Algebraic functions, calculus style

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I was a little disturbed to learn that $|x|$ was an algebraic function, but it makes perfectly good sense because the function $|x|$ is a root of the polynomial $X^2 - x^2$ in accordance with the standard definition of an algebraic function as one that satisfies a polynomial (in X) whose coefficients are polynomials (in x) with real (or complex) coefficients (see [1] for example). However, an algebraic function is normally considered to be an element of an algebraic function *field*, and if $|x|$ is an algebraic function, then it, together with $-|x|$, x, and $-x$, constitute four distinct roots of the quadratic polynomial $X^2 - x^2$ which should only have two roots in a field. Another of looking at this is that $(|x| + x)(|x| - x) = 0$ so we have zero divisors in the ring of algebraic functions if $|x|$ is an algebraic function.

On the other hand, the signum function defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

is a root of the polynomial $X^3 - X$, yet is *not* an algebraic function! Why not? In the context of calculus, the consensus is that a (real) function is algebraic if it can be constructed from polynomials using addition, subtraction, multiplication, division, and extraction of roots (see [2] for example). In particular, $|x| = \sqrt{x^2}$ is an algebraic function in this sense. If we are going to show that the signum function is not algebraic, we need to spell out this definition a little more carefully:

- The constant function $f(x) = r$ is an algebraic function for each real number r.

- The identity function $f(x) = x$ is an algebraic function.
- If f and g are algebraic functions so are
  - $f + g$
  - $f \cdot g$
  - f/g
  - $\sqrt[m]{f}$ where m is a positive integer.

We have to worry about domains because algebraic functions are not defined everywhere: for example, $\sqrt{x}$ is only defined when $x \geq 0$. Define the domain, $\operatorname{dom} f$, of an algebraic function f inductively by

- $\operatorname{dom}(f+g) = \operatorname{dom} f \cdot g = \operatorname{dom} f \cap \operatorname{dom} g$
- $\operatorname{dom} f/g = \{x \in \operatorname{dom} f \cap \operatorname{dom} g : g(x) \neq 0\}$
- $\operatorname{dom} \sqrt[m]{f} = \operatorname{dom} f$, if m is odd, and $\{x \in \operatorname{dom} f : f(x) \geq 0\}$ if m is even.

So, for example, the domain of the function $\sqrt{x} + \sqrt{1-x}$ is the interval $[0, 1]$, while the domain of the function $0/(x-1)$ consists of those real numbers that are different from 1. The function $0/(x-1)$ differs from the function 0 because it is not defined at 1.

The signum function above, which is not continuous at 0, is not algebraic because an algebraic function is continuous at each point of its domain, as the following theorem shows.

Theorem 1 *If f is an algebraic function, then there exist points $t_1 < t_2 < \cdots < t_n$ cutting up $\mathbf{R}$ into $n+1$ open intervals $I_0, I_1, \ldots, I_n$ such that on each I_i*

- *f is undefined or*
- *f is analytic (and algebraic) and never 0 (hence positive or negative) or*
- *f is identically 0.*

Moreover, f is continuous at each point of its domain (within that domain).

Proof. The constant functions and the function $f(x) = x$ satisfy these conditions with $n = 1$ and $t_1 = 0$. We need to show that if f and g satisfy these conditions, then so do $f + g$, $f \cdot g$, f/g, and $\sqrt[m]{f}$. Note that if the conditions hold, and we add finitely many more points to the nodes $t_1, \ldots, t_n$, then the conditions still hold. For $\sqrt[m]{f}$ we simply adjoin 0 to the nodes $t_1, \ldots, t_n$, if it isn't there already. Note that $\sqrt[m]{\cdot}$ is analytic on $(0, \infty)$, and analytic on $(-\infty, 0)$ if m is odd. For the others we take the union of the nodes for f and the nodes for g, so the resulting open intervals are totally contained in intervals associated with both f and g. Continuity follows from the continuity of the four algebraic operations on their domains. ■

Call the points (or the open intervals) in Theorem 1 a **witnessing grid** for f. Note that f may or may not be defined on those points.

It is also true that f satisfies, on its domain, a nontrivial polynomial with coefficients in $\mathbf{R}[x]$. One way to see this is to go generic. The key identities are that if

$$a_m x^m + \cdots + a_1 x + a_0 = 0$$

and

$$b_n y^m + \cdots + b_1 y + b_0 = 0$$

then

$$c_k (x + y)^k + \cdots + c_1 (x + y) + c_0 = 0$$

where the c's are polynomials in the a's and b's, and c_k is a power of $a_m b_n$. Similarly for xy in place of $x + y$. This can be proved by looking within the polynomial ring over $\mathbf{Z}$ in the indeterminates $a_0, \ldots, a_m, b_0, \ldots, b_n, x, y$. We could also consider the ring of continuous real valued functions on some infinite subset of $\mathbf{R}$, note that the nonzero elements of $\mathbf{R}[x]$ within that ring are regular, pass to the localization of the entire ring with respect to $a_m b_n$, and use the fact that integral elements over the localization of $\mathbf{R}[x]$ with respect to $a_m b_n$ form a ring. (The passage from x to $1/x$ is no problem.)

> The converse of the statement at the start of the previous paragraph must be false even for everywhere defined continuous functions, but I don't think I ever came up with an ironclad counterexample. The reason is that algebraic functions correspond to solving polynomials by radicals, which presumably can't always be done. Except conceivably it could if we restricted ourselves to polynomials that admit everywhere defined continuous functions

> as roots. I recall that I was morally certain that I knew how to construct an example, even knew more or less what it looked like (it wasn't degree 5), but I'm pretty sure I never actually carried that out to the bitter end.

If we leave the signum function undefined at 0, then it *is* algebraic: it can be written as $x/|x|$. However, consider the function $f(x)$ that, for each integer n, is equal to $(-1)^n$ on the open interval $(n, n+1)$, and is undefined elsewhere. It is continuous on its domain and satisfies the polynomial $X^2 - 1$ yet it is not algebraic because it has no *finite* witnessing grid, as required by Theorem 1.

We want to prove a converse for Theorem 1, that is, we want to show that a continuous, piecewise algebraic function (with a finite number of pieces) is algebraic. With this converse we see how to construct lots of algebraic functions. For example, the function that is equal to $\sqrt{|x|}$ on the interval $[-1, 1]$, to x^3 on the interval $[-1, \infty)$, and to $2 + x$ on $(-\infty, -1]$ is algebraic. We also see exactly what we have to do to show that a function is *not* algebraic.

First of all we will determine what the domain of an algebraic function can be. From Theorem 1 we know that the domain must be a finite union of intervals. The next theorem constructs an algebraic function whose domain is an arbitrary finite union of intervals. These intervals can be finite or infinite, open, closed, or half and half. They can also be degenerate (a single point). Note that the domain of the algebraic function $f(x) = \sqrt{-1}$ is empty and that the domain of the algebraic function $f(x) = b + \sqrt{a - x} + \sqrt{x - a}$ consists of a single point a where f takes on the value b.

Theorem 2 *Let J be a finite union of intervals. Then the function φ_J, whose domain is J and is equal to 0 on J, is algebraic.*

Proof. Note that J is also the complement of a finite union of intervals, hence a finite intersection of complements of intervals. So it suffices to prove the theorem for J the complement of an interval I because algebraic functions are closed under addition. For I the finite interval $(a, b]$, set

$$\varphi_J(x) = 0 \cdot \left(\sqrt{b - x} + \sqrt{x - a} + \frac{1}{x - a}\right)$$

and similarly for the other finite intervals. For I the infinite interval $(-\infty, b)$, set

$$\varphi_J(x) = 0 \cdot \left(\sqrt{b-x} + \frac{1}{b-x}\right)$$

and similarly for the other infinite intervals. ■

A corollary is that if f is an algebraic function, and J is a finite union of intervals, then the function g that is equal to f on J and undefined elsewhere is also algebraic. Indeed, $g = f + \varphi_J$. We refer to g as the **restriction** of f to J.

Lemma 3 *Let f be an algebraic function and I an interval that has a smallest element or a largest element. If a is the smallest element of I, let $J = (-\infty, a)$; if a is the largest element of I, let $J = (a, \infty)$. Then there is an algebraic function h such that*

- $\operatorname{dom} h \subset I \cup J$,
- $h = f$ *on* I,
- *if* $f(a)$ *is defined, then* $h = f(a)$ *on* J,
- *if* $f(a)$ *is not defined, then* $h = 0$ *on* J.

Proof. By symmetry it suffices to do the case that a is the smallest element of I, and by translating, we may assume that $a = 0$. If f is defined at 0, then the function

$$f\left(\frac{x+|x|}{2}\right),$$

restricted to $I \cup J$, has the desired properties. If f is not defined at 0, there are two cases. If f is defined for some $b > 0$, let

$$g(x) = b + \frac{(x/|x|)+1}{2}(x-b)$$

so that

$$g(x) = \begin{cases} x & \text{if } x > 0 \\ b & \text{if } x < 0 \end{cases}.$$

Then the algebraic function

$$\frac{(x/|x|)+1}{2} f(g(x)),$$

restricted to $I \cup J$, works. If f is not defined for any $b > 0$, then the function

$$\frac{0}{x + |x|},$$

restricted to $I \cup J$, works. ■

If a is the smallest (largest) element of I, we call the function constructed in the lemma the **canonical left (right) extension** of f. With this lemma we can show that piecewise algebraic functions, that agree where the pieces join, are algebraic.

Theorem 4 (splicing) *Let $t_1 < t_2 < \cdots < t_n$ be real numbers. These points break up R into $n+1$ closed intervals $I_0, I_1, \ldots, I_n$. If f_i is algebraic for $i = 0, \ldots, n$, and $f_{i-1}(t_i) = f_i(t_i)$ (possibly both undefined) for $i = 1, \ldots, n$, then there is an algebraic function that is equal to f_i on each I_i.*

Proof. By induction we can find an algebraic function g that is equal to f_i on I_i for $i = 0, \ldots, n-2$ and equal to f_{n-1} on $I_{n-1} \cup I_n$. Let h_0 be the canonical right extension of g restricted to the interval $I_0 \cup \cdots \cup I_{n-1}$ and h_1 the canonical left extension of f_n restricted to I_n. If h_0 is defined at t_n, then $h_0 + h_1 - h_0(t_n)$ is equal to f_i on each I_i. If h_0 is not defined at t_n, then $h_0 + h_1$ is equal to f_i on each I_i. ■

The derivative of an algebraic function f is defined and algebraic on the interior of $\operatorname{dom} f$, with the possible exception of a finite number of points (for functions like $\sqrt[3]{x}$).

Theorem 5 *Let f be an algebraic function, U the union of those open intervals of a witnessing grid for f on which f is defined, and $(\operatorname{dom} f)^o$ the interior of the domain of f. Then $\operatorname{dom} f$ is equal to U plus possibly a finite number of points, and for $i = 0, 1, 2, \ldots$ the i-th derivative $f^{(i)}$ of f is algebraic and $U \subset \operatorname{dom} f^{(i)} \subset (\operatorname{dom} f)^o$.*

The domain of the derivative of f is contained in the interior of the domain of f by definition. The domain of $f^{(i)}$ may get a little smaller as i gets larger (for example, $|x|^3$). Even if it doesn't, the witnessing grid may not stay the same because $f^{(i)}$ could change sign on an interval of U.

Of course an antiderivative of an algebraic function need not be algebraic: an antiderivative of $1/x$ is $\ln x$, and an antiderivative of $1/\left(1 + x^2\right)$ is $\arctan x$.

References

[1] Bliss, Gilbert Ames, *Algebraic functions*, AMS colloquium publication **16**, 1933.

[2] Stewart, James, *Calculus*, Thomson Brooks/Cole, 2008.