_0 will always be zero, so nothing gets lost in passing to $\mathbf{F}'_+$.

What does $\operatorname{fd} A > n$ mean? For discrete rings it undoubtedly should mean that there exists a complex (1), and an element $\theta \in F_n$ that goes to zero in F_{n-1}, so that whenever we have a commutative diagram (2), then θ goes to a nonzero element in F'_n. For arbitrary rings it's not clear what the good meaning of $\operatorname{fd} A > n$ is. We will formulate all our theorems in terms of $\operatorname{fd} A \leq n$.

So what is the flat Hilbert syzygy theorem? We want to show that $\operatorname{fd} A \leq n$ for all R-modules A, if and only if $\operatorname{fd} A \leq n + 1$ for all $R[x]$-modules A. The main tool is a lemma relating the flat dimensions of modules in a short exact sequence. In proving this we will use the following construction of a map into a mapping cylinder.

Lemma 1. *Let* $\mathbf{F}$, $\mathbf{F}'$ *and* $\mathbf{F}''$ *be complexes. Suppose we are given maps* $f : \mathbf{F} \to \mathbf{F}'_-$ *and* $g : \mathbf{M}(f) \to \mathbf{F}''$. *Let* g' *be the restriction of* g *to* $\mathbf{F}'$, *viewed as a map from* $\mathbf{F}'_- \to \mathbf{F}''_-$. *Then there is a map* $\mathbf{F} \to \mathbf{M}(g')$ *that takes* $x \in F_i$ *to* $(-1)^i(-fx, gx) \in F'_{i-1} \oplus F''_i$.

Proof. We have to show that the square

$$\begin{array}{ccc} F_{i+1} & \xrightarrow{\partial} & F_i \\ \downarrow & & \downarrow \\ F'_i \oplus F''_{i+1} & \to & F'_{i-1} \oplus F''_i \end{array}$$

commutes, where the maps down take $x \in F_{i+1}$ to $(-1)^i(fx, -gx)$ and $x \in F_i$ to $(-1)^i(-fx, gx)$, and the bottom map takes (x, y) to $(-\partial x, \partial y + gx)$.

Let $x \in F_{i+1}$. Going over and down takes x to ∂x to $(-1)^i(-f\partial x, g\partial x)$. Going down and over takes x to $(-1)^i(fx, -gx)$ to $(-1)^i(-\partial fx, -\partial gx + gfx)$. But $f\partial x = \partial fx$ because f is a map of complexes, and $g\partial x = -\partial gx + gfx$ because $g : \mathbf{M}(f) \to \mathbf{F}''$ is a map of complexes, and the boundary of x in $\mathbf{M}(f)$ is $(-\partial x, fx)$, so $\partial gx = g(-\partial x, fx) = -g\partial x + gfx$. ■

Here is the main lemma.

Lemma 2. *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence of* R*-modules. If* $\operatorname{fd} A \le n$ *and* $\operatorname{fd} B \le n+1$*, then* $\operatorname{fd} C \le n+1$*. If* $\operatorname{fd} B \le n$ *and* $\operatorname{fd} C \le n+1$*, then* $\operatorname{fd} A \le n$*.*

Proof. Throughout the proof, use of the bold letter $\mathbf{F}$ will indicate a complex of finite-rank free modules. We will consider A to be a submodule of B.

To show that $\operatorname{fd} C \le n+1$, suppose we have a map $\mathbf{F} \to C$. Lift the map $F_0 \to C$ to a map $\beta : F_0 \to B$. The composition $F_1 \to F_0 \to B$ maps F_1 into A, giving a map $\beta\partial : \mathbf{F}_+ \to A$. As $\operatorname{fd} A \le n$, there is a complex $\mathbf{F}'$, with $\deg \mathbf{F}' \le n$, and maps $\mathbf{F}_+ \to \mathbf{F}'$ and $\alpha : \mathbf{F}' \to A$, so that the diagram

$$\begin{array}{ccc} \mathbf{F}_+ & \stackrel{\beta\partial}{\to} & A \\ \downarrow & & \| \\ \mathbf{F}' & \stackrel{\alpha}{\to} & A \end{array} \tag{3}$$

commutes. The mapping cylinder $\mathbf{M}(f)$ of the map $f : \mathbf{F} \to \mathbf{F}'_-$ corresponding to $\mathbf{F}_+ \to \mathbf{F}'$ maps into B via the map $F_0 \oplus F_0' \to B$ that is the sum of the given maps $\beta : F_0 \to B$ and $\alpha : F_0' \to A$. We need to show that $\mathbf{M}(f) \to B$ is a complex at $F_0 \oplus F_0'$. But $(x, y) \in F_1 \oplus F_1'$ goes to $(-\partial x, \partial y + fx) \in F_0 \oplus F_0'$ which goes to $-\beta\partial x + \alpha f x \in B$ which is zero as $\beta\partial = \alpha f$ from diagram (3).

As $\operatorname{fd} B \le n+1$ there is a commutative diagram

$$\begin{array}{ccc} \mathbf{M}(f) & \stackrel{\beta+\alpha}{\to} & B \\ g\downarrow & & \| \\ \mathbf{F}'' & \to & B \end{array} \tag{4}$$

with $\deg \mathbf{F}'' \le n+1$. Let $g' : \mathbf{F}'_- \to \mathbf{F}''_-$ be the map induced by the restriction of g to $\mathbf{F}'$. Lemma 1 gives us a map $\mathbf{F} \to \mathbf{M}(g')$ that is equal to g on F_0. We can map $\mathbf{M}(g')$ to C by the composite $F_0'' \to B \to C$. This makes the diagram

$$\begin{array}{ccc} \mathbf{F} & \to & C \\ \downarrow & & \| \\ \mathbf{M}(g') & \to & C \end{array}$$

commute because diagram (4) commutes. The bottom row is a complex because the last three terms are

$$F_0' \oplus F_1'' \to F_0'' \to C$$

and $(x, y) \in F_0' \oplus F_1''$ goes to $-\partial y + gx \in F_0''$. The term $-\partial y$ goes to 0 already in B, while x is taken by α into A, so gx goes to 0 in C—see diagram (4). As $\deg \mathbf{M}(g') \le n+1$, this proves the first part of the lemma.

To show that $\mathrm{fd}\, A \le n$, suppose we are given a map $\alpha : \mathbf{F} \to A$. As $\mathrm{fd}\, B \le n$ we can construct a commutative diagram

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{\alpha} & A \\ \downarrow & & \downarrow \\ \mathbf{F}' & \xrightarrow{\lambda} & B \end{array}$$

with $\deg \mathbf{F}' \le n$. Let $f : \mathbf{F}_- \to \mathbf{F}'_-$ be the map induced by the map $\mathbf{F} \to \mathbf{F}'$. Let π denote the given map from B onto C. The mapping cylinder $\mathbf{M}(f)$ maps into C via the composite $F'_0 \xrightarrow{\lambda} B \xrightarrow{\pi} C$. We need to show that $\mathbf{M}(f) \xrightarrow{\pi\lambda} C$ is a complex at F'_0. The element $(x, y) \in F_0 \oplus F'_1$ goes to $\partial y + fx \in F'_0$, and $\lambda \partial y = 0$, because $\mathbf{F}' \to B$ is a complex, while $\lambda f x = \alpha x \in A$.

As $\mathrm{fd}\, C \le n + 1$ there is a commutative diagram

$$\begin{array}{ccc} \mathbf{M}(f) & \xrightarrow{\pi\lambda} & C \\ g\downarrow & & \| \\ \mathbf{F}'' & \to & C \end{array}$$

where $\deg \mathbf{F}'' \le n + 1$. The last two squares in this diagram are

$$\begin{array}{ccccc} F_0 \oplus F'_1 & \to & F'_0 & \xrightarrow{\pi\lambda} & C \\ \downarrow & & \downarrow & & \| \\ F''_1 & \xrightarrow{\partial} & F''_0 & \to & C \end{array}$$

from which we see that if $x \in F_0$, then $gfx = \partial gx$.

Let $g' : \mathbf{F}'_- \to \mathbf{F}''_-$ be the map induced by g restricted to $\mathbf{F}'$. Lemma 1 gives us a map $\mathbf{F}_- \to \mathbf{M}(g')$, and hence a map $\mathbf{F} \to \mathbf{M}(g')_+$, that takes $x \in F_0 = (\mathbf{F}_-)_1$ to $(fx, -gx)$. Lift the map $F''_0 \to C$ to a map $\mu : F''_0 \to B$. The difference $\xi = \mu g - \lambda$ of the composites $F'_0 \xrightarrow{g} F''_0 \xrightarrow{\mu} B$ and $F'_0 \xrightarrow{\lambda} B$ goes into A because it goes to 0 in C by the last square in the diagram above. The composite $\eta : F''_1 \xrightarrow{\partial} F''_0 \xrightarrow{\mu} B$ also goes into A, because $\mathbf{F}'' \to C$ is a complex, so we get a commutative diagram of complexes

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{\alpha} & A \\ \downarrow & & \| \\ \mathbf{M}(g')_+ & \xrightarrow{-\xi-\eta} & A \end{array}$$

The only problems are observing that the bottom row is a complex at $F'_0 \oplus F''_1$, and that the right-hand square commutes. The last two squares of this diagram look like:

$$\begin{array}{ccccc} F_1 & \to & F_0 & \xrightarrow{\alpha} & A \\ \downarrow & & \downarrow & & \| \\ F'_1 \oplus F''_2 & \to & F'_0 \oplus F''_1 & \xrightarrow{-\xi-\eta} & A \end{array}$$

In the bottom row, $(x, y) \in F_1' \oplus F_2''$ goes to $(-\partial x, \partial y + gx)$ which goes to $\mu g \partial x - \lambda \partial x + \mu \partial \partial y + \mu \partial g x = 0$ as g commutes with ∂ and $\lambda \partial = 0$. To see that the last square commutes, let x be in F_0. Down and right takes x to $(fx, -gx)$ to $-\xi fx + \eta gx = -\mu gfx + \lambda fx + \mu \partial gx = \lambda fx = \alpha x$. ■

We leave to the reader to prove that if $\operatorname{fd} A \leq n$ and $\operatorname{fd} C \leq n$, then $\operatorname{fd} B \leq n$. We will not need this fact.

For the construction of a free module on an arbitrary set, see the construction of an external direct sum in [11, p. 54]. Free resolutions provide an alternative definition of flat dimension, as the following theorem shows. However, as free modules need not be projective, the development of the theory from this definition would have to rely on their flatness, so would probably involve constructions like the ones in the proof of the preceding lemma.

Theorem 3. *Let $0 \to K \to F_{n-1} \to \cdots \to F_0 \to A \to 0$ be an exact sequence with each F_i a flat module. Then $\operatorname{fd} A \leq n$ if and only if K is flat.*

Proof. If K is flat, then repeated application of Lemma 2, starting by proving that the image of F_{n-1} in F_{n-2} has flat dimension at most one, shows that $\operatorname{fd} A \leq n$. Conversely, if $\operatorname{fd} A \leq n$, then repeated application of Lemma 2, starting by proving that the kernel of the map $F_0 \to A$ has flat dimension at most $n-1$, shows that K is flat. ■

Corollary 4. *If A is a finitely presented module over a coherent ring, then $\operatorname{fd} A \leq n$ if and only if the projective dimension of A is at most n.*

Proof. As finitely presented modules are flat if and only if they are projective, this is an immediate consequence of Theorem 3. ■

4. The flat Hilbert syzygy theorem

If R is a ring, then $R[x]$ denotes the polynomial ring in one variable x with coefficients in R, where x commutes with each element of R. If M is an R-module, then $M[x]$ is the set of polynomials with coefficients in M. Clearly $M[x]$ is an R-module, and an $R[x]$-module, in a natural way.

In [11, Theorem III.5.3] it is shown that a (left) module M is flat if and only if whenever A is a submodule of a right module B, then the natural map from $A \otimes M$ to $B \otimes M$ is one-to-one—a standard classical result. As a consequence, if $R \to S$ is a map of rings, and M is a flat R-module, then $S \otimes_R M$ is a flat S-module. In particular, using this observation with the natural ring maps $R \to R[x]$ and $R[x] \to R$, we see that M is a flat R-module if and only if $M[x]$ is a flat $R[x]$-module.

Theorem 5. *If A is an R-module, then $\operatorname{fd}_R A \leq n$ if and only if $\operatorname{fd}_{R[x]} A[x] \leq n$.*

Proof. We observed the case $n = 0$ above. For $n > 0$, take a free resolution $0 \to K \to F \to A \to 0$ of A, giving a free resolution $0 \to K[x] \to F[x] \to A[x] \to 0$ of $A[x]$. By induction, $\operatorname{fd}_R K \leq n-1$ if and only if $\operatorname{fd}_{R[x]} K[x] \leq n-1$, so the theorem follows from Theorem 3. ■

The following result uses the key construction in the proof (by contradiction) of [13, Theorem 9.33]. By $\operatorname{fd}_R A < \infty$ we mean that $\operatorname{fd}_R A \leq n$ for some n.

Lemma 6. *Let A be an $R[x]$-module such that $xA = 0$ and $\operatorname{fd}_R A < \infty$. If $\operatorname{fd}_{R[x]} A \leq n+1$, then $\operatorname{fd}_R A \leq n$.*

Proof. Let $0 \to K \to F \to A \to 0$ be a free $R[x]$-resolution of A. Then we have an exact sequence

$$0 \to \frac{xF}{xK} \to \frac{K}{xK} \to \frac{F}{xF} \to A \to 0.$$

If $n = 0$, then K is flat over $R[x]$, so $K/xK \cong R \otimes_{R[x]} K$ is flat over R. As A is isomorphic to xF/xK, if $\operatorname{fd}_R A \leq m$, then $\operatorname{fd}_R A \leq \max(0, m-2)$, so A is a flat R-module.

If $n > 0$, then $\operatorname{fd}_{R[x]} K \leq n-1$ and, as F is also a free R-module, $\operatorname{fd}_R K < \infty$. By induction, $\operatorname{fd}_R K \leq n-1$, so $\operatorname{fd}_R A \leq n$. ■

The next theorem is [13, Lemma 9.29].

Theorem 7. *If A is an $R[x]$-module, then the sequence*

$$0 \to A[x] \to A[x] \to A \to 0$$

is exact, where the second map takes ax^i to $x^i a$, and the first map takes ax^i to $xax^i - ax^{i+1}$.

Proof. The first map takes $\sum_{i=0}^{k} a_i x^i$ to

$$xa_0 + (xa_1 - a_0)x + (xa_2 - a_1)x^2 + \cdots - a_k x^{k+1}$$

The kernel of the first map is zero, and the composition is zero. Suppose $\sum x^i a_i = 0$. For $i \geq 0$ define

$$-b_i = a_{i+1} + xa_{i+2} + x^2 a_{i+3} + \cdots$$

Then $xb_i - b_{i-1} = a_i$ for $i \geq 1$, and $xb_0 = a_0$. ■

Corollary 8 [Flat Hilbert Syzygy]. *Let R be a ring and n a nonnegative integer. Then the following conditions are equivalent.*

1. *Every R-module has flat dimension at most n.*
2. *Every $R[x]$-module has flat dimension at most $n+1$.*

Proof. Suppose (1) holds and A is an $R[x]$-module. Then $\mathrm{fd}_R A \leq n$, so $\mathrm{fd}_{R[x]} A[x] \leq n$. Hence $\mathrm{fd}_{R[x]} A \leq n+1$ by Theorem 7 and Lemma 2. Conversely, suppose (2) holds and A is an R-module. Then A becomes an $R[x]$-module if we set $xA = 0$, and $\mathrm{fd}_{R[x]} A \leq n+1$. Hence $\mathrm{fd}_R A \leq n$ by Lemma 6. ■

5. Flat dimension via Tor

Another definition of flat dimension is that $\mathrm{Tor}_n(A,B) = 0$ for all modules A and B, where the functor Tor_n is the one constructed without projective resolutions by MacLane [10, V.7]. The elements of MacLane's Tor_n are pairs of complexes

$$F_n^* \to F_{n-1}^* \to \cdots \to F_0^* \to A$$

$$F_n \to F_{n-1} \to \cdots \to F_0 \to B$$

where the F_i are finite-rank free, and $F_i^* = \mathrm{Hom}_R(F_i, R)$. Thus the basic structures involved in the this definition of Tor_n are the same as in our definition of flat dimension. MacLane's treatment appears to go through without a constructive hitch.

Alternatively we could define Tor_n by taking a free resolution of B, tensoring it with A, and showing that the homology of the resulting complex is a functor (and so does not depend on the resolution of B). The usual constructions use arbitrary projective resolution. The existence of projective resolutions derives from the existence of free resolutions, which is constructive, and the projectivity of free modules, which is not. We can circumvent this by taking *canonical* free resolutions: for any R-module B there is a canonical free module mapping onto it, namely the free module $F_1(B)$ on the underlying set of B. It follows from the universal property of free modules that $F_1(B)$ is a functor (see [11] for constructive details).

Set $F_0(B) = B$ and $F_{-1}(B) = 0$, and inductively define $F_i(B)$ to be F_1 of the kernel of the map from $F_{i-1}(B)$ to $F_{i-2}(B)$ for $i > 0$. The result is a functorial free resolution of B. Given a map $B \to B'$ we get a commutative diagram of exact sequences

$$\begin{array}{ccccccccccc} \cdots & \to & F_3(B) & \to & F_2(B) & \to & F_1(B) & \to & B & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \to & F_3(B') & \to & F_2(B') & \to & F_1(B') & \to & B' & \to & 0 \end{array}$$

hence a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \to & A\otimes F_3(B) & \to & A\otimes F_2(B) & \to & A\otimes F_1(B) & \to & A\otimes B \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & A\otimes F_3(B') & \to & A\otimes F_2(B') & \to & A\otimes F_1(B') & \to & A\otimes B'
\end{array}$$

from which it's easy to see that $\mathrm{Tor}_n(A,B)$ is a functor. The question of dependence on resolution does not come up because we take a canonical one. Thus we can construct $\mathrm{Tor}_n(A,B)$, but its key properties are more difficult to verify.

Clearly $\mathrm{Tor}_n(A,B)=0$ if A is flat, but it's not so clear that $\mathrm{Tor}_n(A,B)=0$ if B is flat. This can be established by showing that a short exact sequence $0\to K\to F\to F/K\to 0$ remains exact after tensoring provided that F/K is flat, and that K is flat if F and F/K are (the latter is part of Lemma 2).

We also have to show that, conversely, if $\mathrm{Tor}_1(A,B)=0$ for every A, then B is flat. This may be deduced from the long exact sequence

$$\cdots\to \mathrm{Tor}_1(A_2,B)\to \mathrm{Tor}_1(A_3,B)\to A_1\otimes B\to A_2\otimes B\to A_3\otimes B\to 0$$

which is easy to construct because the free resolution of B consists of flat modules, so tensoring with an exact sequence of modules gives an exact sequence of complexes from which to derive the long exact sequence of homology. However to prove Lemma 2 in this development we need the long exact sequence

$$\cdots\to \mathrm{Tor}_1(A,B_2)\to \mathrm{Tor}_1(A,B_3)\to A\otimes B_1\to A\otimes B_2\to A\otimes B_3\to 0$$

which is also what we would use to prove that if $\mathrm{Tor}_1(A,B)=0$ for every B, then A is flat. Here the connecting homomorphism $\mathrm{Tor}_1(A,B_3)\to A\otimes B_1$ is defined in the classical case by building a resolution of B_2 from the resolutions of B_1 and B_3, an option not available to us. Perhaps we can prove that if $\mathrm{Tor}_1(A,B)=0$ for every B, then A is flat, without using this latter sequence, in which case we could define the flat dimension of A in terms of the vanishing of $\mathrm{Tor}_n(A,B)$ for all B. But we would want both exact sequences eventually.

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