FILTERED MODULES OVER DISCRETE VALUATION DOMAINS

Fred Richman
Florida Atlantic University
Boca Raton FL 33431

Elbert A. Walker
New Mexico State University
Las Cruces NM 88003

Abstract. We consider a unified setting for studying local valuated groups and coset-valuated groups, emphasizing the associated filtrations rather than the values of elements. Stable exact sequences, projectives and injectives are identified in the encompassing category, and in the category corresponding to coset-valuated groups.

1. Introduction

Throughout, R will denote a discrete valuation domain with prime p, and *module* will mean R-module. In the motivating example, R is the ring of integers localized at a prime p. In that case, a module is simply an abelian group for which multiplication by any integer prime to p is an automorphism—a **p-local** abelian group. The indecomposable, divisible, torsion module Q/R, where Q is the quotient field of R, will be denoted by R_{p^∞}.

The notion of a valuated module (v-module) arises from considering a submodule A of a module B, together with the height function on B restricted to A. The dual notion of a coset-valuated module (c-module) comes up when considering the quotient module B/A with a valuation related to the height function on B. Traditionally, [2], [4], one sets

$$v(b + A) = \sup\{\mathrm{ht}(b + a) + 1 : a \in A\}.$$

For finite abelian p-groups, the v-group A tells all about how the subgroup A sits inside the group B in the sense that if the subgroups A and A' are isomorphic as v-groups, then there is an automorphism of B taking A to A' [6]. For isotype subgroups A of simply presented p-groups B, the c-group B/A tells all about how A sits inside B [4].

In this paper we consider these two notions in terms of filtered modules, focusing on the submodules

$$B(\alpha) = \{b \in B : vb \leq \alpha\}$$

rather than on the valuations themselves. This has the virtue, if you are so inclined, that the structure is defined in terms of submodules, not elements, so can be dealt with in purely categorical terms. Independent of that, or possibly because of that, many of the ideas take a more natural form when the valuations are suppressed. In particular, the relationship between v-modules and c-modules appears more natural, and we are not forced to consider the somewhat artificial traditional definition of the coset valuation.

We consider a category of filtered modules that includes both v-modules and c-modules. Every object in this category is both a quotient of a v-module and a submodule of a c-module. The stable exact sequences, the elements of Ext, are identified in this category and in the category of c-modules, as are the projectives and injectives.

2. Height

A general setting for height is a forest with a unique zero, which we will call simply a **forest**. This consists of a set X together with a function $\pi : X \to X$ such that π has a unique periodic point, which is a fixed point, called 0. In the motivating example, X is a p-local abelian group, and $\pi x = px$. The elements of a forest are often called **nodes**. A **map** between two forests is a function f such that $f(\pi x) = \pi f(x)$ for all nodes x.

If $\pi x = y$, then we say that y is the **parent** of x and that x is a **child** of y. If $\pi^n x = y$, where n can be 0, then we say that x is an **ancestor** of y, and that y is a **descendant** of x. A nonzero node whose parent is 0 is called a **root**, a childless node a **leaf**.

A subset S of a forest X is a subforest if $\pi S \subset S$. If S is a subforest of a forest X, then so is πS, the set of all parents of nodes in S. For each ordinal α define $\pi^\alpha S$ inductively by

$$\pi^\beta S = \bigcap_{\alpha<\beta} \pi(\pi^\alpha S).$$

In particular, $\pi^{\alpha+1} S = \pi(\pi^\alpha S)$, and, if β is a limit ordinal, then $\pi^\beta S = \bigcap_{\alpha<\beta} \pi^\alpha S$.

If $\pi^\alpha X = \pi^{\alpha+1} X$, then $\pi^\alpha X = \pi^\beta X$ for each $\beta > \alpha$. The **length** of X is the least α such that $\pi^\alpha X = \pi^{\alpha+1} X$. A forest is **torsion** if for each x there is n such that $\pi^n x = 0$. If x is a node in a forest, then the **order**, or **exponent**, of x is the smallest nonnegative integer n such that $\pi^n x = 0$. If no such n exists, then x is said to have **infinite order**.

A module becomes a forest upon setting $\pi x = px$—we forget all its structure except multiplication by p. Conversely, if X is a forest, then we can construct a module $S(X)$ by taking the free module on X modulo the submodule generated by

$$\{y - px : y = \pi x\}.$$

Note that 0 in X becomes 0 in $S(X)$ because $1 - p$ is invertible. The functor S from forests to modules is the left adjoint of the forgetful functor from the category of modules to the category of forests. A module isomorphic to some $S(X)$ is said to be **simply presented**.

As an example of a forest, which we will use later, consider the forest $F_{\beta,n}$ constructed from ordinals β and n, where $n \leq \omega$. A node of the forest $F_{\beta,n}$ is either finite, strictly increasing, string $a_1 a_2 \ldots a_m$ of ordinals less than β, or the symbol t_k, where $k \leq n$ is a nonnegative integer, and t_0 is the empty string. The function π is defined by

$$\begin{aligned} \pi(a_1 a_2 \ldots a_m) &= a_2 \ldots a_m \text{ if } m \geq 1, \\ \pi(t_k) &= t_{\min(n,k+1)}. \end{aligned}$$

Clearly $F_{\beta,n}$ is a forest of length $\beta + n$. If $1 \leq n < \omega$, then $F_{\beta,n}$ is torsion with the unique root t_{n-1} and zero t_n. If $n = \omega$, then $F_{\beta,n}$ has no roots or zeros.

Related forests are $F_{\infty,1}$ and $F_{\infty,\omega}$. The nonzero nodes of $F_{\infty,1}$ are x_n, with n a nonnegative integer, satisfying $\pi x_0 = 0$ and $\pi x_{n+1} = x_n$. This is a torsion forest, and $S(F_{\infty,1})$ is isomorphic to R_{p^∞}. In $F_{\infty,\omega}$ the nonzero nodes x_n are indexed by the integers, and $\pi x_{n+1} = x_n$ throughout. The module $S(F_{\infty,\omega})$ is isomorphic to the quotient field of R.

A node x in a forest X is said to have **height** α if $x \in \pi^\alpha X \setminus \pi^{\alpha+1} X$. If $x \in \pi^\lambda X$, where λ is the length of X, then x is said to have **height** ∞. In $F_{\beta,n}$, the node $a_1 a_2 \ldots a_m k$ has height a_1, if $m \geq 1$, and the node k has height $\beta + k$. The length of $F_{\beta,n}$ is $\beta + 1 + n$.

3. O-modules

We are interested in modules, and forests, with descending filtrations indexed by the ordinals. For any index class $\mathcal{I}$, not just the ordinals, we may consider an **$\mathcal{I}$-module** to be a module G together with a family of submodules $G(\alpha)$ indexed by $\mathcal{I}$. A map $f : A \to B$ of $\mathcal{I}$-modules is a module homomorphism such that $f(A(\alpha)) \subset B(\alpha)$ for each α in $\mathcal{I}$.

The category of $\mathcal{I}$-modules is **preabelian**: every map has a kernel and a cokernel. The **kernel** of a map $f : B \to C$ of $\mathcal{I}$-modules is $A = \{b \in B : f(a) = 0\}$ with $A(\alpha) = A \cap B(\alpha)$. It is easy to see that this is the categorical kernel, that is, if g is a map from an $\mathcal{I}$-module into B such that $fg = 0$, then g factors uniquely through A.

$$\begin{array}{ccccc} A & \subset & B & \xrightarrow{f} & C \\ & \nwarrow & \uparrow g & & \\ & & \bullet & & \end{array}$$

The **cokernel** of a map $f : A \to B$ of $\mathcal{I}$-modules is $C = B/f(A)$ with $C(\alpha)$ equal to the image of $B(\alpha)$ in C. This is the categorical cokernel: if g is a map from B into an $\mathcal{I}$-module such that $gf = 0$, then g factors uniquely through C.

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \rightarrow & C \\ & & \downarrow g & \swarrow & \\ & & \bullet & & \end{array}$$

If the class $\mathcal{I}$ has some structure, like the class of ordinals, we would normally want the family of submodules $G(\alpha)$ to reflect that structure (for example, to be a descending filtration in the case of ordinals). These conditions will be relatively harmless if whenever A and B are objects in the more restrictive category, and $f : A \to B$ is a map, then the kernel and cokernel of f in the larger category are in the smaller one. Taking $\mathcal{I}$ to be the ordinal numbers, we put on three such harmless restrictions.

An **o-module** is a module G with a family of submodules $G(\alpha)$ indexed by the ordinals such that

- If $\alpha < \beta$, then $G(\alpha) \supset G(\beta)$,
- $G(0) = G$,
- $pG(\alpha) \subset G(\alpha + 1)$.

(In general we denote $p^\theta(G(\alpha))$ by $p^\theta G(\alpha)$.) Call such a family of submodules an **$\mathcal{O}$-filtration**. Set $G(\infty) = \bigcap G(\alpha)$. We say that G is **value reduced** if $G(\infty) = 0$. There is an ordinal λ such that $G(\lambda) = G(\infty)$. The smallest such ordinal is called the **value length** of G.

Note that we have not referred to the additive structure of G, except that the $G(\alpha)$ are submodules. The same definition goes through if G is a forest and the $G(\alpha)$ are subforests. In this case we speak of an **o-forest**. Obviously an o-module is an o-forest. Moreover if F is an o-forest, then $A = S(F)$ becomes an o-module if we let $A(\alpha)$ be the submodule generated by $F(\alpha)$, and S is the left adjoint of the forgetful functor from o-modules to o-forests. The left adjoint property says that any o-forest map from an o-forest F to an o-module C extends uniquely to an o-module map $S(F) \to C$.

If we put on two more conditions, we have characterized the submodules $p^\alpha G$, the **height filtration**.

1. $G(\beta) \supset \bigcap_{\alpha<\beta} G(\alpha + 1)$ (continuity),
2. $G(\alpha + 1) \subset pG(\alpha)$ (divisibility).

We say that an ordinal β is a **limit ordinal** if $\beta = \sup\{\alpha : \alpha < \beta\}$, so a limit ordinal is an ordinal with no immediate predecessor (0 is a limit ordinal). We might as well restrict (1) to limit ordinals β. For $\beta = 0$ it says that $G(0) = G$. An o-module that satisfies (1) is called a **valuated module**, one that satisfies (2) is called a **c-valuated module** (Hill and Megibben [4]). We will use the shorter terminology **v-module** and **c-module**. If an o-module G is both a v-module and a c-module, then $G(\alpha) = p^\alpha G$ and we say that G is a **module** (with the natural height filtration), or an **h-module**, for emphasis.

Note again that the same definitions can be applied to o-forests, resulting in definitions of c-forests and v-forests.

The continuity and divisibility conditions are not harmless in the way that the other three conditions are. The cokernel in the category of o-modules of a map between two v-modules need not be a v-module, and the kernel in the category of o-modules of a map between two c-modules need not be a c-module.

Finite (Jordan-Hölder) length c-modules are h-modules, but finite length o-modules need not be v-modules. The idea of a c-module was introduced by Fuchs in [1] as a group with a *coset valuation*, whence the "c". He showed that every torsion c-group is the quotient of a simply presented torsion group. Note that the data for a c-module A can be provided by specifying the submodules $A(\alpha)$ only for α a limit ordinal, subject to the condition that $A(\alpha + \omega) \subset p^\omega A(\alpha)$.

If B is an h-module and A is a submodule of B, then A is an o-module kernel exactly when A is an **isotype submodule** of B, that is, if $p^\alpha A = A \cap p^\alpha B$ for each ordinal α. Alternatively, the o-submodule A of B is an isotype submodule exactly when A is a c-module. That is because A is already a v-module, being a submodule of the v-module B, so A is a c-module if and only if it is an h-module—but because A is an o-submodule of the module B, this is the same as saying A is isotype.

The next theorem characterizes when a submodule of a c-module is a c-module. First a lemma isolating a familiar maneuver used with isotype subgroups. (Here the usual convention is that $p^m C[p^n] = (p^m C)[p^n]$.)

Lemma 1. *Let m and n be nonnegative integers, and $A \subset B \xrightarrow{f} C$ a short exact sequence of modules. Then the following conditions are equivalent.*

1. $p^m C[p^n] \subset f(B[p^n])$,
2. $A \cap p^{m+n} B \subset p^n A$.

Proof. Suppose (1) holds and let $a \in A \cap p^{m+n} B$. Then $a = p^{m+n} b$ for $b \in B$. So

$$f(p^m b) \in p^m C[p^n] \subset f(B[p^n])$$

whence $p^m b - b' \in A$ for some $b' \in B[p^n]$, so $a = p^n(p^m b - b') \in p^n A$.

Now suppose (2) holds and $c \in p^m C[p^n]$. Then $c = p^m c'$ for $c' \in C$, so $p^{m+n} c' = 0$. Choose b so that $f(b) = c'$. Then

$$p^{m+n} b \in A \cap p^{m+n} B \subset p^n A$$

whence $p^{m+n} b = p^n a$ for $a \in A$. So $p^n(p^m b - a) = 0$ and $c = f(p^m b - a) \in f(B[p^n])$. ■

We will use this lemma again later. For now we only need the case $m = 0$.

Theorem 2. *Let* $0 \to A \to B \xrightarrow{f} C \to 0$ *be an exact sequence of o-modules, where* B *is a c-module. Then the following are equivalent.*

- A *is a c-module,*
- $f(B(\alpha)[p^n]) = C(\alpha)[p^n]$ *for each ordinal* α *and nonnegative integer* n*,*
- $f(B(\alpha)[p]) = C(\alpha)[p]$ *for each ordinal* α*.*

Proof. If A is a c-module, then

$$A(\alpha) \cap p^n B(\alpha) = A(\alpha) \cap B(\alpha + n) = A(\alpha + n) = p^n A(\alpha)$$

so $C(\alpha)[p^n] = f(B(\alpha)[p^n])$ by the lemma. If $f(B(\alpha)[p]) = C(\alpha)[p]$ for each ordinal α, then, by the lemma,

$$pA(\alpha) = A(\alpha) \cap pB(\alpha) = A(\alpha) \cap B(\alpha + 1) = A(\alpha + 1)$$

so A is a c-module. ■

This is just the condition that $B(\alpha)$ maps purely onto $C(\alpha)$ for each α.

A one-to-one map $A \to B$ of o-modules is said to be an **embedding** if the inverse image of $B(\alpha)$ is $A(\alpha)$ for each ordinal α. So every kernel is an embedding, and vice versa. If A is a submodule of a module B, then the module B/A is an o-module cokernel if and only if A is a nice submodule of B.

It will be convenient to introduce the symbol β^-, for each nonzero limit ordinal β, to serve as a predecessor to β with the properties that $\beta^- < \beta$ and $\alpha < \beta^-$ for any ordinal $\alpha < \beta$. Moreover—and this is the purpose of the notation—we set

$$G(\beta^-) = \bigcap_{\alpha < \beta} G(\alpha)$$

so the continuity condition is simply $G(\beta^-) = G(\beta)$ for any nonzero limit ordinal β. It will be convenient to let $G(0^-) = G$, which is reasonable for an empty intersection, even though we don't want or need a symbol to precede 0. Finally, we set $\beta - 1 = \beta^-$ for β a nonzero limit ordinal. So, for *any* ordinal $\beta > 0$ we have

$$G(\beta - 1) = \bigcap_{\alpha<\beta} G(\alpha).$$

A sequence $0 \to A \to B \to C \to 0$ of o-modules is exact if $0 \to A(\alpha) \to B(\alpha) \to C(\alpha) \to 0$ is exact for all ordinals α. Every short exact sequence o-modules is **stable**: pushouts of kernels are kernels, and pullbacks of cokernels are cokernels. This is exactly what is needed for the short exact sequence to represent an element of Ext [5].

Theorem 3. *In the category of $\mathcal{I}$-modules, pushouts of kernels are kernels, and pullbacks of cokernels are cokernels.*

Proof. Consider the pushout diagram

$$\begin{array}{ccc} A & \subset & B \\ g\downarrow & & \downarrow \\ A' & \to & B' \end{array}$$

where $A(\alpha) = A \cap B(\alpha)$ for each α in $\mathcal{I}$. We construct B' as

$$B' = \frac{A' \oplus B}{\{(g(a), -a) : a \in A\}}$$

so the map $A' \to B'$ is one-to-one. We want to show that $A'(\alpha) = A' \cap B'(\alpha)$. Because $A' \to B'$ is a map, $A'(\alpha) \subset A' \cap B'(\alpha)$. Now $B'(\alpha)$ is the image of $A'(\alpha) \oplus B(\alpha)$. If $(x, 0)$ represents an element of $B'(\alpha)$, then $(x, 0) = (a', b) + (g(a), -a)$ with $(a', b) \in A'(\alpha) \oplus B(\alpha)$. Then $b = a$, so $a \in A(\alpha)$, and $x = a' + g(a)$, so $x \in A'(\alpha)$.

For the second part, consider the pullback diagram

$$\begin{array}{ccc} B' & \to & C' \\ \downarrow & & \downarrow g \\ B & \stackrel{f}{\to} & C \end{array}$$

where $C(\alpha) = f(B(\alpha))$ for each α in $\mathcal{I}$. We construct B' as

$$B' = \{(b, c') \in B \oplus C' : f(b) = g(c')\}.$$

We want to show that each element c' of $C'(\alpha)$ is in the image of $B'(\alpha)$. As $g(c') \in C(\alpha)$, we can choose $b \in B(\alpha)$ such that $f(b) = g(c')$. Then $(b, c') \in B'(\alpha)$ and $f(b, c') = c'$. ■

The set of all $\mathcal{O}$-filtrations on a module A is closed under intersection. So it is also closed under join: if $A_i(\alpha)$ is a family of $\mathcal{O}$-filtrations on A, then the smallest $\mathcal{O}$-filtration containing them all is given by setting $A(\alpha) = \sum_i A_i(\alpha)$.

The largest $\mathcal{O}$-filtration on a module A is obtained by setting $A(\alpha) = A$ for each ordinal α. The smallest is given by setting $A(n) = p^n A$ for $n < \omega$, and $A(\omega) = 0$. Each c-module structure on A lies below the h-module structure on A, and each v-module structure on A lies above it. Note that the largest filtration gives a v-module structure, which is generally not a c-module structure, and the smallest filtration gives a c-module structure, which is not generally a v-module structure. The set of c-module structures is closed under suprema, the set of v-module structures under infima.

Given an o-module A, there exists a v-module A', the **reflection** of A, and a map $\varphi : A \to A'$ of o-modules such that if B is a v-module, then any map $A \to B$ factors uniquely through φ. This is a **left adjoint** of the inclusion functor F from v-modules to o-modules:

$$\mathrm{Hom}(A, FB) = \mathrm{Hom}(A', B).$$

Thus the category of v-modules is a full reflective subcategory of the category of o-modules, like the category of torsion-free abelian groups in the category of abelian groups. This construction is described in [6, Theorem 3]: you simply take the intersection of all continuous $\mathcal{O}$-filtrations on A that contain the given one.

Dually, there exists a c-module A'', the **coreflection** of A, and a map $\varphi : A'' \to A$ of o-modules such that if B is a c-module, then any map $B \to A$ factors uniquely through φ:

$$\mathrm{Hom}(FB, A) = \mathrm{Hom}(B, A''),$$

so this is a right adjoint of the inclusion functor F from c-modules to o-modules. The filtration on A'' is the sum of all the $\mathcal{O}$-filtrations on A that give c-modules and are contained in the given filtration. So A'' is just A with the filtration given inductively by

$$A''(\beta) = A(\beta) \cap \bigcap_{\alpha<\beta} pA''(\alpha).$$

Clearly v-modules are closed under submodules, and c-modules are closed under quotients. Both are closed under extensions.

Theorem 4. *Let $0 \to A \to B \xrightarrow{f} C \to 0$ be an exact sequence of o-modules. If A and C are v-modules (c-modules), then B is a v-module (c-module).*

Proof. Suppose A and C are v-modules, β is a limit ordinal, and $b \in \bigcap_{\alpha<\beta} B(\alpha)$. Then $f(b) \in \bigcap_{\alpha<\beta} C(\alpha) = C(\beta)$, so $f(b) = f(b')$ where $b' \in B(\beta)$. So $b - b' \in \bigcap_{\alpha<\beta} A(\alpha) = A(\beta)$. Therefore $b \in B(\beta)$.

Now suppose A and C are c-modules and $b \in B(\alpha+1)$. Then $f(b) \in C(\alpha+1) \subset pC(\alpha)$, so $f(b) = pf(b')$ with $b' \in B(\alpha)$. Thus $b - pb' \in A(\alpha+1) = pA(\alpha)$, whence $b \in pB(\alpha)$. ■

Each o-module A is the quotient of a v-module in a canonical way. There is a canonical v-forest F_A associated with A. For each ordinal α not exceeding the length λ of A, and each element t in $A(\alpha)$, let $x_{t,\alpha}$ have order the same as t, and $v\pi^n x_{t,\alpha} = \alpha + n$ if $p^n t \neq 0$ and $\alpha < \lambda$, and $v\pi^n x_{t,\lambda} = \infty$. Then we have a natural pure quotient map

$$S(F_A) \to A$$

that takes $x_{t,\alpha}$ to t. Note that $S(F_A)$ is torsion if A is. The following theorem generalizes [6, Theorem 1] which shows how to obtain a nice embedding of a v-module in a module.

Theorem 5. *Every o-module can be embedded in a c-module with a simply presented torsion module quotient.*

Proof. Let A be an o-module. Clearly we can embed F_A in a forest F made up of the forests $F_{\beta,n}$. Consider the push-out

$$\begin{array}{ccccccccc} & & K & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & S(F_A) & \to & S(F) & \to & T & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & A & \to & \bullet & \to & T & \to & 0 \end{array}$$

Here K is the kernel of the quotient map $S(F_A) \to A$. The o-module $\bullet$ is a c-module because it's a quotient of the h-module $S(F)$ (the cokernel of the map $K \to S(F)$—the push-out of a quotient map is a quotient map). ■

In particular, every v-module can be embedded in a module with a simply presented torsion module quotient, because if A is a v-module, then $\bullet$ is both a v-module and a c-module, hence a module. This gives [6, Theorem 1] because the sequence is exact in the category of o-modules, hence is nice.

Theorem 6. *Every (torsion) o-module C is a quotient of a (torsion) v-module B by a module A so that $B(\alpha)$ maps purely onto $C(\alpha)$ for each ordinal α.*

Proof. Let C be an o-module and $S(F_C) \to C$ the canonical quotient map from the v-module $V = S(F_C)$. Note that $V(\alpha)$ maps purely onto $C(\alpha)$. Embed the kernel K of this map in a module A with A/K a simply-presented torsion module [6, Theorem 1]. Consider the pushout

$$\begin{array}{ccccccccc} 0 & \to & K & \to & V & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

The inclusion $K \subset A$ is nice, so B is a v-module because V and A are. Moreover $B(\alpha)$ maps purely onto $C(\alpha)$ because $V(\alpha)$ does. ■

If C is a c-module, then B is a module, so every c-module is isomorphic to the quotient of a module. As A is a module, and an o-module kernel, it is isotype in B. We see that the o-modules are characterized as quotients of v-modules and as submodules of c-modules. They are exactly what you get if you start with modules, and repeatedly take submodules and quotient modules with the induced $\mathcal{O}$-filtrations.

We can get a little finer information about writing a c-module C as a quotient of a module B. For example, Hill and Megibben show that we can take B to be simply presented torsion if C is a p-group [4, Theorem 2.8]. That is a consequence of the following characterization of c-modules among o-modules.

Theorem 7. *Let C be a c-module and F a v-forest. Then any o-forest map φ from a subforest F' of F to C can be extended to an o-forest map from F to C.*

Proof. If $\pi^n x \notin F'$ for each positive integer n, then set $\varphi(x) = 0$. Otherwise induct on the smallest n such that $\pi^n x \in F'$. So we may assume that φ is defined on πx but not on x. If $vx = \alpha$, then $\pi x \in F(\alpha + 1)$ so $\varphi(\pi x) \in C(\alpha + 1)$. As C is a c-module, there exists $c \in C(\alpha)$ such that $pc = \varphi(\pi x)$. Set $\varphi(x) = c$. Much the same argument works if $vx = \infty$. ■

A **KT-module** (balanced projective module) is a module of the form $S(F)$ where F is a forest with the property that for each node x there exists a positive integer n such that either $\operatorname{ht} p^n x = \infty$ or $\operatorname{ht} p^{n+i} x = \operatorname{ht} p^n x + i$ for each positive integer i. A torsion module is a KT-module if and only if it is simply presented.

Corollary 8. *Every (torsion) c-module C is the quotient of a (torsion) KT-module B by an isotype submodule.*

Proof. Embed the v-forest F_C in a forest F made up of the forests $F_{\beta,n}$. The canonical o-forest map $F_C \to C$ extends to an o-forest map $F \to C$ by the theorem. The induced map from $B = S(F)$ to C is the desired quotient map. To say that the kernel A is an isotype submodule is to say that A is a c-module. This follows from Theorem 2 because $B(\alpha)[p]$ maps onto $C(\alpha)[p]$, for each α. ■

4. Valuations

An alternative way to view an o-module is by means of a **valuation** v. If G is an o-module, and $x \in G(\alpha) \setminus G(\alpha+1)$, then we set $vx = \alpha$. If $x \in G(\alpha)$ for all ordinals α, then we set $vx = \infty$. If β is a limit ordinal, and $x \in G(\beta^-) \setminus G(\beta)$, then we set $vx = \beta^-$. Thus vx is either an ordinal, the symbol ∞, or β^- where β is a nonzero limit ordinal. Note that v-modules are characterized by the property that vx is always either an ordinal or ∞.

The valuation v on an o-module has the following properties.

- vx is either an ordinal, ∞, or β^- for a nonzero limit ordinal β,
- $vpx > vx$ if vx is an ordinal,
- $v(x+y) \geq \min(vx, vy)$,
- $vux = vx$ if p does not divide u,
- $v0 = \infty$.

For G an o-module and α an ordinal, we have $G(\alpha) = \{x \in G : vx \geq \alpha\}$, so the valuation and the filtration are equivalent ways of viewing an o-module.

It is instructive to compare the coset valuation as defined in [1] and [4] with how we view it here. The setting is an epimorphism $\varphi : B \to C$ of modules. They set

$$\begin{aligned} vc &\leq \alpha+1 \text{ if } c \in \varphi(p^\alpha B), \text{ for } \alpha \text{ any ordinal, and} \\ vc &\leq \beta \text{ if } vc \leq \alpha+1 \text{ for some } \alpha < \beta, \text{ for } \beta \text{ a nonzero limit ordinal.} \end{aligned}$$

So knowledge of the submodules $\varphi(p^\alpha B) = \{c \in C : vc \leq \alpha+1\}$ suffices to specify the coset valuation. From the filtration point of view, we simply filter C with the submodules $\varphi(p^\alpha B)$, indexing $\varphi(p^\alpha B)$ by α—end of story. But if we demand a description in terms of an ordinal-valued function v, then we are forced to make an artificial shift by 1 in order to make room for a value for the elements of $\bigcap_{\alpha<\beta} \varphi(p^\alpha B)$ that are not in $\varphi(p^\beta B)$, if β is a limit ordinal. This has the annoying consequence that if φ is the identity map, then the coset valuation on C is not the same as the valuation on B. It could be argued that this is a small price to pay to avoid introducing elements β^-, but there is no need introduce such elements from the filtration point of view, and it is certainly more natural to index $\varphi(p^\alpha B)$ by α than by $\alpha+1$.

Another illustration of the advantage of the filtration point of view is provided by Hill's notion [3] of *compatible submodules* H and K. The valuation definition is

- For all $h \in H$ and $k \in K$, there exists $x \in H \cap K$ such that $v(h+x) \geq v(h+k)$.

The filtration definition is

- $(H+K)(\alpha) = H(\alpha) + K(\alpha)$ for all ordinals α.

Not only is the filtration definition simpler and easier to remember, but it makes it obvious that compatibility is a symmetric relation.

A submodule A of an o-module B is **nice** if every coset $b+A$ has an element of maximum value. Equivalently, if $(b+A) \cap B(\alpha)$ is nonempty for all ordinals α in some set S, then $(b+A) \cap \bigcap_{\alpha \in S} B(\alpha)$ is nonempty. That is, the submodule A has the **best approximation property**: if b is any element of B, then there exists a in A such that any ball $b + B(\alpha)$ centered at b that intersects A contains a. An alternative way of expressing this is to say that, for any set S of ordinals,

$$\bigcap_{\alpha \in S} (B(\alpha) + A) = \bigcap_{\alpha \in S} B(\alpha) + A.$$

This definition has the advantage that it is expressed purely in terms of submodules, not elements. If A is a submodule of a v-module B, then the quotient B/A is a v-module if and only if A is nice.

A nonzero element of the divisible part of a module really has height ∞^-, and only 0 has height ∞, but the use of ∞ for the height of elements in the divisible part is well established. Note that with this usage ∞ behaves much more like α^- than like an ordinal.

5. Projective and injective o-modules

The projectives in the category of o-modules are easily characterized: they are the valuated modules that are free on a valuated set.

Theorem 9. *The projectives in the category of o-modules are direct sums of torsion-free cyclic valuated modules such that $vpx = vx + 1$ for each nonzero element x, or $vx = \infty$ for all x. Every o-module is the quotient of a projective.*

Proof. These o-modules are obviously projective. Conversely, it suffices to show that every o-module is the quotient of such a direct sum because, by the Azumaya theorem in an additive category [8, Theorem 5], such direct sums are closed under summands as the endomorphism ring of a cyclic o-module is local. Let C_α denote a torsion-free valuated module with generator c such that $vp^n c = \alpha + n$, if α is an ordinal, and $vp^n c = \infty$ if $\alpha = \infty$. If A is an o-module of value length λ, construct the direct sum

$$F = \bigoplus_{\substack{\alpha < \lambda \\ \alpha = \infty}} \bigoplus_{x \in A(\alpha)} C_{\alpha,x}$$

where $C_{\alpha,x}$ is a copy of C_α. Map F onto A by taking the generator of $C_{\alpha,x}$ to x. ∎

We turn to the injective o-modules.

Lemma 10. *Let I be an injective o-module. Then I is a c-module and $I(\beta^-)$ is a divisible summand of I for each nonzero limit ordinal β.*

Proof. If $x \in I(\alpha+1)$, then the cyclic submodule generated by x can be embedded in a cyclic o-module C generated by y such that $py = x$ and $y \in C(\alpha)$. As I is injective, we get a map from C to A that fixes x, so $x \in pA(\alpha)$. So I is a c-module.

Now let D be a divisible hull of $I(\beta^-)$, filtered by setting $D(\beta^-) = D$ and $D(\alpha) = I(\alpha)$ for $\alpha \geq \beta$. This gives an embedding of $I(\beta^-)$ in D which must extend to a map $D \to I$ that is the identity on $I(\beta^-)$. As $D = D(\beta^-)$, this map goes into $I(\beta^-)$. Therefore $I(\beta^-)$ is divisible, hence splits out of I as a module. But since for all α, either $I(\alpha) \supset I(\beta^-)$ or $I(\alpha) \subset I(\beta^-)$, this is a splitting of o-modules. ∎

Note that as $I(\infty) = I(\beta^-)$ for any limit ordinal beyond the value length of I, the submodule $I(\infty)$ is a divisible summand of I. Note also that $p^\infty I = I(\omega^-)$.

Lemma 11. *Let D be a c-module and α a limit ordinal, possibly zero. If $p^\omega D(\alpha) = 0$ and $D(\alpha^-) = D$, then D is an injective o-module if and only if the underlying module of $D(\alpha)$ is algebraically compact.*

Proof. Note that $p^\omega D(\alpha) = D((\alpha+\omega)^-)$. Suppose $D(\alpha)$ is algebraically compact. Given o-modules $A \subset B$ and a map $f : A \to D$, we have to extend f to B. As $f(A((\alpha+\omega)^-)) = 0$, we may assume that $A((\alpha+\omega)^-) = 0 = B((\alpha+\omega)^-)$. First extend f from $A(\alpha) = A \cap B(\alpha)$ to $B(\alpha)$. We can do this because $A(\alpha)$ and $B(\alpha)$ are v-modules, and $D(\alpha)$ is algebraically compact, hence injective in the category of v-modules. This gives an extension of f to $A + B(\alpha)$. If $\alpha = 0$, we are done. If not, then the c-module D is divisible as $D = D(\alpha^-)$, so we can extend f to a homomorphism from B to D. The result is a map of o-modules because $D = D(\alpha^-)$.

Conversely, suppose D is an injective o-module. Let A be a pure submodule of a module B, and $f : A \to D(\alpha)$ a homomorphism. If we show that we can extend f to a homomorphism from B to $D(\alpha)$, we will have shown that $D(\alpha)$ is pure injective, hence algebraically compact. We may assume that $p^\omega B = 0$ as $p^\omega D(\alpha) = 0$. Put an o-module structure on B by setting $B(\alpha+n) = p^n B$. Then f is an o-module map, hence extends to an o-module map from B to D, which must take B into $D(\alpha)$ as $B = B(\alpha)$. ∎

We will be especially interested in the rank-one torsion divisible o-module $I_{\beta+n}$, where β is a nonzero limit ordinal, $I_{\beta+n}(\beta^-) = I_{\beta+n}$, and $I_{\beta+n}(\beta+i) = p^i I_{\beta+n}[p^n]$.

Also the rank-one divisible o-module I_∞ where $I_\infty(\alpha) = I_\infty$ for each α, and $I_n = R/p^n R$.

Theorem 12. *The torsion cyclic c-modules are the modules I_n. The value reduced quasicyclic c-modules are the c-modules $I_{\beta+n}$ for β a nonzero limit ordinal. The torsion cyclic and quasicyclic c-modules are injective o-modules.*

Proof. If G is a c-module, and $p^n G = 0$, then $G(\omega^-) = 0$, so G is a v-module, hence a module. So the torsion cyclic c-modules are the modules I_n. If G is a quasicyclic c-module, let x be a generator of $G[p]$. If $vx = \beta^-$, then $G = I_\beta$. If $vx = \beta + n$, then $G = I_{\beta+n+1}$. The modules $I_{\beta+n}$ and I_n are injective by Lemma 11. ■

We can now show that the category of o-modules has enough injectives.

Theorem 13. *Each (value reduced) o-module can be embedded as a submodule of a product of (value reduced) cyclic and quasicyclic torsion c-modules.*

Proof. Let A be an o-module of value length λ. Let F be the set of all o-module maps

$$f : A \to C_f$$

where C_f is I_α for some $\alpha \leq \lambda$, or I_∞ if A is not value reduced. Then the evaluation map

$$\varphi : A \to P = \prod_{f \in F} C_f$$

is defined by $\varphi(x)_f = f(x)$. We shall show that φ is an embedding

Suppose $x \in A$. If $x \notin A(\alpha)$, for some $\alpha \leq \lambda$, then there is $f \in F$ taking A to I_α such that $f(x)$ generates $I_\alpha[p]$, so $f(x) \notin I_\alpha(\alpha) = 0$, whence $\varphi(x) \notin P(\alpha)$. Contrapositively, if $\varphi(x) \in P(\alpha)$, then $x \in A(\alpha)$. To complete the proof that φ is an embedding, we need only verify that it is one-to-one. If $\varphi(x) = 0$, then $x \in A(\lambda)$. If $x \in A(\lambda)$ is nonzero, in which case A is not value reduced, then there is $f : A \to I_\infty$ with $f(x) \neq 0$, so $\varphi(x) \neq 0$. ■

So an o-injective I is a summand of a product of copies of the c-modules I_α. It turns out that they are also products, but not necessarily products of copies of I_α. To show this we first prove a general theorem about writing modules as products.

Theorem 14. *Let A be a module with a continuous descending filtration A_α indexed by ordinals α, such that $A_0 = A$ and $A_\lambda = 0$ for some limit ordinal λ. Suppose A_α is an absolute direct summand of A for each $\alpha < \lambda$, and set*

$$P_\beta = \prod_{\beta \leq \alpha < \lambda} A_\alpha / A_{\alpha+1}$$

for each $\beta < \lambda$. Then there is a monomorphism $\varphi : A \to P_0$ such that $\varphi(A_\beta) = \varphi(A) \cap P_\beta$ for each β, and the composite of φ with the α-th coordinate map from P_0 to $A_\alpha / A_{\alpha+1}$ restricts to the natural projection map on A_α. Moreover, if $P_0 = \varphi(A) \oplus K$, with $P_\beta = \varphi(A_\beta) \oplus (P_\beta \cap K)$ for each β, then $\varphi(A) = P_0$.

Proof. Inductively construct an ascending chain B_α of submodules of A so that $A = B_\alpha \oplus A_\alpha$ for each ordinal α, using the fact that A_α is an absolute direct summand. Map $A = B_\alpha \oplus A_\alpha$ to $A_\alpha / A_{\alpha+1}$ by taking B_α to zero and using the natural projection map on A_α. This defines a map $\varphi : A \to P_0$ taking A_β into P_β, such that the composite of φ with the α-th coordinate map from P_0 to $A_\alpha / A_{\alpha+1}$ restricts to the natural projection map on A_α. If x is a nonzero element of A, then, by the continuity of the filtration, there exists α such that $x \in A_\alpha \setminus A_{\alpha+1}$, so $\varphi(x) \in P_\alpha \setminus P_{\alpha+1}$. This shows that φ is a monomorphism and that $\varphi(A_\beta) = \varphi(A) \cap P_\beta$.

Now suppose $P_0 = \varphi(A) \oplus K$, with $P_\beta = \varphi(A_\beta) \oplus (P_\beta \cap K)$ for each β. We want to show that $K = 0$. Note that φ induces an isomorphism from $A_\beta / A_{\beta+1}$ to $P_\beta / P_{\beta+1}$. So

$$\frac{P_\beta}{P_{\beta+1}} = \frac{A_\beta}{A_{\beta+1}} \oplus \frac{P_\beta \cap K}{P_{\beta+1} \cap K}$$

from which it follows that $P_\beta \cap K = P_{\beta+1} \cap K$ for all β. So $K = 0$. ■

Corollary 15. *An o-module is injective if and only if it is a product of a reduced algebraically compact module, a divisible module, and an o-module of the form $\prod K_\alpha$, where the product is over a set of nonzero limit ordinals, and K_α is a divisible c-module such that the underlying module of $K_\alpha(\alpha)$ is reduced algebraically compact, and $K_\alpha(\alpha^-) = K_\alpha$.*

Proof. The reduced algebraically compact modules, and the o-modules K_α, are injective o-modules by Lemma 11. The divisible modules are clearly injective o-modules. So the product is injective.

Conversely, suppose A is an injective o-module. As $A(\infty)$ is a divisible summand of A, we may assume that $A(\infty) = 0$. Set $A_\alpha = A((\omega\alpha)^-)$ for each ordinal α. Note that this is a continuous filtration of A. Each A_α (including $A_0 = A$) is an injective o-group by Lemma 10. To see that A_α is an absolute summand of A, suppose $K \subset A$ and $K \cap A_\alpha = 0$. Then $K \oplus A_\alpha$ is a direct sum of o-modules because A_α is comparable to $A(\beta)$ for each ordinal β. So the projection of $K \oplus A_\alpha$ onto A_α is an o-module map, hence extends to A. Thus A satisfies the hypotheses of Theorem 14.

So there is a monomorphism $\varphi : A \to P_0$ such that $\varphi(A_\beta) = \varphi(A) \cap P_\beta$ for each β, and the composite of φ with the α-th coordinate map from P_0 to $A_\alpha / A_{\alpha+1}$ restricts to the natural projection map on A_α. It is easy to see that φ is an embedding, so

$\varphi(A)$ is a summand of P_0, and therefore equal to P_0. The factors $D_\alpha = A_\alpha/A_{\alpha+1}$ are injective o-modules, hence c-modules by Lemma 10. As $p^\theta D_\alpha = D_\alpha$ for each $\theta < \omega\alpha$, and $p^\omega D_\alpha(\omega\alpha) = D_\alpha(\omega\alpha + \omega) = 0$, the underlying module of $D_\alpha(\omega\alpha)$ is algebraically compact by Lemma 11. ■

Fuchs [2] shows that injective valuated vector spaces are products. He does this in the context where the values lie in a linearly ordered set, and the product is the *Hahn product*—elements with well-ordered support. In our situation, every element has well-ordered support.

Although o-exactness is not the right concept in the category of c-modules, as it was in the category of v-modules, we can still investigate what happens in the category of c-modules relative to that notion of exactness.

Theorem 16. *The projectives in the category of c-modules relative to o-exactness are the KT-modules. There are enough.*

Proof. Let $C = S(T)$ be a rank-one KT-module and $0 \to A \to B \xrightarrow{\varphi} C \to 0$ an exact sequence of o-modules with A a c-module (so B is also a c-module). Let x be a node in T such that $v\pi^n x = \alpha + n$ for each nonnegative integer n. Then there is $b \in B(\alpha)$ such that $\varphi(b) = x$, whence $vp^n b = \alpha + n$. So we can lift the tree X generated by x and 0 back to B. If there is no such x, just let $X = 0$.

Now suppose we have lifted back a subtree X of T to B, and t is a node in $T \setminus X$ such that $pt = x \in X$. We want to extend the lifting to t. Suppose $vt = \theta < \alpha = vx$ and $b \in B(\alpha)$ is the lift of x. Then there is $b' \in B(\theta)$ such that $\varphi(b') = t$. As $\varphi(pb' - b) = 0$ we have $pb' - b \in A(\theta + 1)$, so $pb' - b = pa$ for some $a \in A(\theta)$. So $p(b' - a) = b$ and $b' - a \in B(\theta)$. Moreover, $\varphi(b' - a) = \varphi(b') = t$, so we can extend the lifting to t.

Corollary 8 shows that there are enough projectives. ■

6. Stable sequences of c-modules

In a preabelian category, the short exact sequences $0 \to A \to B \to C \to 0$ are generally not suitable for forming the functor $\mathrm{Ext}(C, A)$. The reason is that the pushout of a kernel need not be a kernel, and the pullback of a cokernel need not be a cokernel. Those short exact sequences for which these properties do hold are the ones that constitute $\mathrm{Ext}(C, A)$, the **stable** exact sequences. In the category of o-modules, every short exact sequence is stable. In the category of v-modules, the stable short exact sequences are those that are short exact in the category of o-modules ([6, Theorem 6]). This does not carry over to the category of c-modules.

A map $f : A \to B$ in a preabelian category is a **semistable kernel** if, for any pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B' \end{array}$$

the map f' is a kernel. Dually, a **semistable cokernel** is one for which each pullback is a cokernel.

It follows from Theorem 3 that, in the category of o-modules, every kernel is semistable: pushouts of kernels are kernels. This fails for the category of v-modules: if $A \subset B$ is a semistable kernel in that category, then every coset of finite order in B/A contains an element of maximum value (but not conversely) [6, Theorem 7]. Stanton [7] characterized semistable kernels for v-modules.

If A is an o-module, we define the **coreflection** $K = F_c(A)$, a c-module, to have the same underlying module as A with the filtration given inductively by

$$\begin{aligned} K(\beta+1) &= pK(\beta) \\ K(\beta) &= A(\beta) \cap \bigcap_{\alpha<\beta} K(\alpha) \text{ for } \beta \text{ a limit.} \end{aligned}$$

The identity is an o-module map $F_c(A) \to A$, and any o-module map from a c-module to A factors uniquely through this map. Note that F_c depends only on $A(\beta)$ for β a limit.

Cokernels in the category of c-modules are cokernels in the category of o-modules. Kernels in the category of c-modules are obtained by applying F_c to the kernel in the category of o-modules. If $\varphi : B \to C$ is a map of c-modules, and $K = \{b \in B : \varphi(b) = 0\}$, then the kernel of φ in the category of c-modules is K equipped with the filtration $K(\beta)$, for limit ordinals β, given inductively by

$$K(\beta) = (K \cap B(\beta)) \cap \bigcap_{\alpha<\beta} p^{\omega} K(\alpha)$$

where α ranges over limit ordinals (although you could let α range over all ordinals). This is the biggest c-group structure on K that respects the inclusion $K \subset B$.

Theorem 17. *A kernel $A \subset B$, in the category of c-modules, is semistable if and only if $A(\alpha) = A \cap B(\alpha)$ for each limit ordinal α.*

Proof. Note that if $A' \subset B'$ are c-modules, and $A'(\alpha) = A' \cap B'(\alpha)$ for each limit ordinal α, then $A' \subset B'$ is a kernel in the category of c-modules. Conversely, if $A' \subset B'$

is a kernel in the category of c-modules, and $A'(\alpha^-) = A'$, then $A'(\alpha) = A' \cap B'(\alpha)$ by the inductive definition of the filtration on a kernel.

Let α be a limit ordinal and consider the pushout diagram

$$\begin{array}{ccc} A & \subset & B \\ g\downarrow & & \downarrow \\ A' & \subset & B' \end{array}$$

where B' is a quotient of $A' \oplus B$, so $B'(\alpha)$ is the image of $A'(\alpha) \oplus B(\alpha)$. Suppose $A(\alpha) = A \cap B(\alpha)$. We will show that $A'(\alpha) = A' \cap B'(\alpha)$, which suffices to prove the "if" part of the theorem. If $x \in A' \cap B'(\alpha)$, then

$$(x, 0) + (g(a), -a) \in A'(\alpha) \oplus B(\alpha)$$

for some $a \in A$. So $a \in B(\alpha)$, whence $a \in A(\alpha)$ and $g(a) \in A'(\alpha)$. Thus $x \in A'(\alpha)$.

Conversely, suppose $a \in A \cap B(\alpha)$. Let A' be R_{p^∞} with $A'(\alpha^-) = A'$ and $A'(\alpha) = 0$. Then $g(a) \in B'(\alpha)$, so if $A' \subset B'$ is a kernel, then $g(a) \in A'(\alpha) = A' \cap B'(\alpha)$, so $g(a) = 0$. Hence any module homomorphism from A to R_{p^∞} that kills $A(\alpha)$ kills a, whence $a = 0$. ■

It is easy to see that the condition of the theorem is equivalent to $A \cap B(\alpha + \omega) \subset p^\omega(A \cap B(\alpha))$ for each limit ordinal α. It is interesting to compare this characterization with that of semistable cokernels $f : B \to C$ in the category of v-modules. There the condition is that f be **semi-nice**: $f(B(\alpha)) = C(\alpha)$ for each *nonlimit* ordinal α [6, Lemma 5].

Corollary 18. *If $A \subset B$ is a semistable kernel in the category of c-modules, then the sequence*

$$0 \to A(\infty) \to B(\infty) \to \frac{B}{A}(\infty) \to 0$$

is a split short exact sequence of divisible modules.

Proof. Let λ be a limit ordinal greater than the value lengths of A and B. Then the above sequence is the same as

$$0 \to A(\lambda) \to B(\lambda) \to \frac{B}{A}(\lambda) \to 0$$

hence is short exact. Moreover, as $A(\infty) = A(\lambda+1) = p(A(\lambda)) = p(A(\infty))$, it follows that $A(\infty)$ is divisible, and similarly for B and B/A. ■

A **stable kernel** $A \subset B$ is a semistable kernel such that the associated cokernel $B \to B/A$ is also semistable. That is, the short exact sequence $A \subset B \to B/A$ is stable. Similarly for **stable cokernels**. We will characterize the stable cokernels, hence the stable exact sequences, without characterizing the semistable cokernels.

We say that a homomorphism $f : B \to C$ with kernel A is **isotop** if for each n there exist m such that either of the two equivalent conditions of Lemma 1 are met:

$$\begin{aligned} p^m C[p^n] &\subset f(B[p^n]) \\ A \cap p^{m+n} B &\subset p^n A. \end{aligned}$$

The name comes from the fact that the map f is isotop exactly when the p-adic topology on A is induced by the p-adic topology on B. This condition played a role as a condition in [5, Theorem 15]. Here it gives a sufficient condition for a cokernel to be semistable, that turns out to be necessary for stability.

Theorem 19. *Let $f : B \to C$ be a cokernel in the category of c-modules such that the restriction*

$$f : B(\alpha) \to C(\alpha)$$

is isotop for each limit ordinal α. Then f is a semistable cokernel.

Proof. Let

$$\begin{array}{ccc} B' & \xrightarrow{f'} & C' \\ \downarrow & & \downarrow g \\ B & \xrightarrow{f} & C \end{array}$$

be a pullback diagram, with $B' = \{(b, c') \in B \oplus C' : f(b) = g(c')\}$. We will show that

$$B'(\beta) = B' \cap (B(\beta) \oplus C'(\beta))$$

for each limit ordinal β. Thus f' is a cokernel because $f'(B'(\beta)) = C'(\beta)$ for each limit ordinal β, so $f'(B'(\alpha)) = C'(\alpha)$ for each ordinal α.

Induct on the limit ordinal β. Clearly $B'(\beta) \subset B' \cap (B(\beta) \oplus C'(\beta))$ so suppose $(b, c') \in B(\beta) \oplus C'(\beta)$ and $f(b) = g(c')$. We want to show that $(b, c') \in p^\omega B'(\alpha)$ for each limit ordinal $\alpha < \beta$, so it suffices to show that $(b, c') \in p^n B'(\alpha)$ for each positive integer n.

As f is isotop, there is m such that

$$p^m C(\alpha)[p^n] \subset f(B(\alpha)[p^n]).$$

Because $b \in B(\beta)$ and $c' \in C'(\beta)$, we can write $b = p^n b_0$ with $b_0 \in p^m B(\alpha)$, and $c' = p^n c'_0$ with $c'_0 \in p^m C'(\alpha)$. Then

$$p^n(f(b_0) - g(c'_0)) = f(b) - g(c') = 0 \quad \text{and} \quad f(b_0) - g(c'_0) \in p^m C(\alpha)$$

so there exists $b_1 \in B(\alpha)[p^n]$ such that $f(b_1) = f(b_0) - g(c'_0)$. Then $p^n(b_0 - b_1) = p^n b_0 = b$ and $f(b_0 - b_1) = g(c'_0)$. So $p^n(b_0 - b_1, c'_0) = (b, c')$ and

$$(b_0 - b_1, c'_0) \in B' \cap (B(\alpha) \oplus C'(\alpha))$$

which is equal to $B'(\alpha)$ by induction. ■

Corollary 20. *If $f : B \to C$ is a cokernel in the category of c-modules, with C bounded, then f is semistable. If, in addition, $B(\omega) = 0$, then f is stable.*

Corollary 21. *Any exact sequence of bounded modules is stable in the category of c-modules.*

The condition of Theorem 19 does not characterize semistable cokernels. Let B and C be R_{p^∞} with $B(\omega) = B[p]$ and $C(\omega) = 0$, and consider the map $B \xrightarrow{f} C$ which is multiplication by p. Clearly f is not isotop, but it is semistable because of the following theorem.

Theorem 22. *If B is a c-group, and $B(\omega)$ is of finite length, then the map $f : B \to C = B/B(\omega)$ is a semistable cokernel.*

Proof. Consider the pullback diagram

$$\begin{array}{ccccc} B(\omega) & \to & B' & \xrightarrow{f'} & C' \\ \| & & \downarrow & & \downarrow g \\ B(\omega) & \to & B & \xrightarrow{f} & C \end{array}$$

We first show that $f'(B'(\omega)) = C'(\omega)$, so $f'(B'(\omega + m)) = C'(\omega + m)$ for each positive integer m. Let $c' \in C'(\omega)$. Because $B(\omega)$ has finite length, $B(\omega)$ is nice as a submodule of B', so $f'(p^\omega B') = p^\omega C'$. Thus there exists $b \in B$ such that $(b, c') \in p^\omega B'$. As $f(b) = g(c') = 0$, we have $b \in B(\omega)$, so $(b, c') \in (B(\omega) \oplus C'(\omega)) \cap p^\omega B' = B'(\omega)$ and $f'(b, c') = c'$.

Now choose n so that $p^n B(\omega) = 0$. We claim that $B'(\omega + n) = C'(\omega + n)$ in $B \oplus C'$, so $f'(B'(\alpha)) = C'(\alpha)$ for each $\alpha \geq \omega + n$, and therefore the map f' is a cokernel. To see this, suppose $(b, c') \in B'(\omega + n)$. Then $(b, c') = p^n(b, c'_0)$ where $(b_0, c'_0) \in B'(\omega)$. As $f(b_0) = g(c'_0) = 0$, we have $b_0 \in B(\omega)$, so $b = p^n b_0 = 0$. Clearly $C'(\omega + n) \subset B'(\omega + n)$. ■

We want to show that the converse of Theorem 19 holds for *stable* cokernels. First we look at some conditions that assure that f is isotop. Note that $p(B[p^n]) = pB[p^{n-1}]$.

Lemma 23. *Let $A \subset B \xrightarrow{f} C$ be a short exact sequences of modules. Then f is isotop if and only if*

1. *$p^\omega C[p^n] \subset f(p^k B[p^n])$ for each k and n, and*
2. *if $K \subset C$ is a countable direct sum of torsion cyclics, then for each n there exists m such that $p^m K[p^n] \subset f(B[p^n])$.*

Proof. Suppose f is isotop. Then there exists m such that $p^m C[p^{n+k}] \subset f(B[p^{n+k}])$. Multiplying by p^k we get $p^{m+k}C[p^n] \subset f(p^k B[p^n])$, so (1) holds. Condition (2) holds for any submodule K of C: use the m such that $p^m C[p^n] \subset f(B[p^n])$.

Conversely, suppose (1) and (2) hold. We want to show that for each n there exists m such that $p^m C[p^n] \subset f(B[p^n])$. First we show that, for $n > 1$, there exists m such that if $c \in p^m C[p^n]$ and $p^{n-1}c \in p^\omega C$, then $c \in f(B[p^n])$. By induction on n there exists m such that

$$p^m C[p^{n-1}] \subset f(B[p^{n-1}]).$$

Now

$$p^{n-1}c \in p^\omega C[p] \subset f(p^{m+n-1}B[p])$$

by (1). So $p^{n-1}c = p^{n-1}f(p^m b)$ with $b \in B[p^{m+n}]$. Then

$$c - f(p^m b) \in p^m C[p^{n-1}] \subset f(B[p^{n-1}]),$$

and clearly $f(p^m b) \in f(B[p^n])$.

Suppose now, by way of contradiction, that $p^m C[p^n]$ is not contained in $f(B[p^n])$ for any m. Then we could construct $c_i \in p^i C[p^n]$ inductively, with $\operatorname{ht} p^{n-1}c_i < \operatorname{ht} p^{n-1}c_{i+1} < \omega$ for each i, and $c_i \notin f(B[p^n])$. This contradicts (2). ∎

We want to go from stability to condition (2) of Lemma 23. For $n \geq 1$ define $B_n = \bigoplus_{i \geq n} Rx_i$ where x_i has order p^i and let P_n be B_{n+1} with all the $p^{i-1}x_i$ identified. Then $p^\omega P_n$ is cyclic of order p, and there is a natural map π_n taking P_n onto B_n with kernel $p^\omega P_n$. We can think of B_1 as being the submodule $p^{n-1}B_n$ of B_n. Any automorphism of B_1 extends to an automorphism of B_n. The module B_n is the p^{n-1}-extension of B_1 and may be thought of as $\{x \in D : p^{n-1}x \in B_1\}$ where D is a divisible hull of B_1.

Lemma 24. *If $B_n[p^{n-1}] \subset S \subset B_n[p^n]$, and S does not contain $p^m B_n[p^n]$ for any m, then there is an automorphism θ of B_n such that $S \subset \theta\pi_n P_n[p^n]$.*

Proof. Consider the case $n = 1$, endowing $B_1[p]$ with the topology given by the submodules $p^m B_1[p]$. First we find a proper dense submodule D of $B_1[p]$ that contains S. Let $\overline{S}$ be the closure of S in $B_1[p]$. As $B_1[p]$ is countably generated, $B_1[p] = \overline{S} \oplus T$ as a valuated vector space. If $S \neq \overline{S}$, then set $D = S + T$. If $S = \overline{S}$, then T cannot be finitely generated because S contains no $p^m B_1[p]$. Thus T contains a proper dense submodule T'. In this case set $D = S + T'$.

Let $e_i = p^{i-1}x_i$, so $e_1, e_2, \ldots$ is the standard basis for $B_1[p]$. There is a nonzero linear functional φ on $B_1[p]$ with $\varphi(D) = 0$. Because D is dense and proper, it cannot contain $p^m B_1[p]$ for any m, so the set $K = \{k : \varphi(e_k) \neq 0\}$ is infinite. For each k in K, choose a unit $u_k \in R$ such that $\varphi(u_k e_k) = 1$. Define an automorphism θ of B_1 by setting $\theta(x_k) = u_k x_k$ if $k \in K$, and $\theta(x_j) = x_j + p^{k-j}u_k x_k$ if $j \notin K$, and k is the smallest element of K bigger than j.

We see that $\varphi(\theta e_j) = 1$ for all j, so $\varphi(\theta\pi_1 P_1[p]) = 0$ as $\pi_1 P_1[p]$ is generated by the elements $e_i - e_j$. But $\pi_1 P_1[p]$ has codimension 1 in $B_1[p]$, so $\operatorname{Ker}\varphi = \theta\pi_1 P_1[p]$, whence $D \subset \theta\pi_1 P_1[p]$.

For $n > 1$ note that $p^{n-1}S \subset B_1[p]$, and that if $p^{n-1}S \supset p^{m+n-1}B_1[p]$, then

$$S = S + B[p^{n-1}] \supset p^m B_1[p^n]$$

which is not so. So the case $n = 1$ obtains for $p^{n-1}S$, so there is an automorphism θ_1 of B_1 such that $p^{n-1}S \subset \theta_1\pi_1(P_1[p])$. Extend θ_1 to an automorphism θ of B_n, and consider the commutative diagram.

$$\begin{array}{ccccc} P_n & \xrightarrow{\pi_n} & B_n & \xrightarrow{\theta} & B_n \\ \downarrow & & \downarrow & & \downarrow \\ P_1 & \xrightarrow{\pi_1} & B_1 & \xrightarrow{\theta_1} & B_1 \end{array}$$

where the maps down are multiplication by p^{n-1}. Because $p \ker \pi_n = 0$, we have $\pi_n P_n[p^n] \supset B[p^{n-1}]$, so, as

$$p^{n-1}\theta\pi_n P_n[p^n] = \theta_1\pi_1 P_1[p] \supset p^{n-1}S$$

we have $S \subset \theta\pi_n(P_n[p^n])$. ■

We now derive a couple of consequences of the stability of a short exact sequence. The first is a bit technical.

Lemma 25. *Let $A \subset B \xrightarrow{f} C$ be a stable short exact sequence in the category of c-modules, and $g : C' \to C$. Let α be an ordinal, and n and k positive integers. If $x \in C'(\alpha + \omega)$ and $g(x) = 0$, then there exists y in $C'(\alpha + k)$ such that $p^n y = x$ and $g(y) \in f(B(\alpha + k)[p^n])$.*

Proof. Consider the pullback diagram

$$\begin{array}{ccccc} A & \subset & B' & \to & C' \\ \| & & \downarrow & & \downarrow g \\ A & \subset & B & \xrightarrow{f} & C \end{array}$$

with $B' = \{(b, c') \in B \oplus C' : f(b) = g(c')\}$. As f is a semistable cokernel, $B'(\alpha + \omega)$ maps onto $C'(\alpha + \omega)$, so there is an element $(a, x) \in B'(\alpha + \omega) \subset p^{k+n}B'(\alpha)$. Thus $(a, x) = p^n(b, y)$ with y in $C'(\alpha + k)$ and $b \in B(\alpha + k)$. As $g(x) = 0$, we have $a \in A$, and since $A \subset B$ is a semistable kernel, $a \in A(\alpha+\omega)$, so $a = p^n a'$ with $a' \in A(\alpha+k)$. Thus $(0, x) = p^n(b - a', y)$ and $f(b - a') = g(y)$. But $b - a' \in B(\alpha + k)[p^n]$. ∎

Lemma 26. *Let $A \subset B \xrightarrow{f} C$ be a stable short exact sequence in the category of c-modules, α an ordinal, and k and n nonnegative integers. If $K \subset C(\alpha)$ is a direct sum of torsion cyclics, then there exists m such that $p^m K[p^n] \subset f(B(\alpha + k)[p^n])$.*

Proof. If K is bounded, then the conclusion clearly holds. We may assume that K is *standard*: one cyclic summand of each length. Indeed, if the conclusion fails for each m, then it fails on a standard submodule of K.

By induction on n, there exists m' such that

$$p^{m'}K[p^{n-1}] \subset f(B(\alpha + k)[p^{n-1}]).$$

We may drop the cyclics of lengths less than $m' + n$ from K, without affecting the conclusion of the theorem. Thus we may assume that $p^{m'}K[p^{n-1}] = K[p^{n-1}]$ so

$$K[p^{n-1}] \subset f(B(\alpha + k)[p^n]).$$

Suppose, by way of contradiction, that $p^m K[p^n]$ is not contained in $f(B(\alpha+k)[p^n])$ for any m. Then Lemma 24 gives a homomorphism g of P_n onto K, with kernel $p^\omega P_n = \langle x\rangle$, so that $f(B(\alpha + k)[p^n]) \cap K \subset g(P_n[p^n])$. Write P_n as $S(F)$ for a forest F, and extend F to a forest F' such that $F = p^\alpha F'$. Let $C' = S(F')$. Then g is an o-module map from $P_n = C'(\alpha)$, to the c-module C. We can extend g from F to F', because C is a c-module, and so to an o-module map $C' \to C$, By Lemma 25 there exists $y \in C'(\alpha + k) \subset P_n$ such that $p^n y = x$ and $g(y) \in f(B(\alpha + k)[p^n])$, so $g(y) \in g(P[p^n])$. But if $g(y) = g(t)$, and $p^n t = 0$, then $y - t \in \langle x\rangle$ so $p^n y = 0$, a contradiction. ∎

Theorem 27. *Let $A \subset B \xrightarrow{f} C$ be a stable short exact sequence in the category of c-modules, and α a limit ordinal. Then the restriction $f : B(\alpha) \to C(\alpha)$ is isotop.*

Proof. We first show that $p^\omega C(\alpha)[p^n] \subset f(B(\alpha+k)[p^n])$ for all n and k, which is condition (1) of Lemma 23. Suppose the torsion submodule of $C(\alpha)$ has infinite final rank, and $c \in p^\omega C(\alpha)[p^n]$. Then there exists a sequence c_i of independent elements of $C(\alpha)[p^n]$ such that $c_i \in C(\alpha+i)$ has order p^n, and $\langle c\rangle \cap \langle c_0, c_1, \ldots\rangle = 0$. Construct a direct sum $K = \sum Re_i$ of torsion cyclics in $C(\alpha)$ such that $p^i e_i = c_i$. By Lemma 26, the sequence c_i is eventually in $f(B(\alpha+k)[p^n])$. The sequence $c_i + c$ is such another such sequence, and we can construct another direct sum $K' = \sum Re'_i$ of torsion cyclics in $C(\alpha)$ so that $p^i e'_i = c_i + c$. Then $c \in f(B(\alpha+k)[p^n])$ by Lemma 26. That is, $p^\omega C(\alpha)[p^n] \subset f(B(\alpha+k)[p^n])$.

Now suppose the torsion submodule of $C(\alpha)$ has finite final rank, so it is bounded plus finite-rank divisible. This implies that $p^\omega C(\alpha)[p^n] = p^\infty C[p^n]$. We will show that $p^\infty C(\alpha)[p^n] \subset f(B(\alpha+k)[p^n])$. To do this we let C' be R_{p^∞} with c-module structure given by $C'(\alpha+\omega) = C'[p]$. Let z generate $C'[p^{n+1}]$. Given $c \in p^\infty C[p^n]$, let $g : C' \to C$ with $g(z) = c$. It follows from Lemma 25, with $x = p^n z$, that $c \in f(B(\alpha+k)[p^n])$ (the y you get there is a unit times z). That completes the proof that $p^\omega C(\alpha)[p^n] \subset f(B(\alpha+k)[p^n])$ for all n and k.

The result now follows from Lemmas 26 and 23. ■

Corollary 28. *A short exact sequence $A \subset B \xrightarrow{f} C$ in the category of c-modules is stable if and only if $A(\alpha) = A \cap B(\alpha)$, and the restriction $f : B(\alpha) \to C(\alpha)$ is isotop, for each limit ordinal α.*

7. Projective and injective c-modules

The full characterization of stability in the category of c-modules is not needed to describe the projectives and injectives.

Theorem 29. *A c-module is projective in the category of c-modules exactly when it is the direct sum of a free module and a divisible module.*

Proof. Clearly free modules are projective in the category of c-modules. To see that divisible modules are projective, let $A \subset B \to C$ be a stable exact sequence of c-modules. Then $A(\infty) \subset B(\infty) \to C(\infty)$ is a split exact sequence (Corollary 18). As any map from a divisible module into C goes into $C(\infty)$, it follows that divisible modules are projective.

For the converse, we must show that any value-reduced projective c-module A is a free module. By Theorems 6, 17 and 19, A is a stable quotient of a module, so A is a summand of a module, hence a module. The torsion submodule of A must be zero for otherwise A would have a torsion cyclic summand, and torsion cyclics are not projective (Corollary 21). So A is a reduced torsion-free module, whence $A(\omega) = 0$. Any map from a free module onto A is pure, hence a stable cokernel by Theorems 17 and 19, so A is free. ■

Note that there are not enough projectives in the category of c-modules. In fact, C is the quotient of a c-projective exactly when $C(\omega) = C(\infty)$.

What are the c-injectives?

Theorem 30. *If D is a divisible o-module such that either $D(\infty) = D$, or there is a limit ordinal α such that $D(\alpha) = 0$ and $D(\alpha^-) = D$, then D is injective in the category of c-modules.*

Proof. Let $A \subset B$ be a semistable kernel of c-modules, and $f : A \to D$. If $D(\infty) = D$, we can extend f to a homomorphism $B \to D$ because D is divisible, and the result will be a map of o-modules. In the second case, $f(A(\alpha)) = 0$. As $A \subset B$ is semistable, $A(\alpha) = A \cap B(\alpha)$. As D is divisible, we can extend f to a homomorphism from B to D that kills $B(\alpha)$. Because $D(\alpha^-) = D$, this map is a map of o-modules. ■

Call the c-modules of Theorem 30 **elementary c-injectives of type** α. We will show that the c-injectives are products of elementary c-injectives.

Theorem 31. *If G is an injective in the category of c-modules, then $G(\beta - 1)$ is divisible for each nonzero ordinal β (including limit ordinals).*

Proof. Let $\beta = \alpha + n$ for α a limit ordinal. For $\alpha = 0$ (so $n > 0$), consider the inclusion $pR \subset R$ with the height valuation on both. This is a stable kernel because $R(\omega) = 0$ and R/pR is bounded (Corollary 20). We can map $p \in pR$ to any element of G, so when we extend to R we see that $G = pG$.

Now suppose α is a nonzero limit ordinal, and $n > 0$. Set $A = S(F_{\alpha,\omega})$. Let F be the forest obtained by taking $F_{\alpha,\omega}$ and $F_{\alpha,1}$ and letting the (root) node of height α in $F_{\alpha,1}$ be another child of the node x of height α in $F_{\alpha,\omega}$. This has the effect of giving the node $p^i x$ height $\alpha + i + 1$ in F for each i. Set $B = S(F)$. It is not hard to see that $A(\gamma) = A \cap B(\gamma)$ for each ordinal $\gamma < \alpha$, so the inclusion $A(\gamma) \subset B(\gamma)$ is pure. As $(B/A)(\alpha)$ is bounded, the inclusion $A \subset B$ is stable. As G is a c-module, any element of $G(\alpha)$ is the image of the node x of value α in A under some map $A \to G$, so any element of $G(\beta - 1)$ is the image of $p^{n-1}x$, hence is in $G(\beta)$, as $p^{n-1}x$ has value β in B.

If $n = 0$, construct A and B as above, but value each $p^i x$, and the root of $F_{\alpha,1}$, with α^-. Now any element of $G(\alpha^-)$ is the image of x under some map $A \to G$, hence is in $pG(\alpha^-)$ because $x \in pB(\alpha^-)$. ■

Corollary 32. *If G is injective in the category of c-modules, then $G(\beta - 1)$ is a divisible summand for each nonzero ordinal β.*

Proof. We can write $G = G(\beta - 1) \oplus K$ as a module. But as $G(\beta - 1)$ is comparable to $G(\alpha)$ for each α, this is a direct sum in the category of o-modules. ■

Theorem 33. *Every c-injective is a product of elementary c-injectives.*

Proof. Let A be a c-injective. As $A(\infty) = A(\lambda)$ for sufficiently large λ, we can split off $A(\infty)$ by Corollary 32, so we may assume that A is reduced. Set $A_\alpha = A((\omega\alpha)^-)$ for each ordinal α. From Corollary 32, each A_α is a summand of A, hence a c-injective and therefore an absolute summand of A. Thus A satisfies the hypotheses of Theorem 14, so A is isomorphic to the product $\prod A_\alpha / A_{\alpha+1}$. Clearly $A_\alpha / A_{\alpha+1}$ is an elementary c-injective of type $\omega\alpha$. ■

There are not enough injectives in the category of c-modules. If $A \subset B$ is a stable kernel, and B is a c-injective, then $A(\alpha)$ is divisible for each limit ordinal α.

References

[1] Fuchs, László, Extension of isomorphisms between subgroups. *Abelian group theory* (Oberwolfach, 1981), pp. 289–296, Lecture Notes in Math., **874**, Springer, Berlin-New York, 1981. MR 83k:20065

[2] _______________, Vector spaces with valuations, *J. Algebra*, **35**(1975), 23–38. MR 51 #8212

[3] Hill, Paul, Criteria for freeness in groups and valuated vector spaces, *Abelian group theory*, Lecture Notes in Mathematics, **616**(1977), Springer-Verlag, 140–157. MR 58 #5978

[4] Hill, Paul and Charles Megibben, On the theory and classification of abelian p-groups. *Math. Z.* **190**(1985), no. 1, 17–38. MR 86k:20049

[5] Richman, Fred and Elbert A. Walker, Ext in pre-abelian categories, *Pac. J. Math.*, **71**(1977), 521–535. MR 56 #3091

[6] ______________________________, Valuated groups, *Journal of Algebra*, **56**(1979), 145–167.MR 80k:20053

[7] Stanton, Robert O., Semistable kernels of valuated groups. *Rocky Mountain J. Math.* **10**(1980), no. 3, 629–635. MR 81j:20079

[8] Walker, Carol L. and Robert B. Warfield, Jr., Unique decomposition and isomorphic refinement theorems in additive categories, *J. Pure Appl. Alg.*, **7**(1976), 347–359. MR 54 #2756